Identification of conductivity imperfections of small diameter by boundary measurements. Continuous dependence and computational reconstruction.

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Abstract

We derive an asymptotic formula for the steady state voltage potential in the presence of a finite number of diametrically small inhomogeneities with conductivity different from the background conductivity. We use this formula to establish continuous dependence estimates and to design an effective computational identification procedure.

1 Introduction

The non-destructive inspection technique known as electrical impedance imaging has recently received considerable attention in the mathematical- as well as in the engineering literature [2, 4, 10, 14, 17]. Using this technique one seeks to determine information about the internal conductivity- (or impedance) profile of an object based on boundary information about the applied steady-state currents and corresponding voltage potentials. The goal could well be to image an entirely unknown internal conductivity profile, but frequently it may be somewhat more limited in scope: a priori one has some knowledge of the overall form of the conductivity profile and one then seeks to determine very specific

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features. Examples of the latter type are found in connection with the identification of cracks [12, 15] and the identification of one or more inhomogeneities [1, 6, 7, 8]. Following this line of investigation the purpose of the present paper is to design an efficient method to determine the location and size of diametrically small conductivity imperfections inside a conductor of known background conductivity. The imperfections (inhomogeneities) have constant conductivities. These conductivies may be known or unknown, depending on the application. Unlike the inhomogeneities treated in [1, 6, 8] the ones considered here are of small size, and this shall allow for the design of a very effective identification procedure. The identification of small inhomogeneities has previously been analysed in [7], but whereas those inhomogeneities were either perfectly insulating (voids) or perfectly conducting (merely a dual) the ones considered here are just required to have a (finite) conductivity different from the background conductivity.

The fundamental step in the design of our identification procedure is the derivation of an asymptotic formula for the steady-state voltage potential for a conductor with a finite number of well separated, small inhomogeneities. In one particular version this asymptotic formula states that the voltage potential has the form

$$\begin{aligned} u(y) = \epsilon^n \ \Sigma_{i=1}^m \frac{\gamma(z_i)(\gamma(z_i)-k_i)}{k_i} \nabla_x N(z_i,y) \cdot M_i \nabla_x U(z_i) \\ + \int_{\partial\Omega} \psi(x) N(x,y)\, d\sigma_x + O(\epsilon^{n+\frac{1}{2}}) \quad . \end{aligned} \tag{1}$$

Here Ω is the bounded $n-$dimensional domain occupied by the conductor, m is the number of inhomogeneities, z_i, $1 \le i \le m$, represents the "centers" of the inhomogeneities (which are not necessarily balls) and ϵ is the common order of magnitude of their "diameters". The function γ is the background conductivity and k_i is the (constant) conductivity of the i^{th} inhomogeneity. $N(x,y)$ is the Neumann function corresponding to the domain Ω, ψ is the applied boundary current, and U is the voltage potential corresponding to the background conductivity γ. Indeed with this notation

$$U(y) = \int_{\partial\Omega} \psi(x) N(x,y)\, d\sigma_x \quad .$$

M_i (a symmetric, positive definite $n \times n$ matrix) is a "polarization" tensor associated with the i^{th} inhomogeneity. Finally, the notation $O(\epsilon^{n+\frac{1}{2}})$ signifies a term which goes to zero like $\epsilon^{n+\frac{1}{2}}$, uniformly in y when y is bounded away from the inhomogeneities. For our identification procedure we shall only make use of the asymptotic representation formula (1) for $y \in \partial\Omega$. Two special cases of the formula (1) corresponding to inhomogeneities of conductivity 0 and of conductivity ∞ were previously derived in [7]. The analysis leading to the general formula is considerably different from that used in those two special cases. The two special formulas may be recovered by letting k_i tend to 0 and ∞ respectively in the general formula derived here. At this point it should be noted that the

the "polarization" tensor M_i in addition to the shape of the i^{th} inhomogeneity also depends on the conductivity ratio $\gamma(z_i)/k_i$. The formula (1) asserts that the expression

$$\epsilon^n \ \Sigma_{i=1}^m \frac{\gamma(z_i)(\gamma(z_i)-k_i)}{k_i} \nabla_x N(z_i,y) \cdot M_i \nabla_x U(z_i) \\ + \int_{\partial\Omega} \psi(x) N(x,y) \, d\sigma_x \tag{2}$$

is a good approximation to the voltage potential for inhomogeneities that are well separated and for ϵ that are sufficiently small; however, numerical evidence strongly indicate that it is also a quite reasonable approximation even for inhomogeneities that are relatively close and for ϵ that are fairly large.

The asymptotic formula (1) and a slight variation thereof are used for two different purposes. (a): Given two different sets of inhomogeneities such a formula allows us estimate the difference in the location and relative size of the inhomogeneities in terms of the difference of the two corresponding boundary voltage potentials (the two prescribed currents are the same). This estimate is similar to one derived earlier for the case of inhomogeneities of extreme conductivity. (b): Such a formula suggests a very effective procedure to identify, for instance, the location and the size of a finite number of inhomogeneities based on knowledge of the boundary voltage potential. Namely, to select the parameters of an approximate expression like (2) so that it best fits the the (measured) boundary voltage potential. We construct a procedure to obtain a (nearly) best fit by employing a fairly straightforward least-squares approach, and we demonstrate by numerical examples the viability of this procedure. If the size and the location of the inhomogeities were known, then it is natural to try to use an approximate expression, such as (2), to reconstruct the conductivities k_i. We are currently undertaking work to develop an effective procedure for this purpose.

2 The steady state electric problem

In the introduction we have briefly made reference to some of the ingredients of the steady state conductivity problem. In this section we shall provide more concise definitions. We suppose the conducting component occupies a bounded, smooth subdomain of $\Re^n$, $n = 2$ or 3. For simplicity we take $\partial\Omega$ to be C^∞, but this assumption could be considerably weakened. Let $\gamma(\cdot)$ denote the smooth background conductivity, that is the conductivity in the absence of any inhomogeneities. We suppose that

$$0 < c_0 \le \gamma(x) \le C_0 < \infty, \quad x \in \Omega$$

for some fixed constants c_0 and C_0. For simplicity, we assume that γ is $C^\infty(\bar{\Omega})$, but this latter assumption could be considerably weakened. The function ψ denotes the imposed boundary current. It suffices that $\psi \in H^{1/2}(\partial\Omega)$, with

$\int_{\partial\Omega} \psi \ d\sigma \ = 0$. The background voltage potential, U, is the solution to the boundary value problem

$$\begin{aligned} \nabla \cdot (\gamma(x) \nabla U) = 0 \quad \text{in } \Omega \ , \\ \gamma \frac{\partial U}{\partial \nu} = \psi \ \text{ on } \partial\Omega \quad . \end{aligned} \tag{3}$$

Here ν denotes the unit outward normal to the domain Ω. Let m denote the number of inhomogeneities and suppose that each inhomogeneity has the form $z_i + \epsilon B_i$, where $B_i \subset \Re^n$ is a bounded, smooth domain containing the origin. We assume that each B_i is strictly star-shaped (meaning there exists $y_0 \in B_i$ such that $(y - y_0) \cdot \nu > 0$ for $y \in \partial B_i$, ν denoting the outer normal to B_i). For simplicity we assume that B_i is a C^∞ domain, but this assumption could be considerably weakened. The points $z_i \in \Omega$, $i = 1, ...m$, determine the location of the inhomogeneities; we shall assume they satisfy

$$\begin{aligned} 0 < d_0 \leq |z_i - z_j|, \ \ \forall \ i \neq j \\ 0 < d_0 \leq dist(z_i, \partial\Omega), \ \ \forall i \quad . \end{aligned} \tag{4}$$

We also assume that ϵ, the common order of magnitude of the diameters of the inhomogeneities, is sufficiently small so that the inhomogeneities are disjoint and their distance to $\Re^n \backslash \Omega$ is larger than $d_0/2$.

Let $\hat{\gamma}_\epsilon$ denote the conductivity profile in the presence of the small inhomogeneities. The function $\hat{\gamma}_\epsilon$ is equal to γ except on the inhomogeneities; on the i^{th} inhomogeneity, $z_i + \epsilon B_i$, we have $\hat{\gamma}_\epsilon = k_i$, where k_i , $i = 1 \ldots m$, is a set of positive constants. Let

$$\omega_\epsilon = \bigcup_{i=1}^{m} (z_i + \epsilon B_i)$$

denote the total collection of inhomogeneities. With this notation we have

$$\hat{\gamma}_\epsilon(x) = \begin{cases} \gamma(x), & x \in \Omega \setminus \bar{\omega}_\epsilon \\ k_i, & x \in z_i + \epsilon B_i, \ i = 1 \ldots m \end{cases} \tag{5}$$

The voltage potential in the presence of the inhomogeneities is denoted $u_\epsilon(x)$. It is the solution to

$$\begin{aligned} \nabla \cdot (\hat{\gamma}_\epsilon(x) \nabla u_\epsilon) = 0 \quad \text{in } \Omega \ , \\ \hat{\gamma}_\epsilon \frac{\partial u_\epsilon}{\partial \nu} = \psi \ \text{ on } \partial\Omega \quad . \end{aligned} \tag{6}$$

We normalize both U and u_ϵ by requiring that

$$\int_{\partial\Omega} U d\sigma = 0 \ , \quad \text{and} \quad \int_{\partial\Omega} u_\epsilon d\sigma = 0 \quad .$$

To ensure the validity of our continuous dependence results and to guarantee the success of our identification procedure it becomes necessary to assume that the following non-degeneracy conditions hold:

$$\nabla_x U(x) \neq 0 \quad \forall x \in \Omega \ , \quad \text{and} \quad \gamma(z_i) \neq k_i \quad i = 1, ..., m \ .$$

These conditions are necessary and sufficient to guarantee that each of the terms

$$\frac{\gamma(z_i)(\gamma(z_i) - k_i)}{k_i} \nabla_x N(z_i, y) \cdot M_i \nabla_x U(z_i)$$

from the expression (2) is nontrivial. If this were not true then any continuous dependence result for the i^{th} inhomogeneity would depend on a higher order term in the asymptotic expansion of u_ϵ, and it would not be as strong. Similarly the ability to effectively identify the i^{th} inhomogeneity would depend on finding the explicit form of this (higher order, nontrivial) term. As a result the identification would be considerably more complicated and not nearly as accurate.

In the computational identification experiments we present at the end of this paper, the inhomogeneities are assumed to be of the form $z_i + \epsilon\rho_i B$ or $z_i + \epsilon Q_i B$ (or even $z_i + \epsilon\rho_i Q_i B$) for a common, known domain B, but for unknown locations, z_i, dilatation parameters, $\epsilon\rho_i > 0$, and rotations Q_i. The conductivities k_i are also assumed to be known, and we seek to identify specific values of the unknown parameters associated with the inhomogeneities.

3 An energy estimate

We start the derivation of the asymptotic formula for u_ϵ with the following estimate of the $H^1(\Omega)$ norm of $U - u_\epsilon$.

Lemma 1 . *There exists a constant C, independent of ϵ, such that*

$$\int_\Omega (|\nabla(U - u_\epsilon)|^2 + |U - u_\epsilon|^2) \, dx \leq C\epsilon^n. \tag{7}$$

Proof . Since $\int_\Omega |U - u_\epsilon|^2 \, dx \ \leq \ C(\int_\Omega |\nabla(U - u_\epsilon)|^2 \, dx + |\int_{\partial\Omega}(U - u_\epsilon) \, d\sigma|^2) = C \int_\Omega |\nabla(U - u_\epsilon)|^2 \, dx$, it suffices to show that

$$\int_\Omega |\nabla(U - u_\epsilon)|^2 \, dx \leq C\epsilon^n.$$

For $v \in H^1(\Omega)$, let $E_\epsilon(v)$ be the energy defined by

$$E_\epsilon(v) = \frac{1}{2} \int_\Omega \hat{\gamma}_\epsilon |\nabla v|^2 \, dx - \int_{\partial\Omega} \psi v \, d\sigma \quad .$$

Then, since $\int_\Omega \gamma |\nabla U|^2 \, dx = \int_{\partial\Omega} \psi U \, d\sigma$ and $\hat{\gamma}_\epsilon = \gamma$ on $\Omega \setminus \omega_\epsilon$,

$$\begin{aligned} E_\epsilon(U) &= \frac{1}{2} \int_\Omega \hat{\gamma}_\epsilon |\nabla U|^2 \, dx - \int_{\partial\Omega} \psi U \, d\sigma \\ &= -\frac{1}{2} \int_\Omega \gamma |\nabla U|^2 \, dx + \frac{1}{2} \int_{\omega_\epsilon} (\hat{\gamma}_\epsilon - \gamma) |\nabla U|^2 \, dx \quad . \end{aligned}$$

Expanding, integrating by parts, and using the formula for $E_\epsilon(U)$ given above, we see that

$$\begin{aligned}
\int_\Omega \hat{\gamma}_\epsilon |\nabla(U-u_\epsilon)|^2\,dx &= \int_\Omega \hat{\gamma}_\epsilon |\nabla U|^2\,dx - 2\int_\Omega \hat{\gamma}_\epsilon \nabla u_\epsilon \nabla U\,dx + \int_\Omega \hat{\gamma}_\epsilon |\nabla u_\epsilon|^2\,dx \\
&= \int_\Omega \hat{\gamma}_\epsilon |\nabla U|^2\,dx - 2\int_{\partial\Omega} \psi U\,d\sigma + \int_\Omega \hat{\gamma}_\epsilon |\nabla u_\epsilon|^2\,dx \\
&= 2E_\epsilon(U) + \int_\Omega \hat{\gamma}_\epsilon |\nabla u_\epsilon|^2\,dx \\
&= -\int_\Omega \gamma |\nabla U|^2\,dx + \int_{\omega_\epsilon} (\hat{\gamma}_\epsilon - \gamma)|\nabla U|^2\,dx \qquad (8) \\
&\qquad + \int_\Omega \hat{\gamma}_\epsilon |\nabla u_\epsilon|^2\,dx\ .
\end{aligned}$$

We note that $\int_\Omega \hat{\gamma}_\epsilon |\nabla u_\epsilon|^2\,dx = \int_\Omega \gamma \nabla U \nabla u_\epsilon\,dx$ and $\int_\Omega \gamma |\nabla U|^2\,dx = \int_\Omega \hat{\gamma}_\epsilon \nabla u_\epsilon \nabla U\,dx$, which together yield

$$\int_\Omega \hat{\gamma}_\epsilon |\nabla u_\epsilon|^2\,dx - \int_\Omega \gamma |\nabla U|^2\,dx = \int_{\omega_\epsilon} (\gamma - \hat{\gamma}_\epsilon)\nabla u_\epsilon \nabla U\,dx \quad . \tag{9}$$

Substituting (9) into (8) and simplifying we obtain

$$\begin{aligned}
\int_\Omega \hat{\gamma}_\epsilon |\nabla(U-u_\epsilon)|^2\,dx &= \int_{\omega_\epsilon} (\hat{\gamma}_\epsilon - \gamma)(\nabla U - \nabla u_\epsilon)\nabla U\,dx \\
&\le \Big(\int_{\omega_\epsilon} |\nabla(U-u_\epsilon)|^2\,dx\Big)^{\frac{1}{2}} \Big(\int_{\omega_\epsilon} (\hat{\gamma}_\epsilon - \gamma)^2 |\nabla U|^2\,dx\Big)^{\frac{1}{2}} \quad .
\end{aligned}$$

Thus, since $\hat{\gamma}_\epsilon$ is positive and uniformly bounded away from zero, and since γ and $\hat{\gamma}_\epsilon$ are both uniformly bounded, it follows that

$$\begin{aligned}
\int_\Omega |\nabla(U-u_\epsilon)|^2\,dx &\le C\int_\Omega \hat{\gamma}_\epsilon |\nabla(U-u_\epsilon)|^2\,dx \\
&\le C\Big(\int_\Omega |\nabla(U-u_\epsilon)|^2\,dx\Big)^{\frac{1}{2}} \Big(\int_{\omega_\epsilon} |\nabla U|^2\,dx\Big)^{\frac{1}{2}} \quad .
\end{aligned}$$

Combining this last estimate with the fact that $|\nabla U|$ is uniformly bounded on ω_ϵ (due to elliptic regularity) we finally obtain

$$\Big(\int_\Omega |\nabla(U-u_\epsilon)|^2\,dx\Big)^{\frac{1}{2}} \le C\Big(\int_{\omega_\epsilon} |\nabla U|^2\,dx\Big)^{\frac{1}{2}} \le C\epsilon^{\frac{n}{2}} \quad ,$$

from which the lemma immediately follows.

□

4 Some additional preliminary estimates

We begin by making the change of variables,

$$y = x/\epsilon$$

and restating the energy estimate from the last section in terms of y. Let

$$\tilde{\Omega} = \left\{ \frac{x}{\epsilon} : x \in \Omega \right\}$$

be the corresponding rescaled domain. From Lemma 1 it follows that,

$$\|u_\epsilon(\epsilon y) - U(\epsilon y)\|_{L^2(\tilde{\Omega})} \leq C, \tag{10}$$

$$\|\nabla_y (u_\epsilon(\epsilon y) - U(\epsilon y))\|_{L^2(\tilde{\Omega})} \leq C\epsilon, \tag{11}$$

for some constant C, independent of ϵ. To obtain a higher order estimate, we need to subtract another function from $u_\epsilon - U$.

For the remainder of this section we consider the case of a single inhomogeneity ϵB, which for simplicity we have assumed to be "centered" at the origin, with conductivity k. Let $v_\epsilon(y)$ denote the unique solution to

$$\begin{gathered} \Delta_y v_\epsilon = 0 \quad \text{in} \quad B \ , \qquad \nabla_y \cdot \gamma(\epsilon y) \nabla_y v_\epsilon = 0 \quad \text{in} \quad \tilde{\Omega} \setminus \bar{B} \\ v_\epsilon(y) \text{ is continuous across } \partial B \\ \gamma(\epsilon y) \frac{\partial v_\epsilon}{\partial \nu_y}^{+} - k \frac{\partial v_\epsilon}{\partial \nu_y}^{-} = -(\gamma(0) - k) \nabla_x U(0) \cdot \nu \quad \text{on} \quad \partial B \\ \gamma(\epsilon y) \frac{\partial v_\epsilon}{\partial \nu_y} = 0 \quad \text{on} \quad \partial \tilde{\Omega} \ , \end{gathered} \tag{12}$$

with $\int_{\partial \tilde{\Omega}} v_\epsilon \, d\sigma = 0$. The notation ν (or frequently ν_y) is used for the outward unit normal to both $\tilde{\Omega}$ and B. The existence and uniqueness of v_ϵ is most easily established by variational means. Elliptic regularity estimates then guarantee that it is indeed a classical solution. Let $v(y)$ denote a solution to

$$\begin{gathered} \Delta_y v = 0 \quad \text{in} \quad B \ , \qquad \Delta_y v = 0 \quad \text{in} \quad \Re^n \setminus \bar{B} \\ v \text{ is continuous across } \partial B \\ \gamma(0) \frac{\partial v}{\partial \nu_y}^{+} - k \frac{\partial v}{\partial \nu_y}^{-} = -(\gamma(0) - k) \nabla_x U(0) \cdot \nu \quad \text{on} \quad \partial B \\ \lim_{|y| \to \infty} v(y) = 0 \ . \end{gathered} \tag{13}$$

The existence of v is most easily established by representing it as a single layer potential on ∂B for a suitably chosen smooth density (that solves an associated boundary integral equation, cf. [5] or [11]). A single layer potential always vanishes at infinity for $n \geq 3$, for $n = 2$ the vanishing at infinity of this particular representation relies on the fact that $\int_{\partial B} \nabla_x U(0) \cdot \nu_y \, d\sigma_y = 0$. A maximum principle argument immediately gives that the solution to (13) is unique. We shall use the fact (cf. [5]) that the function v satisfies the following bounds (decay estimates) at infinity

$$v(y) = \begin{cases} O(|y|^{-1}), & n = 2 \\ O(|y|^{2-n}), & n \geq 3 \end{cases} \tag{14}$$

and

$$\nabla_y v(y) = \begin{cases} O(|y|^{-2}), & n = 2 \\ O(|y|^{1-n}), & n \geq 3. \end{cases} \tag{15}$$

This decay ensures that for any n,

$$\|v\|_{L^2(\tilde{\Omega})} \leq C/\epsilon^{1/2}. \tag{16}$$

We now proceed with the analysis which eventually leads to an estimate for $\nabla_y(u_\epsilon(\epsilon y) - U(\epsilon y) - \epsilon v(y))$ (Theorem 1) an estimate that plays a crucial role in the derivation of our representation formula. The proof of Theorem 1 we provide holds for two and three dimensions, but with minimal changes it could be extended to higher dimensions as well. We shall later point out the single place where the fact that the dimension is either two or three is used. We first prove the following lemma concerning $\nabla_y(u_\epsilon(\epsilon y) - U(\epsilon y) - \epsilon v_\epsilon(y))$

Lemma 2. *There exists a constant C, independent of ϵ, such that*

$$\|\nabla_y(u_\epsilon(\epsilon y) - U(\epsilon y) - \epsilon v_\epsilon(y))\|_{L^2(\tilde{\Omega})} \leq C\epsilon^2.$$

Proof. Define

$$z_\epsilon(y) = u_\epsilon(\epsilon y) - U(\epsilon y) - \epsilon v_\epsilon(y) - c_\epsilon$$

where the constant c_ϵ is chosen so that

$$\int_{\partial B} z_\epsilon \, d\sigma_y = 0.$$

Using the equations (3), (6) and (12) we compute that z_ϵ solves

$$\Delta_y z_\epsilon = -\Delta_y U(\epsilon y) \quad \text{in} \quad B \quad , \qquad \nabla_y \cdot \gamma(\epsilon y) \nabla_y z_\epsilon = 0 \quad \text{in} \quad \tilde{\Omega} \setminus \bar{B}$$

$$z_\epsilon \text{ is continuous across } \partial B \ , \quad \text{and} \quad \gamma(\epsilon y) \frac{\partial z_\epsilon}{\partial \nu_y} = 0 \quad \text{on} \quad \partial\tilde{\Omega} \ ,$$

along with the jump condition

$$\gamma(\epsilon y) \frac{\partial z_\epsilon}{\partial \nu_y}^{+} - k \frac{\partial z_\epsilon}{\partial \nu_y}^{-} = \epsilon(\gamma(0) - k) \nabla_x U(0) \cdot \nu_y - (\gamma(\epsilon y) - k) \nabla_y U(\epsilon y) \cdot \nu_y$$

on ∂B. Now consider the quantity

$$\int_{\tilde{\Omega}} \hat{\gamma}_\epsilon(\epsilon y) \nabla_y z_\epsilon \cdot \nabla_y z_\epsilon \, dy = \int_{\tilde{\Omega} \setminus \bar{B}} \gamma(\epsilon y) \nabla_y z_\epsilon \cdot \nabla_y z_\epsilon \, dy + \int_B k \nabla_y z_\epsilon \cdot \nabla_y z_\epsilon \, dy \quad .$$

Integrating each term on the right by parts

$$\begin{aligned} &\int_{\tilde{\Omega}} \hat{\gamma}_\epsilon(\epsilon y) \nabla_y z_\epsilon \cdot \nabla_y z_\epsilon \, dy \\ &\qquad = \int_{\partial B} \left[k \frac{\partial z_\epsilon}{\partial \nu_y}^{-} - \gamma(\epsilon y) \frac{\partial z_\epsilon}{\partial \nu_y}^{+} \right] z_\epsilon \, d\sigma_y + \int_B k \Delta_y U(\epsilon y) z_\epsilon \, dy \\ &\qquad = \int_{\partial B} \epsilon \left[(\gamma(\epsilon y) - k) \nabla_x U(\epsilon y) - (\gamma(0) - k) \nabla_x U(0) \right] \cdot \nu_y z_\epsilon \, d\sigma_y \\ &\qquad\qquad + \epsilon^2 \int_B k \Delta_x U(\epsilon y) z_\epsilon \, dy \quad . \end{aligned}$$

Using Taylor's theorem, the smoothness of γ, and the fact that U and all of its first and second derivatives are uniformly bounded on ϵB, we get that

$$(\gamma(\epsilon y) - k)\nabla_x U(\epsilon y) - (\gamma(0) - k)\nabla_x U(0)$$

is $O(\epsilon)$ in $L^\infty(\partial B)$ (hence $O(\epsilon)$ in $H^{-1/2}(\partial B)$) and that $\triangle_x U(\epsilon y)$ is bounded in $L^2(B)$. Therefore, by the trace theorem

$$\|\nabla_y z_\epsilon\|^2_{L^2(\tilde{\Omega})} \leq C\epsilon^2 \|z_\epsilon\|_{H^{1/2}(\partial B)} + C\epsilon^2 \|z_\epsilon\|_{L^2(B)} \leq C\epsilon^2 \|z_\epsilon\|_{H^1(B)} \quad .$$

Note that in defining z_ϵ we have subtracted the constant c_ϵ so that we can apply Poincaire's inequality on B. In doing so we obtain

$$\|\nabla_y z_\epsilon\|^2_{L^2(\tilde{\Omega})} \leq C\epsilon^2 \|\nabla_y z_\epsilon\|_{L^2(B)} \leq C\epsilon^2 \|\nabla_y z_\epsilon\|_{L^2(\tilde{\Omega})}$$

which, upon division, implies the lemma.

□

The following corollaries now follow from Poincare's inequality and a change of variables.

Corollary 1. *There exists a constant C, independent of ϵ, such that*

$$\|u_\epsilon(\epsilon y) - U(\epsilon y) - \epsilon v_\epsilon(y)\|_{L^2(\tilde{\Omega})} \leq C\epsilon \quad .$$

Corollary 2. *There exists a constant C, independent of ϵ, such that*

$$\|u_\epsilon(x) - U(x) - \epsilon v_\epsilon(x/\epsilon)\|_{H^1(\Omega)} \leq C\epsilon^{n/2+1} \quad .$$

We shall need a somewhat special Poincare-type inequality:

Lemma 3 . *Let $f \in L^\infty(\partial B)$ with $0 < \alpha \leq f \leq \beta$. Then there exists a constant C such that*

$$\left(\int_B |u|^2 \, dy\right)^{\frac{1}{2}} \leq C\Big(\left(\int_B |\nabla_y u|^2 \, dy\right)^{\frac{1}{2}} + \Big|\int_{\partial B} f u \, d\sigma_y\Big|\Big) \qquad \forall u \in H^1(B) \quad . \qquad (17)$$

The constant C depends on the constants α and β, but is otherwise independent of f.

Proof . We shall prove the above by contradiction. Suppose (17) is not true, then for all m, $\exists \, u_m$ in $H^1(B)$ and $0 < \alpha \leq f_m \leq \beta$ such that

$$\|u_m\|_{L^2(B)} > m(\|\nabla u_m\|_{L^2(B)} + |\int_{\partial B} u_m f_m \, d\sigma_y|) \quad . \qquad (18)$$

We may normalize by taking $\|u_m\|_{L^2(B)} = 1 \;\; \forall m$. Then $\|\nabla u_m\|_{L^2(B)} < \frac{1}{m}$. By compactness, there exists a subsequences, u_m, f_m (with common index, $m \to \infty$)

such that $u_m \rightharpoonup u$ in $H^1(B)$, $u_m \to u$ in $L^2(B)$, $u_m|_{\partial B} \to u|_{\partial B}$ in $L^2(\partial B)$ and $f_m \rightharpoonup f$ in $L^2(\partial B)$, $0 < \alpha \leq f \leq \beta$. Passing to the limit we have

$$0 \leq \|\nabla u\|_{L^2(B)} \leq \liminf \|\nabla u_m\|_{L^2(B)} \leq \lim_{m\to\infty} \frac{1}{m} = 0 \quad .$$

This implies $u \equiv$ constant in B. In addition,

$$\left| \int_{\partial B} uf \, d\sigma_y \right| = \lim \left| \int_{\partial B} u_m f_m \, d\sigma_y \right| \leq \lim_{m\to\infty} \frac{1}{m} = 0.$$

Now, since $| \int_{\partial B} uf \, d\sigma_y| = 0$ and $\int_{\partial B} f \, d\sigma_y \neq 0$, it follows that $u = 0$. However, if $u = 0$ then $\|u\|_{L^2(B)} = 0$ which contradicts the fact that $u_m \to u$ in $L^2(B)$ and $\|u_m\|_{L^2(B)} = 1$.

□

We now come to the only place in our analysis where we shall use the fact that n is either 2 or 3.

Lemma 4. *Suppose that $n = 2$ or 3 and suppose B is strictly star-shaped with respect to the origin. Then there exist constants $0 < \epsilon_0$ and C such that*

$$\|v_\epsilon\|_{H^1(B)} \leq C \quad , \quad 0 < \epsilon < \epsilon_0 \quad .$$

Proof. A combination of Lemma 2 and the estimate (11) immediately yields $\|\nabla_y v_\epsilon\|_{L^2(B)} \leq \|\nabla_y v_\epsilon\|_{L^2(\tilde{\Omega})} \leq C$. From Lemma 3 we know that

$$\left(\int_B |v_\epsilon|^2 dy\right)^{\frac{1}{2}} \leq C\left(\left(\int_B |\nabla_y v_\epsilon|^2 dy\right)^{\frac{1}{2}} + \left|\int_{\partial B} f_\epsilon v_\epsilon d\sigma_y\right|\right)$$

provided $0 < \alpha \leq f_\epsilon \leq \beta$. In order to prove this lemma it thus suffices to show that

$$\left|\int_{\partial B} f_\epsilon v_\epsilon d\sigma_y\right| < C \tag{19}$$

for some appropriate choice of $0 < \alpha \leq f_\epsilon \leq \beta$.

Let $N(x)$ be the Neumann function in Ω corresponding to a Dirac mass at the origin and to coefficient γ. That is, N is the solution to

$$\begin{aligned} -\nabla_x \gamma(x) \nabla_x N &= \delta_0 \quad \text{in} \quad \Omega \\ \gamma(x)\frac{\partial N}{\partial \nu_x} &= -\frac{1}{|\partial\Omega|} \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{20}$$

For $n = 2$ we can express N as the sum of a logarithm and a smoother function [7]. More precisely

$$N(x) = \Phi_2(x) + R_2(x) = -\frac{1}{2\pi\gamma(0)} \log|x| + R_2(x) \qquad (n = 2) \quad ,$$

where $R_2(x) \in W^{2,p}(\Omega)$ for any $1 \le p < 2$ and solves

$$\begin{aligned} \nabla_x \cdot \gamma(x) \nabla_x R_2(x) &= \frac{1}{2\pi\gamma(0)} \frac{\nabla_x \gamma(x) \cdot x}{|x|^2} \qquad x \in \Omega \quad , \\ \gamma(x) \frac{\partial R_2}{\partial \nu_x}(x) &= -\frac{1}{|\partial\Omega|} + \frac{\gamma(x)}{2\pi\gamma(0)} \frac{x \cdot \nu_x}{|x|^2} \qquad x \in \partial\Omega \quad . \end{aligned} \tag{21}$$

For $n = 3$ we can similarly express N as the sum of $\frac{1}{4\pi\gamma(0)}|x|^{-1}$ and a remainder in $W^{2,p}(\Omega)$, $1 \le p < 3/2$, but we shall need a more refined expansion, namely

$$N(x) = \Phi_3(x) + R_3(x) = \frac{1}{4\pi\gamma(0)}|x|^{-1} - \frac{\nabla\gamma(0) \cdot x}{8\pi[\gamma(0)]^2}|x|^{-1} + R_3(x) \qquad (n = 3) \quad ,$$

where the remainder, R_3, is in $W^{2,p}(\Omega)$ for any $1 \le p < 3$ and solves

$$\begin{aligned} \nabla_x \cdot \gamma(x) \nabla_x R_3(x) &= A \cdot \left[|x|^2 \nabla\gamma(x) - (\nabla\gamma(x) \cdot x)\ x + 2(\gamma(0) - \gamma(x))\ x \right] |x|^{-3} \\ &\quad + \frac{1}{4\pi\gamma(0)} [\nabla\gamma(x) - \nabla\gamma(0)] \cdot x\ |x|^{-3} \qquad x \in \Omega \quad , \\ \gamma(x) \frac{\partial R_3}{\partial \nu_x}(x) &= -\frac{1}{|\partial\Omega|} - \gamma(x) \frac{\partial \Phi_3}{\partial \nu_x} \qquad x \in \partial\Omega \quad . \end{aligned} \tag{22}$$

Here the constant vector A is given by $A = \frac{1}{8\pi[\gamma(0)]^2}\nabla\gamma(0)$. From the way we have written the right hand side of (22) it follows immediately that it is of order $|x|^{-1}$ at the origin, and therefore in $L^p(\Omega)$, for any $1 \le p < 3$. It is based on this fact and elliptic regularity that we conclude R_3 is in $W^{2,p}$, $1 \le p < 3$. Since in particular $R_n \in H^1(\Omega)$ (and $n = 2, 3$) we get

$$\epsilon^{2(n-2)} \int_B |\nabla_y R_n(\epsilon y)|^2\ dy = \epsilon^{n-2} \int_{\epsilon B} |\nabla_x R_n(x)|^2\ dx \le C \quad . \tag{23}$$

The fact that R_n is in $W^{2,p}(\Omega)$ for any $p < n$ ($n = 2, 3$) implies that R_n is continuous (and thus uniformly bounded) on $\bar{\Omega}$. Thus

$$\epsilon^{2(n-2)} \int_B |R_n(\epsilon y)|^2\ dy \le C\epsilon^{2(n-2)} \le C \quad .$$

From a combination of these two inequality we conclude that $\epsilon^{n-2}R_n(\epsilon y)$ is bounded in $H^1(B)$ and therefore by the trace theorem

$$\|\epsilon^{n-2}R_n(\epsilon y)\|_{H^{1/2}(\partial B)} \le C \qquad (n = 2, 3) \quad . \tag{24}$$

We introduce the function

$$N_\epsilon(y) = \begin{cases} N(\epsilon y) + \frac{\log\epsilon}{2\pi\gamma(0)} & (n = 2) \\ \epsilon N(\epsilon y) & (n = 3) \end{cases} \quad . \tag{25}$$

Based on the formulas given above for N and the estimate (24) for $\epsilon^{n-2}R_n(\epsilon y)$ it follows immediately that

$$||N_\epsilon||_{H^{1/2}(\partial B)} \leq C \qquad (n=2,3) \quad . \tag{26}$$

In establishing (19) we shall need the normal derivative of N_ϵ on ∂B. A simple calculation yields

$$\nabla_y N_\epsilon = \frac{-1}{2\pi\gamma(0)}\frac{y}{|y|^2} + \nabla_y R_2(\epsilon y) \qquad (n=2) \quad ,$$

and

$$\begin{aligned}\nabla_y N_\epsilon \quad &= \quad -\frac{1}{4\pi\gamma(0)}|y|^{-3}\left(y + \frac{\epsilon}{2\gamma(0)}\nabla\gamma(0)\cdot y^\perp \; y^\perp\right) \\ &\qquad +\epsilon\nabla_y R_3(\epsilon y) \qquad (n=3) \quad .\end{aligned}$$

Thus, on ∂B

$$\gamma(\epsilon y)\frac{\partial N_\epsilon}{\partial \nu_y} = -f_{n,\epsilon} + \epsilon^{n-2}\gamma(\epsilon y)\frac{\partial R_n(\epsilon y)}{\partial \nu_y} \quad , \tag{27}$$

where $f_{n,\epsilon}$, $n=2,3$, is given by

$$f_{2,\epsilon}(y) = \frac{\gamma(\epsilon y)}{2\pi\gamma(0)}|y|^{-2}y\cdot\nu_y \quad ,$$

and

$$f_{3,\epsilon}(y) = \frac{\gamma(\epsilon y)}{4\pi\gamma(0)}|y|^{-3}\left(y + \frac{\epsilon}{2\gamma(0)}\nabla\gamma(0)\cdot y^\perp \; y^\perp\right)\cdot\nu_y \quad .$$

The positivity of γ and the condition that B is strictly star shaped with respect to the origin guarantee that there exist constants $0<\epsilon_0$, α and β such that $0<\alpha\leq f_{n,\epsilon}(y)\leq\beta$ for $0<\epsilon<\epsilon_0$. Rearranging (27) we obtain

$$\left|\int_{\partial B} f_{n,\epsilon}v_\epsilon \, d\sigma_y\right| \leq \left|\int_{\partial B}\gamma(\epsilon y)\frac{\partial N_\epsilon}{\partial\nu_y}v_\epsilon \, d\sigma_y\right| + \epsilon^{n-2}\left|\int_{\partial B}\gamma(\epsilon y)\nabla_y R_n(\epsilon y)\cdot\nu_y v_\epsilon \, d\sigma_y\right| \quad .$$

Integrating by parts, and using equations (12) and (20) we see that

$$\begin{aligned}\int_{\partial B}\gamma(\epsilon y)\frac{\partial N_\epsilon}{\partial\nu_y}v_\epsilon \, d\sigma_y \quad &= \quad -\int_{\tilde\Omega\setminus\bar B}\gamma(\epsilon y)\nabla_y N_\epsilon \nabla_y v_\epsilon \, dy \\ &= \quad \int_{\partial B}\gamma(\epsilon y)\frac{\partial {v_\epsilon}}{\partial\nu_y}^{+} N_\epsilon \, d\sigma_y \quad ,\end{aligned}$$

and therefore

$$\begin{aligned}&\left|\int_{\partial B} f_{n,\epsilon}v_\epsilon \, d\sigma_y\right| \\ &\leq \left|\int_{\partial B}\gamma(\epsilon y)\frac{\partial {v_\epsilon}}{\partial\nu_y}^{+} N_\epsilon \, d\sigma_y\right| + \epsilon^{n-2}\left|\int_{\partial B}\gamma(\epsilon y)\nabla_y R_n(\epsilon y)\cdot\nu_y v_\epsilon \, d\sigma_y\right| \quad .\end{aligned} \tag{28}$$

The first term on the right-hand side is bounded by

$$\begin{aligned}\left|\int_{\partial B}\gamma(\epsilon y)\frac{\partial v_\epsilon}{\partial \nu_y}^{+} N_\epsilon\, d\sigma_y\right| &\leq C\ \|\gamma(\epsilon y)\frac{\partial v_\epsilon}{\partial \nu_y}^{+}\|_{H^{-1/2}(\partial B)}\|N_\epsilon\|_{H^{1/2}(\partial B)} \\ &\leq C\ \|\gamma(\epsilon y)\frac{\partial v_\epsilon}{\partial \nu_y}^{+}\|_{H^{-1/2}(\partial B)}\quad , \qquad (29)\end{aligned}$$

since N_ϵ is bounded in $H^{1/2}(\partial B)$, according to (26). Furthermore, we have that

$$\gamma(\epsilon y)\frac{\partial v_\epsilon}{\partial \nu_y}^{+} = -(\gamma(0)-k)\nabla_x U(0)\cdot \nu_y + k\frac{\partial v_\epsilon}{\partial \nu_y}^{-} \qquad (30)$$

is bounded in $H^{-1/2}(\partial B)$ independently of ϵ. To see the latter, let χ be any function in $H^{1/2}(\partial B)$ and extend χ into B so that

$$\|\chi\|_{H^1(B)} \leq C\|\chi\|_{H^{1/2}(\partial B)}\quad .$$

Then,

$$\begin{aligned}\int_{\partial B} k\frac{\partial v_\epsilon}{\partial \nu_y}^{-}\chi\, d\sigma_y &= \int_B k\nabla_y v_\epsilon \nabla_y \chi\, dy \\ &\leq C\|\nabla_y v_\epsilon\|_{L^2(B)}\|\chi\|_{H^{1/2}(\partial B)}\quad .\end{aligned}$$

Hence

$$\|k\frac{\partial v_\epsilon}{\partial \nu_y}^{-}\|_{H^{-1/2}(\partial B)} \leq C\|\nabla_y v_\epsilon\|_{L^2(B)}\quad ,$$

the right hand side of which, as pointed out at the beginning of this proof, is bounded independently of ϵ. It now immediately follows that

$$\|\gamma(\epsilon y)\frac{\partial v_\epsilon}{\partial \nu_y}^{+}\|_{H^{-1/2}(\partial B)} \leq C\quad , \qquad (31)$$

as stated previously. The Estimates (29) and (31) in combination give that the first term on the right hand side of (28) is bounded independently of ϵ, *i.e.*,

$$\left|\int_{\partial B}\gamma(\epsilon y)\frac{\partial v_\epsilon}{\partial \nu_y}^{+} N_\epsilon\, d\sigma_y\right| \leq C\quad . \qquad (32)$$

We now seek a bound for the second term on the right-hand side of (28). Integrating twice by parts, and using the fact that $\triangle_y v_\epsilon = 0$ in B we get

$$\begin{aligned}&\int_{\partial B}\gamma(\epsilon y)\nabla_y R_n(\epsilon y)\cdot \nu_y v_\epsilon\, d\sigma_y \\ &\quad = \int_B \gamma(\epsilon y)\nabla_y R_n(\epsilon y)\cdot \nabla_y v_\epsilon\, dy + \int_B \nabla_y\cdot\gamma(\epsilon y)\nabla_y R(\epsilon y) v_\epsilon\, dy \\ &\quad = -\epsilon\int_B R_n(\epsilon y)\nabla_x\gamma(\epsilon y)\cdot\nabla_y v_\epsilon\, dy + \int_{\partial B} R_n(\epsilon y)\gamma(\epsilon y)\frac{\partial v_\epsilon}{\partial \nu_y}^{-}\, d\sigma_y \\ &\qquad + \int_B \nabla_y\cdot\gamma(\epsilon y)\nabla_y R_n(\epsilon y) v_\epsilon\, dy\quad . \qquad (33)\end{aligned}$$

From the equations (21) and (22) for R_n it follows that

$$\nabla_y \cdot \gamma(\epsilon y)\nabla_y R_n(\epsilon y) = \epsilon F_{n,\epsilon}(y) \quad ,$$

where

$$\|F_{n,\epsilon}\|_{L^p(B)} \leq C_p \qquad \text{for any } p < n,$$

(independently of ϵ). Based on (33) we may thus estimate

$$\begin{aligned}\left|\int_{\partial B} \gamma(\epsilon y)\nabla_y R_n(\epsilon y) \cdot \nu_y v_\epsilon \, d\sigma_y\right| \quad &\leq \quad \epsilon \|R_n \nabla_x \gamma(\epsilon y)\|_{L^2(B)} \|\nabla_y v_\epsilon\|_{L^2(B)} \\ &\quad + \|R_n \gamma(\epsilon y)\|_{H^{1/2}(\partial B)} \|\frac{\partial v_\epsilon}{\partial \nu_y}^{-}\|_{H^{-1/2}(\partial B)} \\ &\quad + C_q \epsilon \|v_\epsilon\|_{L^q(B)} \quad ,\end{aligned}$$

for any $q > \frac{n}{n-1}$ (the dual of $p < n$). Since we have already verified that $R_n(\epsilon y)$ is uniformly bounded on B (independently of ϵ) and that the three quantities $\|\nabla_y v_\epsilon\|_{L^2(B)}$, $\|\epsilon^{n-2} R_n(\epsilon y)\|_{H^{1/2}(\partial B)}$ and $\|\frac{\partial v_\epsilon}{\partial \nu_y}^{-}\|_{H^{-1/2}(\partial B)}$ are bounded it now follows that

$$\epsilon^{n-2} \left|\int_{\partial B} \gamma(\epsilon y)\nabla_y R(\epsilon y) \cdot \nu_y v_\epsilon \, d\sigma_y\right| \leq C_q \left(1 + \epsilon^{n-1}\|v_\epsilon\|_{L^q(B)}\right) \quad , \tag{34}$$

for any $q > \frac{n}{n-1}$. Recall that v_ϵ integrates to zero on $\partial\tilde{\Omega}$. Since the gradient of v_ϵ is bounded in $L^2(\tilde{\Omega})$, a rescaling of Poincaire's inequality gives $\|v_\epsilon\|_{H^1(\tilde{\Omega})} \leq C/\epsilon$, which then by Sobolev's imbedding theorem implies that

$$\|v_\epsilon\|_{L^q(B)} \leq C_q \|v_\epsilon\|_{H^1(B)} \leq C_q/\epsilon \quad ,$$

for any $q < \frac{2n}{n-2}$ ($q < \infty$ for $n = 2$). Hence, by selecting any $\frac{n}{n-1} < q < \frac{2n}{n-2}$

$$\epsilon^{n-2} \left|\int_{\partial B} \gamma(\epsilon y)\nabla_y R_n(\epsilon y) \cdot \nu_y v_\epsilon d\sigma_y\right| \leq C(1 + \epsilon^{n-2}) \leq C \quad .$$

This gives an ϵ-independent bound for the second term on the right hand side of (28), and thus completes the proof of Lemma 4.

$\square$

We are now in a position to establish the estimate which is crucial for the derivation of our representation formula. This estimate involves the function v, the solution to the problem (13). Since the previous lemma, which was only verified for dimensions 2 and 3, is used in the proof, the same dimensional restriction will be maintained.

Theorem 1. *Suppose $n = 2$ or 3 and suppose B is strictly star-shaped with respect to the origin. Then there exists a constant C, independent of ϵ, such that*

$$\|\nabla_y (u_\epsilon(\epsilon y) - U(\epsilon y) - \epsilon v(y))\|_{L^2(\tilde{\Omega})} \leq C\epsilon^{3/2}.$$

Proof. It is clearly sufficient to prove the desired estimate for ϵ small, *e.g.*, for $0<\epsilon<\epsilon_0$, with ϵ_0 being the constant from Lemma 4. From Lemma 2 it follows that it suffices to show that

$$\|\nabla_y(v_\epsilon - v)\|_{L^2(\tilde\Omega)} \le C\epsilon^{1/2} \qquad 0<\epsilon<\epsilon_0 \quad .$$

Now define

$$w_\epsilon = v_\epsilon - v - c_\epsilon \quad ,$$

where c_ϵ is the constant chosen so that $\int_{\partial\tilde\Omega} w_\epsilon \, d\sigma_y = 0$. Then w_ϵ solves

$$\begin{aligned}
&\Delta_y w_\epsilon = 0 \quad \text{in} \quad B \ , \qquad \nabla_y\cdot\gamma(\epsilon y)\nabla_y w_\epsilon = -\epsilon\nabla_x\gamma(\epsilon y)\cdot\nabla_y v \quad \text{in} \quad \tilde\Omega\setminus\bar B\\
&\qquad\qquad w_\epsilon \quad \text{is continuous across } \partial B\\
&\gamma(\epsilon y)\frac{\partial w_\epsilon}{\partial\nu_y}^+ - k\frac{\partial w_\epsilon}{\partial\nu_y}^- = (\gamma(0)-\gamma(\epsilon y))\frac{\partial v}{\partial\nu_y}^+ \quad \text{on} \quad \partial B\\
&\qquad\gamma(\epsilon y)\frac{\partial w_\epsilon}{\partial\nu_y} = -\gamma(\epsilon y)\frac{\partial v}{\partial\nu_y} \quad \text{on} \quad \partial\tilde\Omega.
\end{aligned}$$

For future reference we note that the condition $\int_{\partial\tilde\Omega} v_\epsilon = 0$ combined with the decay of v guarantee that the constants, c_ϵ, are uniformly bounded (indeed they are bounded by $C\epsilon$). Consider the following integral identity

$$\int_{\tilde\Omega}\hat\gamma_\epsilon(\epsilon y)\nabla_y w_\epsilon\cdot\nabla_y w_\epsilon\,dy = \int_{\tilde\Omega\setminus B}\gamma(\epsilon y)\nabla_y w_\epsilon\cdot\nabla_y w_\epsilon\,dy + \int_B k\nabla_y w_\epsilon\cdot\nabla_y w_\epsilon\,dy \quad ,$$

where $\hat\gamma$ is as defined previously. Integrating each term on the right hand side by parts, and using the equations for w_ϵ, we obtain

$$\begin{aligned}
\int_{\tilde\Omega}\hat\gamma_\epsilon(\epsilon y)\nabla_y w_\epsilon\cdot\nabla_y w_\epsilon\,dy \quad &= \quad \int_{\tilde\Omega\setminus\bar B}\epsilon\nabla_x\gamma(\epsilon y)\cdot\nabla_y v\ w_\epsilon\,dy - \int_{\partial\tilde\Omega}\gamma(\epsilon y)\frac{\partial v}{\partial\nu_y}w_\epsilon d\sigma_y\\
&\quad + \int_{\partial B}(\gamma(\epsilon y)-\gamma(0))\frac{\partial v^+}{\partial\nu_y}w_\epsilon\,d\sigma_y \quad .
\end{aligned}$$

Continued integration by parts of the first term on the right hand side yields

$$\begin{aligned}
&\int_{\tilde\Omega}\hat\gamma(\epsilon y)\nabla_y w_\epsilon\cdot\nabla_y w_\epsilon\,dy\\
&\qquad = -\epsilon^2\int_{\tilde\Omega\setminus\bar B}(\Delta_x\gamma(\epsilon y))v w_\epsilon\,dy - \epsilon\int_{\tilde\Omega\setminus\bar B}(\nabla_x\gamma(\epsilon y))v\cdot\nabla_y w_\epsilon\,dy\\
&\qquad\quad + \int_{\partial\tilde\Omega}\epsilon(\nabla_x\gamma(\epsilon y)\cdot\nu_y)v w_\epsilon\,d\sigma_y - \int_{\partial B}\epsilon(\nabla_x\gamma(\epsilon y)\cdot\nu_y)v w_\epsilon\,d\sigma_y\\
&\qquad\quad - \int_{\partial\tilde\Omega}\gamma(\epsilon y)\frac{\partial v}{\partial\nu_y}w_\epsilon\,d\sigma_y + \int_{\partial B}(\gamma(\epsilon y)-\gamma(0))\frac{\partial v}{\partial\nu_y}^+ w_\epsilon\,d\sigma_y \quad . \qquad (35)
\end{aligned}$$

We shall examine each term on the right hand side of (35) separately. The first term is bounded by

$$\epsilon^2\left|\int_{\tilde\Omega\setminus\bar B}(\Delta_x\gamma(\epsilon y))v w_\epsilon\,dy\right| \le C\epsilon^2\|v\|_{L^2(\tilde\Omega\setminus\bar B)}\|w_\epsilon\|_{L^2(\tilde\Omega\setminus\bar B)} \quad .$$

Application of a rescaled Poincaire's inequality and (16) now yields

$$\begin{aligned}\epsilon^2\left|\int_{\tilde\Omega\setminus\bar B}(\Delta_x\gamma(\epsilon y))vw_\epsilon\,dy\right| &\le C\epsilon\|v\|_{L^2(\tilde\Omega\setminus\bar B)}\|\nabla_y w_\epsilon\|_{L^2(\tilde\Omega)}\\ &\le C\epsilon^{1/2}\|\nabla_y w_\epsilon\|_{L^2(\tilde\Omega)}\ .\end{aligned}\tag{36}$$

The second term is bounded by

$$\begin{aligned}\epsilon\left|\int_{\tilde\Omega\setminus\bar B}(\nabla_x\gamma(\epsilon y))v\cdot\nabla_y w_\epsilon\,dy\right| &\le C\epsilon\|v\|_{L^2(\tilde\Omega\setminus\bar B)}\|\nabla_y w_\epsilon\|_{L^2(\tilde\Omega\setminus\bar B)}\\ &\le C\epsilon^{1/2}\|\nabla_y w_\epsilon\|_{L^2(\tilde\Omega)}.\end{aligned}\tag{37}$$

To obtain a bound on the third integral, we change variables back to the original domain. This yields

$$\begin{aligned}\epsilon\left|\int_{\partial\tilde\Omega}(\nabla_x\gamma(\epsilon y)\cdot\nu_y)vw_\epsilon\,d\sigma_y\right| &= \epsilon^{2-n}\left|\int_{\partial\Omega}(\nabla_x\gamma(x)\cdot\nu_x)v(x/\epsilon)w_\epsilon(x/\epsilon)\,d\sigma_x\right|\\ &\le C\epsilon^{2-n}\|v(x/\epsilon)\|_{L^2(\partial\Omega)}\|w_\epsilon(x/\epsilon)\|_{L^2(\partial\Omega)} \qquad (38)\\ &\le C\epsilon^{2-n}\|v(x/\epsilon)\|_{L^2(\partial\Omega)}\|\nabla_x w_\epsilon(x/\epsilon)\|_{L^2(\Omega)}\ .\end{aligned}$$

For the last inequality we have used the fact that $\int_{\partial\Omega} w_\epsilon(x/\epsilon)\,d\sigma_x = 0$. We have

$$\|\nabla_x w_\epsilon(x/\epsilon)\|_{L^2(\Omega)} = \epsilon^{n/2-1}\|\nabla_y w_\epsilon(y)\|_{L^2(\tilde\Omega)}\ ,\tag{39}$$

and due to the decay of v we also have

$$\|v(x/\epsilon)\|_{L^\infty(\partial\Omega)} \le C\epsilon \quad n=2\ ,$$

$$\|v(x/\epsilon)\|_{L^\infty(\partial\Omega)} \le C\epsilon^{n-2} \ \ n\ge 3\ .$$

These estimates, along with (38) and (39), give

$$\epsilon\left|\int_{\partial\tilde\Omega}(\nabla_x\gamma(\epsilon y)\cdot\nu_y)vw_\epsilon\,d\sigma_y\right| \le C\epsilon^{1/2}\|\nabla_y w_\epsilon\|_{L^2(\tilde\Omega)}\ ,\tag{40}$$

for any $n\ge 2$. For the fourth term, we obtain the estimate

$$\begin{aligned}\left|\int_{\partial B}\epsilon(\nabla_x\gamma(\epsilon y)\cdot\nu_y)vw_\epsilon d\sigma_y\right| &\le C\epsilon\|w_\epsilon\|_{L^2(\partial B)}\\ &\le C\epsilon\|w_\epsilon\|_{H^1(B)}.\end{aligned}\tag{41}$$

To estimate the fifth term, we again change variables and obtain

$$\begin{aligned}\left|\int_{\partial\tilde\Omega}\gamma(\epsilon y)\frac{\partial v}{\partial\nu_y}w_\epsilon\,d\sigma_y\right| &= \epsilon^{1-n}\left|\int_{\partial\Omega}\gamma(x)\frac{\partial v}{\partial\nu_y}w_\epsilon(x/\epsilon)\,d\sigma_x\right| \qquad (42)\\ &\le C\epsilon^{1-n}\|\frac{\partial v}{\partial\nu_y}(x/\epsilon)\|_{L^2(\partial\Omega)}\|w_\epsilon(x/\epsilon)\|_{L^2(\partial\Omega)}\\ &\le C\epsilon^{1-n}\|\frac{\partial v}{\partial\nu_y}(x/\epsilon)\|_{L^2(\partial\Omega)}\|\nabla_x w_\epsilon(x/\epsilon)\|_{L^2(\Omega)}\ .\end{aligned}$$

Now note that from (15) it follows that

$$\|\frac{\partial v}{\partial \nu_y}(x/\epsilon)\|_{L^\infty(\partial\Omega)} \le C\epsilon^2 \quad n = 2 \ ,$$

$$\|\frac{\partial v}{\partial \nu_y}(x/\epsilon)\|_{L^\infty(\partial\Omega)} \le C\epsilon^{n-1} \quad n \ge 3 \ .$$

In combination with (39) and (42) these estimates imply

$$\left|\int_{\partial\tilde{\Omega}} \gamma(\epsilon y)\frac{\partial v}{\partial \nu_y} w_\epsilon \, d\sigma_y\right| \le C\epsilon^{1/2}\|\nabla_y w_\epsilon\|_{L^2(\tilde{\Omega})} \ , \tag{43}$$

for any $n \ge 2$. Finally, applying Taylor's theorem to estimate the sixth integral, we obtain

$$\begin{aligned} \left|\int_{\partial B}(\gamma(\epsilon y) - \gamma(0))\frac{\partial v}{\partial \nu_y}^+ w_\epsilon d\sigma_y\right| &= \epsilon\left|\int_{\partial B} y\cdot\nabla_x\gamma(\zeta_x)\frac{\partial v}{\partial \nu_y}^+ w_\epsilon d\sigma_y\right| \\ &\le C\epsilon\|w_\epsilon\|_{H^1(B)} \end{aligned} \tag{44}$$

where we have used the fact that (by elliptic regularity) $\frac{\partial v^+}{\partial \nu_y}$ is in $L^2(\partial B)$. A Combination of the identity (35) with the estimates (36)– (37), (40)– (41) and (43) –(44) yields

$$\left|\int_{\tilde{\Omega}} \hat{\gamma}_\epsilon \nabla_y w_\epsilon \cdot \nabla_y w_\epsilon \, dy\right| \le C\epsilon^{1/2}\|\nabla_y w_\epsilon\|_{L^2(\tilde{\Omega})} + C\epsilon\|w_\epsilon\|_{H^1(B)} \ .$$

From the previous lemma and that fact that the constants c_ϵ are uniformly bounded it follows that w_ϵ is bounded in $H^1(B)$, $0 < \epsilon < \epsilon_0$. Based on the above estimate we now conclude that

$$\|\nabla_y w_\epsilon\|_{L^2(\tilde{\Omega})} \le C\epsilon^{1/2} \qquad 0 < \epsilon < \epsilon_0 \ ,$$

which implies the theorem.

□

5 An asymptotic formula for the voltage potential

We are now ready to derive an asymptotic formula for $u_\epsilon(z)$ for those points, z, a fixed distance, d, away from the inhomogeneities. We shall initially consider the case in which u_ϵ is the potential corresponding to a conductor having a single inhomogeneity, ϵB, that is "centered" at the origin.

Let $N(\cdot, z)$ denote the Neumann function for Ω corresponding to a Dirac mass at the point $z \in \Omega$ and coefficient γ. That is, $N(x, z)$ is the solution to

$$\begin{aligned} -\nabla_x\cdot\gamma(x)\nabla_x N(x,z) &= \delta_z \quad \text{in } \Omega \ , \\ \gamma(x)\frac{\partial N(x,z)}{\partial \nu_x} &= -\frac{1}{|\partial\Omega|} \quad \text{on } \partial\Omega \ . \end{aligned}$$

We normalize $N(x,z)$ by requiring that $\int_{\partial\Omega} N(x,z)\,d\sigma_x = 0$. Whenever no confusion is possible we shall use the simpler notation $N(x) = N(x,z)$. Integrating by parts and using the fact that $\int_{\partial\Omega} u_\epsilon\, d\sigma_x = 0$, we get

$$\begin{aligned} u_\epsilon(z) &= -\int_\Omega u_\epsilon \nabla_x \cdot (\gamma \nabla_x N)\, dx \\ &= \int_\Omega \gamma \nabla_x u_\epsilon \cdot \nabla_x N\, dx - \int_{\partial\Omega} u_\epsilon \gamma \frac{\partial N}{\partial \nu_x}\, d\sigma_x \\ &= \int_{\Omega\setminus\epsilon B} \gamma \nabla_x u_\epsilon \cdot \nabla_x N\, dx + \int_{\epsilon \bar{B}} \gamma \nabla_x u_\epsilon \cdot \nabla_x N\, dx \quad . \end{aligned} \tag{45}$$

We note that the third identity above is the only point in the derivation of the asymptotic formula for u_ϵ where we use the specific form of the boundary condition for N. If one is content with the presence of the term $-\int_{\partial\Omega} u_\epsilon \gamma \frac{\partial N}{\partial \nu_x}\, d\sigma_x$ then it suffices that $N(x,z)$ be a solution to $-\nabla_x \cdot \gamma(x) \nabla_x N(x,z) = \delta_z$ which is smooth in $\bar{\Omega} \setminus \{z\}$. We shall make use of this observation later. Repeated integration by parts in (45) yields

$$\begin{aligned} u_\epsilon(z) &= \int_{\partial\Omega} \psi N\, d\sigma_x - \int_{\partial(\epsilon B)} \gamma \frac{\partial u_\epsilon}{\partial \nu_x}^{+} N\, d\sigma_x + \int_{\epsilon B} \gamma \nabla_x u_\epsilon \cdot \nabla_x N\, dx \\ &= \int_{\partial\Omega} \psi N\, d\sigma_x - \int_{\partial(\epsilon B)} \gamma \frac{\partial u_\epsilon}{\partial \nu_x}^{+} N\, d\sigma_x \\ &\quad + \int_{\epsilon B} (\gamma(x) - \gamma(0)) \nabla_x u_\epsilon \cdot \nabla_x N\, dx + \int_{\epsilon B} \gamma(0) \nabla_x u_\epsilon \cdot \nabla_x N\, dx \\ &= \int_{\partial\Omega} \psi N\, d\sigma_x - \int_{\partial(\epsilon B)} \gamma \frac{\partial u_\epsilon}{\partial \nu_x}^{+} N\, d\sigma_x \\ &\quad + \int_{\epsilon B} (\gamma(x) - \gamma(0)) \nabla_x u_\epsilon \cdot \nabla_x N\, dx + \int_{\partial(\epsilon B)} \gamma(0) \frac{\partial u_\epsilon}{\partial \nu_x}^{-} N\, d\sigma_x \end{aligned} \tag{46}$$

where ν denotes the outward unit normal to both Ω and ϵB. If we expand $\gamma(x) = \gamma(0) + x \cdot \nabla_x \gamma(\zeta_x)$, $\zeta_x \in \epsilon B$, and make the change of variable $x = \epsilon y$ then the second to last term in (46) may be estimated by

$$\begin{aligned} \left| \int_{\epsilon B} (\gamma(x) - \gamma(0)) \nabla_x u_\epsilon \cdot \nabla_x N dx \right| &= \epsilon^{n+1} \left| \int_B (y \cdot \nabla_x \gamma(\zeta_x)) \nabla_x u_\epsilon(\epsilon y) \cdot \nabla_x N(\epsilon y)\, dy \right| \\ &\le C \epsilon^{n+1} \|\nabla_x u_\epsilon(\epsilon y)\|_{L^2(B)} \|\nabla_x N(\epsilon y)\|_{L^2(B)} \quad . \end{aligned}$$

From Lemma 1 we have

$$\int_{\epsilon B} |\nabla_x (U - u_\epsilon)|^2\, dx \le C \epsilon^n \quad .$$

Changing variables we obtain

$$\int_B |\nabla_x (U - u_\epsilon)(\epsilon y)|^2\, dy \le C \quad ,$$

and therefore,

$$\int_B |\nabla_x u_\epsilon(\epsilon y)|^2 dx \le C \quad .$$

Since z is bounded away from ϵB, in the sense that $dist(z, \epsilon B) \ge d > 0$, it now follows that

$$\left| \int_{\epsilon B} (\gamma(x) - \gamma(0)) \nabla_x u_\epsilon \nabla_x N(x,z)\, dx \right| \le C\epsilon^{n+1} \|\nabla_x N(\epsilon y, z)\|_{L^2(B)} \le C\epsilon^{n+1} \quad .$$

Substituting this into (46) and using the boundary condition $\gamma(x)\frac{\partial u_\epsilon^+}{\partial \nu_x} = k \frac{\partial u_\epsilon^-}{\partial \nu_x}$ on $\partial(\epsilon B)$, we obtain

$$\begin{aligned} u_\epsilon(z) &= \int_{\partial\Omega} \psi N \, d\sigma_x - \int_{\partial(\epsilon B)} \gamma \frac{\partial u_\epsilon}{\partial \nu_x}^+ N \, d\sigma_x + \int_{\partial(\epsilon B)} \gamma(0) \frac{\partial u_\epsilon}{\partial \nu_x}^- N \, d\sigma_x + O(\epsilon^{n+1}) \\ &= \int_{\partial\Omega} \psi N \, d\sigma_x + \int_{\partial(\epsilon B)} (\gamma(0) - k) \frac{\partial u_\epsilon}{\partial \nu_x}^- N \, d\sigma_x + O(\epsilon^{n+1}) \quad . \end{aligned}$$

Now let $r_\epsilon(x) = u_\epsilon(x) - U(x) - \epsilon v(\frac{x}{\epsilon})$, where v is the function defined by (13). Then

$$\begin{aligned} u_\epsilon(z) = \int_{\partial\Omega} \psi N \, d\sigma_x &+ \int_{\partial(\epsilon B)} (\gamma(0) - k) (\frac{\partial U}{\partial \nu_x} + \frac{\partial v}{\partial \nu_y}^- (x/\epsilon)) N \, d\sigma_x \\ &+ \int_{\partial(\epsilon B)} (\gamma(0) - k) \frac{\partial r_\epsilon}{\partial \nu_x}^- N \, d\sigma_x + O(\epsilon^{n+1}) \quad . \end{aligned} \tag{47}$$

Integrating the last term in (47) by parts, we obtain

$$\begin{aligned} \int_{\partial(\epsilon B)} (\gamma(0) - k) \frac{\partial r_\epsilon}{\partial \nu_x}^- N \, d\sigma_x &= \int_{\epsilon B} (\gamma(0) - k) \nabla_x r_\epsilon \nabla_x N \, dx \\ &\quad + \int_{\epsilon B} (\gamma(0) - k) \triangle_x r_\epsilon N \, dx \quad . \end{aligned}$$

Changing variables and using the fact that $\triangle_x r_\epsilon = -\triangle_x U$ in ϵB we have

$$\begin{aligned} \int_{\partial(\epsilon B)} (\gamma(0) - k) \frac{\partial r_\epsilon}{\partial \nu_x}^- N \, d\sigma_x = \epsilon^{n-1} & \int_B (\gamma(0) - k) \nabla_y r_\epsilon(\epsilon y) \nabla_x N(\epsilon y) \, dy \\ & - \epsilon^n \int_B (\gamma(0) - k) \triangle_x U(\epsilon y) N(\epsilon y) \, dy \\ = & \; -\epsilon^n \int_B (\gamma(0) - k) \triangle_x U(\epsilon y) N(\epsilon y) dy + O(\epsilon^{n+\frac{1}{2}}) \quad . \end{aligned}$$

In the last inequality we have used the estimate $\|\nabla_y r_\epsilon(\epsilon y)\|_{L^2(B)} \le C\epsilon^{3/2}$ (which follows immediately from Theorem 1) as well as the fact that $\nabla_x N(\epsilon y) = \nabla_x N(\epsilon y, z)$ is uniformly bounded on B (since z is bounded away from ϵB). Expanding the

Neumann function, N, in a Taylor series about the origin, we see that the last term in equation (47) can be written

$$\int_{\partial(\epsilon B)} (\gamma(0)-k)\frac{\partial r_\epsilon}{\partial \nu_x}^{-} N\, d\sigma_x = -\epsilon^n(\gamma(0)-k)N(0,z)\int_B \triangle_x U(\epsilon y)\, dy + O(\epsilon^{n+\frac{1}{2}}) \ , \tag{48}$$

and that the next to last term in (47) can be written

$$\begin{aligned}\int_{\partial(\epsilon B)} &(\gamma(0)-k\,)\,(\frac{\partial U}{\partial \nu_x}+\frac{\partial v}{\partial \nu_y}^{-}(x/\epsilon)\,)N\, d\sigma_x \\ &= \epsilon^{n-1}\int_{\partial B}(\gamma(0)-k)(\frac{\partial U}{\partial \nu_x}(\epsilon y)+\frac{\partial v}{\partial \nu_y}^{-})N(\epsilon y,z)\, d\sigma_y \\ &= \epsilon^{n-1}(\gamma(0)-k)N(0,z)\int_{\partial B}(\frac{\partial U}{\partial \nu_x}(\epsilon y)+\frac{\partial v}{\partial \nu_y}^{-}\,)\, d\sigma_y \\ &\quad +\epsilon^n(\gamma(0)-k)\nabla_x N(0,z)\cdot\int_{\partial B}(\frac{\partial U}{\partial \nu_x}(\epsilon y)+\frac{\partial v}{\partial \nu_y}^{-})y\, d\sigma_y + O(\epsilon^{n+1}) \ .\end{aligned} \tag{49}$$

Now noting that $\int_{\partial B}\frac{\partial U}{\partial \nu_x}(\epsilon y)\, d\sigma_y = \epsilon\int_B \triangle_x U(\epsilon y)\, dy$ and $\int_{\partial B}\frac{\partial v^-}{\partial \nu_y}\, d\sigma_y = 0$, we have, upon substitution into (49)

$$\begin{aligned}\int_{\partial(\epsilon B)}(\gamma(0)-k)(\frac{\partial U}{\partial \nu_x}+\frac{\partial v}{\partial \nu_y}^{-}(x/\epsilon)\,)N\, d\sigma_x &= \epsilon^n(\gamma(0)-k)N(0,z)\int_B \triangle_x U(\epsilon y)\, dy \\ &\quad +\epsilon^n(\gamma(0)-k)\nabla_x N(0,z)\cdot\int_{\partial B}(\frac{\partial U}{\partial \nu_x}(\epsilon y)+\frac{\partial v}{\partial \nu_y}^{-})y\, d\sigma_y + O(\epsilon^{n+1}) \ .\end{aligned} \tag{50}$$

Inserting (48) and (50) into (47) and envoking the jump condition for the derivative of v, given in (13), we get

$$\begin{aligned}u_\epsilon(z) &= \int_{\partial\Omega}\psi N\, d\sigma_x + \epsilon^n(\gamma(0)-k)\nabla_x N(0,z)\cdot\int_{\partial B}(\frac{\partial U}{\partial \nu_x}(\epsilon y)+\frac{\partial v}{\partial \nu_y}^{-})y\, d\sigma_y + O(\epsilon^{n+\frac{1}{2}}) \\ &= \int_{\partial\Omega}\psi N\, d\sigma_x + \epsilon^n(\gamma(0)-k)\nabla_x N(0,z)\cdot\Bigg[\int_{\partial B}\nabla_x U(0)\cdot\nu_y\, y\, d\sigma_y \\ &\qquad + \int_{\partial B}\frac{1}{k}\left((\gamma(0)-k)\nabla_x U(0)\cdot\nu_y + \gamma(0)\frac{\partial v}{\partial \nu_y}^{+}\right) y\, d\sigma_y\Bigg] + O(\epsilon^{n+\frac{1}{2}}) \\ &= \int_{\partial\Omega}\psi N\, d\sigma_x + \epsilon^n\gamma(0)\frac{(\gamma(0)-k)}{k}\nabla_x N(0,z)\cdot\int_{\partial B}(\nabla_x U(0)\cdot\nu_y + \frac{\partial v}{\partial \nu_y}^{+}\,)\, y\, d\sigma_y \\ &\qquad + O(\epsilon^{n+\frac{1}{2}}) \ .\end{aligned} \tag{51}$$

Let ϕ_j denote the solution to

$$\begin{gathered}\Delta_y\phi_j = 0 \quad \text{in} \quad B \ , \qquad \Delta_y\phi_j = 0 \quad \text{in} \quad \Re^n\setminus\bar{B} \\ \phi_j \text{ is continuous across } \partial B\end{gathered} \tag{52}$$

$$\frac{\gamma(0)}{k}\frac{\partial\phi_j}{\partial\nu_y}^+ - \frac{\partial\phi_j}{\partial\nu_y}^- = -\nu_j \quad \text{on} \quad \partial B$$
$$\lim_{|y|\to\infty} \phi_j(y) = 0 \quad .$$

The existence and uniqueness of ϕ_j is established in a manner completely similar to that of v (cf. [5] or [11]). We note that ϕ_j and its first derivatives decay at ∞ in the same way as v and its first derivatives do (*cf.* (14) and (15)). It follows immediately from the definitions of v and ϕ_j that

$$v(y) = \frac{\gamma(0)-k}{k}\Sigma_{j=1}^n \frac{\partial U}{\partial x_j}(0)\phi_j(y) \quad ,$$

and so (51) may be rewritten

$$\begin{aligned} u_\epsilon(z) = \epsilon^n\gamma(0)\frac{\gamma(0)-k}{k}\,\nabla_x\, N(0,z)\cdot M\nabla_x U(0) \\ + \int_{\partial\Omega} \psi(x)N(x,z)\,d\sigma_x + O(\epsilon^{n+\frac{1}{2}}) \quad , \end{aligned} \tag{53}$$

where the (rescaled) polarization tensor $M = m_{ij}$ is given by

$$\begin{aligned} m_{ij} &= \int_{\partial B}\left(y_i\nu_j + \frac{\gamma(0)-k}{k}y_i\,\frac{\partial\phi_j}{\partial\nu}^+\right)\,d\sigma_y \\ &= |B|\delta_{ij} + \left(\frac{\gamma(0)}{k}-1\right)\int_{\partial B} y_i\,\frac{\partial\phi_j}{\partial\nu}^+\,d\sigma_y \quad . \end{aligned}$$

The term $O(\epsilon^{n+\frac{1}{2}})$ is bounded by $C_d\epsilon^{n+\frac{1}{2}}$ uniformly in $z \in \Omega\cap\{dist(z,\epsilon B) \geq d > 0\}$, and so is its derivatives (the latter fact easily follows from the first by elliptic apriori estimates). By continuity the formula (53) also holds for $z \in \partial\Omega$ (provided $dist(\partial\Omega,\epsilon B) \geq d > 0$). We note that the (rescaled) polarization tensor, M, only depends on the ratio $r = \gamma(0)/k$, not on the individual conductivities $\gamma(0)$ and k. We shall occasionally use the notation $M(r)$ to make this dependence on $r = \gamma(0)/k$ more explicit. If the single inhomogeneity is given by $z_1 + \epsilon B$ (instead of B) then the only effect is to change 0 to z_1. It is not essential that B be strictly star-shaped with respect to the origin, it suffices that B be strictly star-shaped with respect to any point. We simply write $B = p + \tilde{B}$, where $\tilde{B} = B - p$ is now star-shaped with respect to the origin. The inhomogeneity $z_1 + \epsilon B$ may now be written as $z_1 + \epsilon p + \epsilon\tilde{B}$; the order ϵ translation does not materially affect the previous argument and $\tilde{B}$ has the same polarization tensor as B, so the formula (53) stays unchanged. In case of more than one inhomogeneity, say $\omega_\epsilon = \cup_{i=1}^m(z_i + \epsilon B_i)$ (with $z_i + \epsilon B_i$ of conductivity k_i) the previous argument may very simply be changed to proceed inductively one inhomogeneity at a time. In summary we have therefore proven

Theorem 2. *Suppose $n = 2$ or 3 and suppose the domains B_i, $i = 1,...,m$, are strictly star-shaped. Also suppose the points $z_i \in \Omega$, $i = 1\ldots m$, are mutually* *distinct and satisfy (4). Then for any $z \in \bar{\Omega} \setminus \{z_i\}_{i=1}^m$*

$$u_\epsilon(z) = \epsilon^n \Sigma_{i=1}^m \gamma(z_i) \frac{\gamma(z_i) - k_i}{k_i} \nabla_x N(z_i, z) \cdot M_i \nabla_x U(z_i)$$
$$+ \int_{\partial\Omega} \psi(x) N(x, z)\, d\sigma_x + O(\epsilon^{n+\frac{1}{2}}) \quad , \tag{54}$$

for ϵ sufficiently small. The term $O(\epsilon^{n+\frac{1}{2}})$ and its derivatives are uniformly bounded by $C_d \epsilon^{n+\frac{1}{2}}$ on $\bar{\Omega} \cap \{dist(z, \omega_\epsilon) \geq d > 0\}$. The rescaled polarization tensor, M_i, corresponding to the i^{th} inhomogeneity is calculated just as before, only with B replaced by B_i and k by k_i.

Remark 1

An interesting special case is when the background conductivity, γ, is constant, the inhomogeneities have a common conductivity k, and all the domains B_i have been obtained from the same B by a dilatation and a rotation. Let $\rho_i > 0$ be the dilatation parameter and Q_i the rotation corresponding to the i-th inhomogeneity, *i.e.*, $B_i = \rho_i Q_i B$. In this case it is not difficult to compute that $M_i = \rho_i^n Q_i M Q_i^T$, where M is the (rescaled) polarization tensor correponding to B and conductivity ratio $r = \gamma/k$. Thus the representation formula for u_ϵ becomes

$$u_\epsilon(z) = \Sigma_{i=1}^m (\epsilon\rho_i)^n \gamma \frac{\gamma - k}{k} \nabla_x N(z_i, z) \cdot Q_i M Q_i^T \nabla_x U(z_i)$$
$$+ \int_{\partial\Omega} \psi(x) N(x, z)\, d\sigma_x + O(\epsilon^{n+\frac{1}{2}}) \quad .$$

Remark 2

As pointed out earlier representation formuli similar to (54) may be obtained using other fundamental solutions for the operator $-\nabla \cdot \gamma \nabla$ than the Neumann function, N. The only difference will be the presence of a second boundary integral. A particularly simple case is when the background conductivity γ is constant. In that situation it is natural in place of $N(x, y)$ to use

$$\Phi(x, y) = \begin{cases} -\frac{1}{2\pi\gamma} \log|x - y| \quad , & n = 2 \\ \frac{1}{4\pi\gamma} |x - y|^{-1} \quad , & n = 3 \end{cases}$$

For $z \in \Omega \setminus \{z_i\}_{i=1}^m$ the representation formula for u_ϵ now becomes

$$u_\epsilon(z) = \epsilon^n \sum_{i=1}^m \gamma \frac{\gamma - k_i}{k_i} \nabla_x \Phi(z_i, z) \cdot M_i \nabla_x U(z_i)$$
$$+ \int_{\partial\Omega} \psi(x) \Phi(x, z)\, d\sigma_x - \int_{\partial\Omega} u_\epsilon(x) \gamma \frac{\partial \Phi}{\partial \nu_x}(x, z)\, d\sigma_x + O(\epsilon^{n+\frac{1}{2}}) \quad ,$$

or by subtraction of $U(z)$ on both sides

$$u_\epsilon(z) - U(z) = \epsilon^n \sum_{i=1}^{m} \gamma \frac{\gamma - k_i}{k_i} \nabla_x \Phi(z_i, z) \cdot M_i \nabla_x U(z_i)$$
$$- \int_{\partial\Omega} (u_\epsilon(x) - U(x)) \gamma \frac{\partial \Phi}{\partial \nu_x}(x, z)\, d\sigma_x + O(\epsilon^{n+\frac{1}{2}}) \quad , \; z \in \Omega \setminus \{z_i\}_{i=1}^m \quad .$$

The term $O(\epsilon^{n+\frac{1}{2}})$ is bounded by $C\epsilon^{n+\frac{1}{2}}$ uniformly for $z \in \Omega \cap \{dist(z, \omega_\epsilon) \geq d > 0\}$. As z converges to the boundary, the integral

$$\int_{\partial\Omega} (u_\epsilon(x) - U(x)) \gamma \frac{\partial \Phi}{\partial \nu_x}(x, z)\, d\sigma_x$$

approaches

$$-\frac{1}{2}(u_\epsilon(z) - U(z)) + \int_{\partial\Omega} (u_\epsilon(x) - U(x)) \gamma \frac{\partial \Phi}{\partial \nu_x}(x, z)\, d\sigma_x \quad ,$$

and so by continuity we obtain the (rearranged) formula

$$u_\epsilon(z) - U(z) + 2 \int_{\partial\Omega} (u_\epsilon(x) - U(x)) \gamma \frac{\partial \Phi}{\partial \nu_x}(x, z)\, d\sigma_x \tag{55}$$
$$= 2\epsilon^n \sum_{i=1}^{m} \gamma \frac{\gamma - k_i}{k_i} \nabla_x \Phi(z_i, z) \cdot M_i \nabla_x U(z_i) + O(\epsilon^{n+\frac{1}{2}}) \quad , \quad z \in \partial\Omega \quad .$$

This may be thought of as a representation formula for $L(u_\epsilon - U)$ where $L(V)$ is defined by $L(V) = V + 2\int_{\partial\Omega} V \gamma \frac{\partial \Phi}{\partial \nu_x}\, d\sigma_x$. For the inverse problem, when $u_\epsilon - U$ and thus $L(u_\epsilon - U)$ is known on the entire boundary, then the formula (55) becomes a very effective tool for the identification of the z_i and properties of $\frac{\gamma - k_i}{k_i} M_i$ (as we shall see in the last section of this paper). It should be pointed out that the formula (55) is not well suited for the inverse problem when the data $u_\epsilon - U$ is only known on part of $\partial\Omega$, nor is it suited to prove the kind of continuous dependence results discussed in section 7. For those purposes the version of the representation formula stated in Theorem 2 is much more useful.

6 Properties of the polarization tensor

In this section we analyse in some detail the properties of the $n \times n$ matrix M, introduced in the previous section. We have already referred to this matrix as the rescaled polarization tensor associated to B and conductivity ratio $r = \gamma(0)/k$. The polarization tensor itself is defined to be $(r-1)M$. We recall that $M = \{m_{ij}\}$ is given by

$$m_{ij} = |B|\delta_{ij} + (r - 1) \int_{\partial B} y_i \frac{\partial \phi_j}{\partial \nu}^{+} d\sigma_y. \tag{56}$$

We begin by showing

Lemma 5 . *The polarization tensor is symmetric.*

Proof. In order to verify the symmetry, we shall rewrite the integral appearing in the definition of M. Using the jump condition for $\frac{\partial \phi_j}{\partial \nu}$ on ∂B and integrating by parts we obtain

$$\begin{aligned}\int_{\partial B} y_i \frac{\partial \phi_j}{\partial \nu}^+ d\sigma_y &= \frac{1}{r}\int_{\partial B} y_i (\frac{\partial \phi_j}{\partial \nu}^- - \nu_j)\, d\sigma_y \\ &= \frac{1}{r}\Big(\int_{\partial B} \phi_j \nu_i \, d\sigma_y - \int_{\partial B} y_i \nu_j \, d\sigma_y\Big) \\ &= \frac{1}{r}\Big(\int_{\partial B} \phi_j \nu_i \, d\sigma_y - |B|\delta_{ij}\Big) \quad . \end{aligned} \tag{57}$$

Substitution of (57) into (56) and simplification yields

$$m_{ij} = \frac{1}{r}|B|\delta_{ij} + \Big(1 - \frac{1}{r}\Big)\int_{\partial B} \phi_j \nu_i \, d\sigma \quad . \tag{58}$$

To establish the symmetry of M (or $(r-1)M$) it now suffices to show that $\int_{\partial B} \phi_j \nu_i \, d\sigma = \int_{\partial B} \phi_i \nu_j \, d\sigma$, $1 \le i, j \le n$. Using the jump condition for $\frac{\partial \phi_i}{\partial \nu}$ on ∂B once more we get

$$\int_{\partial B} \phi_j \nu_i \, d\sigma = \int_{\partial B} \frac{\partial \phi_i}{\partial \nu}^- \phi_j \, d\sigma - r\int_{\partial B} \frac{\partial \phi_i}{\partial \nu}^+ \phi_j \, d\sigma \quad . \tag{59}$$

The first integral on the right hand side of (59) may immediately be integrated by parts to become $\int_{\partial B} \phi_i \frac{\partial \phi_j^-}{\partial \nu}\, d\sigma$. As previously observed ϕ_j has the following behavior (decay) at infinity

$$\phi_j(y) = \begin{cases} O(|y|^{-1}) & n = 2 \\ O(|y|^{2-n}) & n \ge 3 \end{cases} \qquad \nabla_y \phi_j(y) = \begin{cases} O(|y|^{-2}) & n = 2 \\ O(|y|^{1-n}) & n \ge 3 \quad . \end{cases}$$

This decay implies that the second integral on the right hand side of (59) may also be integrated by parts. Altogether we obtain

$$\begin{aligned}\int_{\partial B} \phi_j \nu_i \, d\sigma &= \int_{\partial B} \frac{\partial \phi_j}{\partial \nu}^- \phi_i \, d\sigma - r\int_{\partial B} \frac{\partial \phi_j}{\partial \nu}^+ \phi_i \, d\sigma \\ &= \int_{\partial B} \phi_i \nu_j \, d\sigma \qquad 1 \le i, j \le n \quad , \end{aligned}$$

and therefore, M is symmetric.

□

We shall use that the polarization tensor is invertible whenever $\gamma(0) \ne k$ $(r \ne 1)$. This follows immediately from

Lemma 6 . *The rescaled polarization tensor, M, is positive definite. Thus, the polarization tensor $(r-1)M$ is positive definite for $r > 1$ $(\gamma(0) > k)$ and it is negative definite for $r < 1$ $(\gamma(0) < k)$.*

Proof. Using (58)and (59) we compute

$$\sum m_{ij}\zeta_i\zeta_j = \frac{1}{r}\Big(|B||\zeta|^2 + (r-1)\Big[\int_{\partial B} \frac{\partial \chi}{\partial \nu}^{-} \chi \, d\sigma - r\int_{\partial B} \frac{\partial \chi}{\partial \nu}^{+} \chi \, d\sigma\Big]\Big) \quad ,$$

where $\chi = \sum \zeta_i \phi_i$, and ν is the outer normal to B. Integrating by parts we obtain

$$\sum m_{ij}\zeta_i\zeta_j = \frac{1}{r}\Big(|B||\zeta|^2 + (r-1)\Big[\int_{B} |\nabla\chi|^2 \, dy + r\int_{\Re^n\setminus\bar{B}} |\nabla\chi|^2 \, dy\Big]\Big) \quad .$$

It follows directly from this representation that if $r \geq 1$ ($\gamma(0) \geq k$) then M is positive definite. Note that if $r = 1$ ($\gamma(0) = k$) then M is simply given by

$$m_{ij} = |B|\delta_{ij} \quad ,$$

however, the polarization tensor $(r-1)M$ vanishes.

We now seek a different representation for M that will allow us to verify its positive definiteness in the case $r < 1$ ($\gamma(0) < k$). Rewriting (56) as a single integral and rearranging terms, we have

$$\begin{aligned} m_{ij} &= \int_{\partial B} \Big(y_i\nu_j + y_i(r-1)\frac{\partial \phi_j}{\partial \nu}^{+} \Big) \, d\sigma_y \\ &= \int_{\partial B} y_i \Big((r-1)\nabla\phi_j^+ + \nabla y_j \Big) \cdot \nu \, d\sigma_y \quad . \end{aligned}$$

Using the jump condition for $\frac{\partial \phi_j}{\partial \nu}$ on ∂B and simplifying we get

$$\begin{aligned} m_{ij} &= \int_{\partial B} y_i \Big((1-\frac{1}{r})(\nabla\phi_j^- - \nabla y_j) \; + \; \nabla y_j \Big) \cdot \nu \, d\sigma_y \\ &= \frac{1}{r}\int_{\partial B} y_i \Big((r-1)\nabla\phi_j^- \; + \; \nabla y_j \Big) \cdot \nu \, d\sigma_y \quad . \end{aligned}$$

Now note that the above can be rewritten as

$$\begin{aligned} m_{ij} &= \frac{1}{r}\Big[\int_{\partial B} \Big((r-1)\phi_i \; + \; y_i \Big) \Big((r-1)\nabla\phi_j^- \; + \; \nabla y_j \Big) \cdot \nu \, d\sigma_y \\ &\qquad + \; (1-r)\int_{\partial B} \phi_i \Big((r-1)\nabla\phi_j^- \; + \; \nabla y_j \Big) \cdot \nu \, d\sigma_y \Big] \quad . \end{aligned} \tag{60}$$

Application of the divergence theorem to the first integral in (60) yields

$$\begin{aligned} m_{ij} &= \frac{1}{r}\Big[\int_{B} \nabla((r-1)\phi_i + y_i) \cdot \nabla((r-1)\phi_j + y_j) \, dy \\ &\qquad + \; (1-r)\int_{\partial B} \phi_i \Big((r-1)\nabla\phi_j^- \; + \; \nabla y_j \Big) \cdot \nu \, d\sigma_y \Big] \quad . \end{aligned} \tag{61}$$

The second integral on the right hand side may also be further simplied. Using the jump condition for $\frac{\partial \phi_j}{\partial \nu}$ on ∂B we obtain

$$
\begin{aligned}
(1-r)\int_{\partial B} \phi_i \Big((r-1)\nabla\phi_j^- \;+\; \nabla y_j\Big)\cdot \nu \, d\sigma_y & \\
&= (1-r)\int_{\partial B} \phi_i \Big((r-1)\nabla\phi_j^- \;+\; \nabla\phi_j^- - r\nabla\phi_j^+\Big)\cdot \nu \, d\sigma_y \\
&= (1-r)r\int_{\partial B} \phi_i \Big(\nabla\phi_j^- \;-\nabla\phi_j^+\Big)\cdot \nu \, d\sigma_y \quad . \qquad (62)
\end{aligned}
$$

Expanding the right hand side of (62) as two separate integrals and integrating by parts (using the decay of ϕ_j) we get

$$
\begin{aligned}
(1-r)\int_{\partial B} \phi_i \Big((r-1)\nabla\phi_j^- \;+\; \nabla y_j\Big)\cdot \nu \, d\sigma_y & \\
&= (1-r)r\Big(\int_{\partial B} \phi_i \frac{\partial \phi_j}{\partial \nu}^- \, d\sigma_y - \int_{\partial B} \phi_i \frac{\partial \phi_j}{\partial \nu}^+ \, d\sigma_y\Big) \\
&= (1-r)r\int_{\Re^n} \nabla\phi_i \cdot \nabla\phi_j \, dy \quad .
\end{aligned}
$$

Note that ν is the outward unit normal to B. Substituting back into (61) we finally have

$$
m_{ij} = \frac{1}{r}\int_B \nabla((r-1)\phi_i + y_i)\cdot\nabla((r-1)\phi_j + y_j)\, dy \;+\; (1-r)\int_{\Re^n} \nabla\phi_i\cdot\nabla\phi_j \, dy \quad ,
$$

or

$$
\sum m_{ij}\zeta_i\zeta_j = \frac{1}{r}\int_B |\nabla\tilde{\chi}|^2 \, dy + (1-r)\int_{\Re^n} |\nabla\chi|^2 \, dy \quad , \qquad (63)
$$

where $\tilde{\chi} = \sum \zeta_i((r-1)\phi_i + y_i)$ and $\chi = \sum \zeta_i\phi_i$, . It follows immediately from (63) that M is also positive definite for $r < 1$ ($\gamma(0) < k$).

□

Remark 3

When B is the unit ball we can explicitly determine M. In this special case

$$
\phi_j = \frac{1}{(n-1)r+1}\, y_j \quad \text{in } B \quad , \qquad \phi_j = \frac{1}{(n-1)r+1}\, \frac{y_j}{|y|^n} \quad \text{in } \Re^n \setminus B \quad ,
$$

and a simple computation then gives

$$
m_{ij} = \frac{\omega_n}{(n-1)r+1}\,\delta_{ij} \quad ,
$$

where ω_n denotes the area of the unit sphere in $\Re^n$. The polarization tensor itself has entries $\delta_{ij}\omega_n(r-1)/((n-1)r+1)$.

□

In [7] we derived an asymptotic expansion for the voltage potential u_ϵ in the special case of inhomogeneities that are either perfectly insulating or perfectly conducting. We shall conclude this section by showing that the approximations we obtained in these cases are entirely the same as those we would obtain by letting k tend towards 0 or ∞ in the approximation obtained here. Polarization tensors corresponding to these two extreme cases have been considered many other places in the literature; we refer the reader to [16] and [9].

6.1 The limit as $k \to 0$.

Let $\phi_j^{(k)} \in C(\Re^n),\ j = 1, ..., n$, denote the unique solutions to

$$\begin{gathered}\triangle\phi_j^{(k)} = 0, \quad \text{in } B \ , \qquad \triangle\phi_j^{(k)} = 0, \quad \text{in } \Re^n \backslash \bar{B} \\ \phi_j^{(k)} \text{ is continuous across } \partial B \\ \frac{\partial \phi_j^{(k)}}{\partial \nu}^+ - \frac{k}{\gamma(0)} \frac{\partial \phi_j^{(k)}}{\partial \nu}^- = -\nu_j, \ \text{ on } \partial B \\ \lim_{|y|\to\infty} \phi_j^{(k)}(y) = 0 \ .\end{gathered} \tag{64}$$

There is a simple relation to the functions ϕ_j introduced earlier (*cf.* (52)), namely $\phi_j^{(k)} = \frac{\gamma(0)}{k}\phi_j$.

Lemma 7. *The sequence of functions $\{\phi_j^{(k)}|_{\partial B}\}$ converges uniformly, as $k \to 0$, to $\phi_j^{(0)}|_{\partial B}$, where $\phi_j^{(0)}$ is the solution to the following exterior Neumann problem,*

$$\begin{gathered}\Delta\phi_j^{(0)} = 0 \quad \text{in } \Re^n \backslash \bar{B} \ , \\ \frac{\partial \phi_j^{(0)}}{\partial \nu}^+ = -\nu_j \quad \text{on } \partial B \ , \quad \lim_{|y|\to\infty} \phi_j^{(0)}(y) = 0 \ .\end{gathered} \tag{65}$$

Proof. We first note that the exterior Neumann problem (65) is uniquely solvable. In $\Re^2$, it owes its solvability to the fact that $\int_{\partial B} \nu_j ds = 0$. Let X be the space of functions

$$X = \{\Psi \in C(\partial B) : \int_{\partial B} \Psi d\sigma = 0\} \ .$$

Let $\Phi(x, y)$ denote the fundamental solution

$$\Phi(x, y) = \begin{cases} -\frac{1}{2\pi} \log|x - y| \ , & n = 2 \\ \frac{1}{(n-2)\omega_n} |x - y|^{2-n} \ , & n \geq 3 \end{cases}$$

$\Phi(x, y)$ is as introduced before, only with $\gamma = 1$. Writing $\phi_j^{(k)}$ as a single layer potential with density $\Psi_j^{(k)} \in X$ we have

$$\phi_j^{(k)}(y) = \int_{\partial B} \Phi(x, y) \Psi_j^{(k)}(x) \, d\sigma_x \ .$$

From classical potential theory (*cf.* [5], [11]) it follows that $\phi_j^{(k)}$ is a solution to (64) iff

$$k\frac{\partial\phi_j^{(k)-}}{\partial\nu_y} - \gamma(0)\frac{\partial\phi_j^{(k)+}}{\partial\nu_y} = (k-\gamma(0))\int_{\partial B}\frac{\partial\Phi(x,y)}{\partial\nu_y}\Psi_j^{(k)}(x)\,d\sigma_x + \frac{k+\gamma(0)}{2}\Psi_j^{(k)}(y)$$
$$= \gamma(0)\nu_j. \tag{66}$$

Multiplying equation (66) by $\frac{2}{k+\gamma(0)}$ we obtain

$$(I-A^{(k)})\Psi_j^{(k)} = f_j^{(k)} \quad ,$$

where $A^{(k)}$ is the compact operator ($X \to X$) defined by

$$A^{(k)}\Psi = -2\left(\frac{k-\gamma(0)}{k+\gamma(0)}\right)\int_{\partial B}\frac{\partial\Phi(x,y)}{\partial\nu_y}\Psi(x)\,d\sigma_x \quad ,$$

and

$$f_j^{(k)} = \frac{2\gamma(0)}{k+\gamma(0)}\,\nu_j \quad .$$

To see that $A^{(k)}$ maps X into X we use the fact that $\int_{\partial B}\frac{\partial\Phi(x,y)}{\partial\nu_y}\,d\sigma_y = -1/2$ for $x \in \partial B$. Similarly writing $\phi_j^{(0)}(y) = \int_{\partial B}\Phi(x,y)\Psi_j^{(0)}(x)\,d\sigma_x$ as a single layer potential with density $\Psi_j^{(0)} \in X$ a necessary and sufficient condition that $\phi_j^{(0)}$ solves (65) is

$$(I-A^{(0)})\Psi_j^{(0)} = f_j^{(0)} \quad . \tag{67}$$

Here $A^{(0)}$ ($X \to X$) is the compact operator given by

$$A^{(0)}\Psi = 2\int_{\partial B}\frac{\partial\Phi(x,y)}{\partial\nu_y}\Psi(x)d\sigma_x \quad ,$$

and

$$f_j^{(0)} = 2\,\nu_j$$

$j = 1,...,n$. To establish the lemma, it now suffices to show that $\Psi_j^{(k)}$ converges uniformly to $\Psi_j^{(0)}$. From the uniqueness associated with (65) it follows that (67) has at most one solution. By the Fredholm Alternative, $(I-A^{(0)})^{-1}$ exists and is bounded. The sequence $A^{(k)}$ converges to $A^{(0)}$ in the operator norm as $k \to 0$, and consequently, the invertible operators $I - A^{(k)}$ have a uniform bound for their inverses

$$\|(I-A^{(k)})^{-1}\| \le C \quad , \quad \forall k \le 1 \quad .$$

A simple estimation now gives

$$\begin{aligned}\|\Psi_j^{(k)} - \Psi_j^{(0)}\| &\le \|(I-A^{(k)})^{-1}\| \;\|(I-A^{(k)})(\Psi_j^{(k)} - \Psi_j^{(0)})\| \\ &\le C\Big(\|(I-A^{(k)})\Psi_j^{(k)} - (I-A^{(0)})\Psi_j^{(0)}\| + \|(A^{(k)}-A^{(0)})\Psi_j^{(0)}\|\Big) \\ &\le C\Big(\|f_j^{(k)} - f_j^{(0)}\| + \|(A^{(k)}-A^{(0)})\Psi_j^{(0)}\|\Big) \quad ,\end{aligned}$$

where all the norms pertain to the space X. The lemma now follows from the facts that $f_j^{(k)} \to f_j^{(0)}$ and $A^{(k)} \to A^{(0)}$ as $k \to 0$.

$\square$

Let $M^{(k)} = \{m_{ij}^{(k)}\}$ denote the polarization tensor $M^{(k)} = (\frac{\gamma(0)}{k} - 1)M(\frac{\gamma(0)}{k})$ (with $M(\frac{\gamma(0)}{k})$ being the rescaled polarization tensor introduced earlier). From (58) we get

$$\begin{aligned} m_{ij}^{(k)} &= (1 - \frac{1}{r})|B|\delta_{ij} + (1 - \frac{1}{r})^2 r \int_{\partial B} \phi_j \nu_i \, d\sigma \\ &= (1 - \frac{1}{r})|B|\delta_{ij} + (1 - \frac{1}{r})^2 \int_{\partial B} \phi_j^{(k)} \nu_i \, d\sigma \quad , \end{aligned}$$

with $r = \frac{\gamma(0)}{k}$. Due to Lemma 7 it follows immediately that the tensor $M^{(k)}$, as $k \to 0$ ($r \to \infty$) converges to the tensor $M^{(0)} = \{m_{ij}^{(0)}\}$, given by

$$m_{ij}^{(0)} = |B|\delta_{ij} + \int_{\partial B} \phi_j^{(0)} \nu_i \, d\sigma \quad .$$

The latter is exactly the tensor defined in (5.7) of [7]. This means that the present asymptotic formula in the extreme case $k \to 0$ (and for simplicity, with one inhomogeneity "centered" at the origin)

$$\begin{aligned} u_\epsilon(z) &\approx \epsilon^n \frac{\gamma(0)(\gamma(0) - k)}{k} \nabla_x N(0, z) \cdot M(\frac{\gamma(0)}{k}) \nabla_x U(0) + U(z) \\ &= \epsilon^n \gamma(0) \nabla_x N(0, z) \cdot M^{(k)} \nabla_x U(0) + U(z) \end{aligned}$$

is entirely consistent with the formula derived in [7] (with $\rho = 1$). The separate proof of the asymptotic validity of this formula (for "$k = 0$") is still needed since we have not shown that the term $O(\epsilon^{n+\frac{1}{2}})$ is uniform in k, as $k \to 0$.

6.2 The limit as $k \to \infty$

To study the limit of the polarization tensor as $k \to \infty$ it is convenient to perform a different rescaling of the functions ϕ_j. In this section we let $\phi_j^{(k)}$ denote the solution to

$$\begin{gathered} \Delta \phi_j^{(k)} = 0, \quad \text{in } B \ , \qquad \Delta \phi_j^{(k)} = 0, \quad \text{in } \Re^n \setminus \bar{B} \\ \phi_j^{(k)} \text{ is continuous across } \partial B \\ \frac{\gamma(0)}{k} \frac{\partial \phi_j^{(k)}}{\partial \nu}^{+} - \frac{\partial \phi_j^{(k)}}{\partial \nu}^{-} = -\frac{k - \gamma(0)}{k} \nu_j, \quad \text{on } \partial B \\ \lim_{|y| \to \infty} \phi_j^{(k)}(y) = 0 \quad . \end{gathered} \tag{68}$$

With this new definition $\phi_j^{(k)} = \frac{k-\gamma(0)}{k} \phi_j$, where ϕ_j is the solution to (52). Once again, we represent $\phi_j^{(k)}$ in the form of a single layer potential

$$\phi_j^{(k)}(y) = \int_{\partial B} \Phi(x, y) \Psi_j^{(k)}(x) d\sigma_x \quad ,$$

$\Psi_j^{(k)} \in X$. In this case

$$\frac{\partial \phi_j^{(k)}{}^-}{\partial \nu_y} = \frac{1}{2}\Psi_j^{(k)}(y) + \int_{\partial B} \frac{\partial \Phi(x,y)}{\partial \nu_y}\Psi_j^{(k)}(x)d\sigma_x \quad , \tag{69}$$

and

$$\frac{\gamma(0)}{k}\frac{\partial \phi_j^{(k)}{}^+}{\partial \nu_y} = -\frac{\gamma(0)}{2k}\Psi_j^{(k)}(y) + \frac{\gamma(0)}{k}\int_{\partial B} \frac{\partial \Phi(x,y)}{\partial \nu_y}\Psi_j^{(k)}(x)d\sigma_x \quad . \tag{70}$$

Insertion of equation (69) and (70) into (68) yields

$$\frac{k+\gamma(0)}{2k}\Psi_j^{(k)}(y) + (1-\frac{\gamma(0)}{k})\int_{\partial B} \frac{\partial \Phi(x,y)}{\partial \nu_y}\Psi_j^{(k)}(x)\, d\sigma_x = \frac{k-\gamma(0)}{k}\nu_j \quad .$$

Multiplication by $2k/(k+\gamma(0))$ therefore gives the following (necessary and sufficient) equation for Ψ_j

$$\Psi_j^{(k)}(y) + 2\frac{(k-\gamma(0))}{(k+\gamma(0))}\int_{\partial B} \frac{\partial \Phi(x,y)}{\partial \nu_y}\Psi_j^{(k)}(x)\, d\sigma_x = 2\frac{k-\gamma(0)}{k+\gamma(0)}\nu_j \quad .$$

Letting $k \to \infty$ we formally obtain

$$\Psi_j^{(\infty)} + A^{(\infty)}\Psi_j^{(\infty)} = 2\nu_j \quad , \quad \Psi_j^{(\infty)} \in X \quad , \tag{71}$$

where

$$A^{(\infty)}\Psi = 2\int_{\partial B} \frac{\partial \Phi(x,y)}{\partial \nu_y}\Psi(x)\, d\sigma_x \quad .$$

Indeed by an argument similar to that in the previous section one can show that $\Psi_j^{(k)}$ converges uniformly to $\Psi_j^{(\infty)}$, and therefore, $\phi_j^{(k)}|_{\partial B}$ converges uniformly to $\phi_j^{(\infty)}|_{\partial B}$, where $\phi_j^{(\infty)}$ is the single layer potential with density function $\Psi_j^{(\infty)}$. It is also easy to prove that $\frac{\partial \phi_j^{(k)}{}^+}{\partial \nu}$ converges uniformly to $\frac{\partial \phi_j^{(\infty)}{}^+}{\partial \nu}$ on ∂B. From (71) it follows that $\phi_j^{(\infty)}|_B$ solves the interior Neumann problem

$$\Delta \phi_j^{(\infty)} = 0 \quad \text{in} \quad B \; ,$$
$$\frac{\partial \phi_j^{(\infty)}{}^-}{\partial \nu} = \nu_j \quad \text{on} \quad \partial B \quad .$$

In other words $\phi_j^{(\infty)} = y_j + c_j$ on B, for some constant c_j. From (71) it follows that $\phi_j^{(\infty)}|_{\Re^n \setminus \bar{B}}$ solves the exterior Neumann problem (for the Laplacian) with boundary data $\frac{\partial \phi_j^{(\infty)}{}^+}{\partial \nu} = \nu_j - \Psi_j^{(\infty)}$, and it satisfies $\phi_j^{(\infty)} \to 0$ as $|y| \to \infty$ (for $n = 2$ this relies on the fact that $\int_{\partial B} \Psi_j^{(\infty)}\, d\sigma = 0$). Furthermore $\phi_j^{(\infty)}$ satisfies the exterior Dirichlet problem (for the Laplacian) with boundary data

$\phi_j^{(\infty)}|_{\partial B} = y_j + c_j$. From this latter observation it follows immediately that $\phi_j^{(\infty)} = -\Phi_j + c_j\phi$ where Φ_j is the solution to the exterior dirichlet problem

$$\begin{aligned} \Delta\Phi_j = 0 \quad &\text{in } \Re^n\backslash\bar{B} \ , \\ \Phi_j = -y_j \quad &\text{on } \partial B \ , \\ \Phi_j \quad &\text{is harmonic at } \infty \ , \end{aligned}$$

and ϕ is the solution to

$$\begin{aligned} \Delta\phi = 0 \quad &\text{in } \Re^n\backslash\bar{B} \ , \\ \phi = 1 \quad &\text{on } \partial B \ , \\ \phi \quad &\text{is harmonic at } \infty \ . \end{aligned}$$

The requirement that a function, which satisfies the equation $\triangle u = 0$ outside $\bar{B}$, be harmonic at ∞ is equivalent to the the requirement that its Kelvin Transform have a removable singularity at 0. For n=2 the function ϕ is identically equal to the constant 1, and so

$$\phi_j^{(\infty)} = -\Phi_j + c_j \ . \tag{72}$$

The fact that $\lim_{|y|\to\infty} \phi_j^{(\infty)}(y) = 0$, uniquely determines c_j. For $n \geq 3$ the requirement that ϕ be harmonic at ∞ is equivalent to the requirement that $\lim_{|y|\to\infty} \phi(y) = 0$, and so the function ϕ is nontrivial.

As noted before we have $\frac{\partial\phi_j^{(\infty)}}{\partial\nu}^{+} = \nu_j - \Psi_j^{(\infty)}$ on ∂B, ; since $\Psi_j^{(\infty)} \in X$ it now follows that $\int_{\partial B} \frac{\partial\phi_j^{(\infty)}}{\partial\nu}^{+} d\sigma = 0$. Therefore

$$0 = \int_{\partial B} \frac{\partial\phi_j^{(\infty)}}{\partial\nu}^{+} d\sigma = -\int_{\partial B} \frac{\partial\Phi_j}{\partial\nu}\, d\sigma + c_j \int_{\partial B} \frac{\partial\phi}{\partial\nu}\, d\sigma \ .$$

For $n \geq 3$ we have that $\frac{\partial\phi}{\partial\nu} < 0$ everywhere on ∂B (by the maximum principle) and we may thus solve the above equation for c_j

$$c_j = \int_{\partial B} \frac{\partial\Phi_j}{\partial\nu}\, d\sigma \left(\int_{\partial B} \frac{\partial\phi}{\partial\nu}\, d\sigma\right)^{-1} \ .$$

Let $M^{(k)} = \{m_{ij}^{(k)}\}$ denote the polarization tensor $M^{(k)} = (\frac{\gamma(0)}{k} - 1)M(\frac{\gamma(0)}{k})$ ($M(\frac{\gamma(0)}{k})$ being the rescaled polarization tensor introduced earlier). From (56) we get

$$m_{ij}^{(k)} = (r-1)|B|\delta_{ij} + (r-1)^2 \int_{\partial B} y_i \frac{\partial\phi_j^{(k)}}{\partial\nu}^{+} d\sigma_y \ ,$$

with $r = \frac{\gamma(0)}{k}$. From the fact that $\frac{\partial\phi_j^{(k)}}{\partial\nu}^{+}$ converges uniformly to

$$\frac{\partial\phi_j^{(\infty)}}{\partial\nu}^{+} = -\frac{\partial\Phi_j}{\partial\nu} + c_j\frac{\partial\phi}{\partial\nu} \ ,$$

we conclude that the tensor $M^{(k)}$, as $k \to \infty$ ($r \to 0$) converges to the tensor $M^{(\infty)} = \{m_{ij}^{(\infty)}\}$, given by

$$\begin{aligned} m_{ij}^{(\infty)} &= -|B|\delta_{ij} + \int_{\partial B} y_i \frac{\partial \phi_j^{(\infty)+}}{\partial \nu} \, d\sigma_y \\ &= -|B|\delta_{ij} - \int_{\partial B} y_i \frac{\partial \Phi_j}{\partial \nu} \, d\sigma_y + c_j \int_{\partial B} y_i \frac{\partial \phi}{\partial \nu} \, d\sigma_y \quad . \end{aligned}$$

For $n = 2$ we substitute $\Phi_i = -y_i$ on ∂B and use that $\frac{\partial \phi}{\partial \nu} = 0$ on ∂B, to get

$$m_{ij}^{(\infty)} = -|B|\delta_{ij} + \int_{\partial B} \Phi_i \frac{\partial \Phi_j}{\partial \nu} \, d\sigma_y \quad ;$$

for $n = 3$ we substitute Φ_i for $-y_i$ on ∂B and integrate the last (nontrivial) integral by parts, to obtain

$$\begin{aligned} m_{ij}^{(\infty)} &= -|B|\delta_{ij} + \int_{\partial B} \Phi_i \frac{\partial \Phi_j}{\partial \nu} \, d\sigma - c_j \int_{\partial B} \frac{\partial \Phi_i}{\partial \nu} \, d\sigma \\ &= -|B|\delta_{ij} + \int_{\partial B} \Phi_i \frac{\partial \Phi_j}{\partial \nu} \, d\sigma - \left(\int_{\partial B} \frac{\partial \phi}{\partial \nu} \, d\sigma \right)^{-1} \int_{\partial B} \frac{\partial \Phi_i}{\partial \nu} \, d\sigma \int_{\partial B} \frac{\partial \Phi_j}{\partial \nu} \, d\sigma \quad . \end{aligned}$$

The latter two formulas are modulo a change of sign exactly those given for the polarization tensor in (3.8) of [7]. This means that the present asymptotic formula in the extreme case $k \to \infty$ (and for simplicity, with one inhomogeneity "centered" at the origin)

$$\begin{aligned} u_\epsilon(z) &\approx \epsilon^n \frac{\gamma(0)(\gamma(0) - k)}{k} \nabla_x N(0, z) \cdot M(\frac{\gamma(0)}{k}) \nabla_x U(0) + U(z) \\ &= -\epsilon^n \gamma(0) \nabla_x N(0, z) \cdot \ (-M^{(k)}) \nabla_x U(0) + U(z) \end{aligned}$$

is entirely consistent with the formula derived in [7] (with $\rho = 1$). The separate proof of the asymptotic validity of this formula (for "$k = \infty$") is still needed since we have not shown that the term $O(\epsilon^{n+\frac{1}{2}})$ is uniform in k, as $k \to \infty$.

7 The continuous dependence of the inhomogeneities

In a fashion completely similar to that in [7] the representation formula (54) may be used to prove asymptotic Lipschitz estimates for certain characteristics of the inhomogeneities in terms of the (rescaled) boundary data. As an example let $\omega_\epsilon = \cup_{i=1}^m (z_i + \epsilon \rho_i B)$ and $\omega'_\epsilon = \cup_{i=1}^{m'} (z'_i + \epsilon \rho'_i B)$ be two sets of inhomogeneities with $0 < d_0 \leq |z_i - z_j|$, $\forall i \neq j$, $0 < d_0 \leq dist(z_i, \partial\Omega)$, $\forall i$, and $0 < d_0 \leq \rho_i \leq D_0$, $\forall i$ (and similar restrictions on the z'_i and ρ'_i). We suppose all the inhomogenieties have the same *known* conductivity $k \in]0, \infty[\backslash [\gamma_{min}, \gamma_{max}]$, where $\gamma_{min} = \min_{x \in \bar{\Omega}} \gamma(x)$ and $\gamma_{max} = \max_{x \in \bar{\Omega}} \gamma(x)$. Let u_ϵ and u'_ϵ denote the voltage potentials corresponding to ω_ϵ and ω'_ϵ for a fixed, nontrivial boundary current ψ. It is crucial that $\nabla U(x) \neq 0$ $\forall x \in \Omega$.

Theorem 3. *Suppose $n = 2$ or 3 and suppose B is strictly star-shaped. Let Γ be an open, nonempty subset of $\partial\Omega$. There exist constants $0 < \epsilon_0$, δ_0 and C such that if $0 < \epsilon < \epsilon_0$ and $\epsilon^{-n}\|u_\epsilon - u'_\epsilon\|_{L^\infty(\Gamma)} < \delta_0$ then*

$$(i) \qquad m = m' \text{ , and, after appropriate reordering,}$$

$$(ii) \qquad \sum_{i=1}^{m} \left(|z_i - z'_i| + |\rho_i - \rho'_i|\right) \le C\left(\epsilon^{-n}\|u_\epsilon - u'_\epsilon\|_{L^\infty(\Gamma)} + \epsilon^{1/2}\right)$$

The constants ϵ_0, δ_0 and C depend on Γ, Ω, γ, ψ, d_0, D_0, B, and k but are otherwise independent of the two sets of inhomogeneities.

The factor ϵ^{-n} in front of $\|u_\epsilon - u'_\epsilon\|_{L^\infty(\Gamma)}$ is best possible; it follows immediately from our analysis (*cf.* Theorem 2) that even if $|z_i - z'_i|$ and $|\rho_i - \rho'_i|$ are all of order 1 the difference in the boundary data is of order ϵ^n. For fixed (small) ϵ the above theorem essentially asserts that the locations of the inhomogeneities and their relative sizes depend Lipschitz-continuously on the rescaled boundary deviation $\epsilon^{-n}\|u_\epsilon - u'_\epsilon\|_{L^\infty(\Gamma)}$. We shall only here provide a sketch of the main ideas of the proof of Theorem 3; for complete details (of the proof of an almost identical result) we refer the reader to [7]. Let us also note that for $n = 2$ and $\Gamma = \partial\Omega$ there is a close relation between Theorem 3 and the continuous dependence results found in [13].

Sketch of the main ideas of proof By combining the representation formula (54) with the fact that $M_i = \rho_i^n M(\gamma(z_i)/k)$, where $M(r)$ is the rescaled polarization tensor associated with B and conductivity ratio r, we get

$$\epsilon^{-n}(u_\epsilon(z) - u'_\epsilon(z)) = \sum_{i=1}^{m} \rho_i^n \gamma(z_i) \frac{\gamma(z_i) - k}{k} \nabla_x N(z_i, z) \cdot M(\frac{\gamma(z_i)}{k}) \nabla_x U(z_i)$$
$$- \sum_{i=1}^{m'} (\rho'_i)^n \gamma(z'_i) \frac{\gamma(z'_i) - k}{k} \nabla_x N(z'_i, z) \cdot M(\frac{\gamma(z'_i)}{k}) \nabla_x U(z'_i) + O(\epsilon^{1/2}) \quad .$$

It is now not hard to see that if $\epsilon^{-n}\|u_\epsilon - u'_\epsilon\|_{L^\infty(\Gamma)}$ and ϵ are both sufficiently small then m and m' must necessarily be equal; for if not one may argue that there exists a function of the form

$$F(z) = \sum_{i=1}^{m} \nabla_x N(z_i, z) \cdot \alpha_i - \sum_{i=1}^{m'} \nabla_x N(z'_i, z) \cdot \alpha'_i$$

(with $\alpha_i \neq 0$ and $\alpha'_i \neq 0$) and $m \neq m'$ such that

$$F(z) = 0 \quad z \in \Gamma \ .$$

To see that α_i as well as α'_i are not 0 we use the fact that ∇U never vanishes and that the tensor M is invertible. Since $N(x, z) = N(z, x)$ we have $\nabla_x N(z_i, z) = \nabla_1 N(z_i, z) = \nabla_2 N(z, z_i)$, where ∇_1 and ∇_2 indicate the gradient with respect to the first and second variable respectively. From the definition of $N(x,z)$ it follows that

$$\gamma(z)\frac{\partial}{\partial \nu_z}\nabla_2 N(z,z_i)=0 \quad z\in\partial\Omega \quad ,$$

and we thus obtain

$$\gamma(z)\frac{\partial}{\partial \nu_z}F(z)=0 \quad z\in\partial\Omega \quad .$$

The function F is also easily seen to be harmonic in $\Omega\setminus(\{z_i\}\cup\{z_i'\})$. From the uniqueness of the solution to the Cauchy problem for the Laplacian we conclude that $F(z)\equiv 0$ in Ω, but this contradicts the fact that $m\neq m'$ (since the points in the set $\{z_i\}$ as well as in the set $\{z_i'\}$ are mutually distinct and since all the vectors α_i and α_i' are different from zero).

When it comes to "verifying" (ii) consider for simplicity the case when $\gamma(x)=\gamma$ (a constant), and suppose also that $U(x)=x_1$ (corresponding to boundary current $\psi=\gamma\nu_1$). Then

$$\begin{aligned}&\epsilon^{-n}(u_\epsilon(z)-u_\epsilon'(z)) \\ &\quad=\frac{\gamma(\gamma-k)}{k}\sum_{i=1}^{m}\left[\rho_i^n\nabla_x N(z_i,z)-(\rho_i')^n\nabla_x N(z_i',z)\right]\cdot\mu_1+O(\epsilon^{1/2}) \quad ,\end{aligned}\tag{73}$$

μ_1 being the first column of $M(\gamma/k)$. A simple calculation gives

$$\begin{aligned}&\sum\left[\rho_i^n\nabla_x N(z_i,z)-(\rho_i')^n\nabla_x N(z_i',z)\right]\cdot\mu_1\\ &\quad=\sum\Big[(\rho_i-\rho_i')n(\bar\rho_i)^{n-1}\nabla_x N(z_i,z)\cdot\mu_1+(\rho_i')^n\left(\nabla_x N(z_i,z)-\nabla_x N(z_i',z)\right)\Big]\cdot\mu_1\\ &\quad=\sum(|\rho_i-\rho_i'|+|z_i-z_i'|)\\ &\qquad\times\sum\Big[d\rho_i\ n(\bar\rho_i)^{n-1}\nabla_x N(z_i,z)\cdot\mu_1+(\rho_i')^n<dz_i,\ D_x^2N(\bar z_i,z)\ \mu_1>\Big] \quad ,\end{aligned}$$

with $\sum(|d\rho_i|+|dz_i|)=1$ for some $\bar\rho_i(z)$ and $\bar z_i(z)$ ("between" ρ_i and ρ_i' and z_i and z_i' respectively). By a fairly simple argument (cf [7]) it follows from (73) that we either (with appropriate reordering) have the desired continuous dependence estimate

$$\sum_{i=1}^{m}(|\rho_i-\rho_i'|+|z_i-z_i'|)\le C\left(\epsilon^{-n}\|u_\epsilon-u_\epsilon'\|_{L^\infty(\Gamma)}+\epsilon^{1/2}\right) \quad ,$$

or there exist (infinitessimal) perturbations $d\rho_i$ and dz_i (with $\sum_{i=1}^m(|d\rho_i|+|dz_i|)=1$), points z_i $(=z_i'=\bar z_i)$ and dilatation parameters ρ_i $(=\rho_i'=\bar\rho_i)$ so that

$$G(z)=\sum_{i=1}^{m}\Big[d\rho_i\ n(\rho_i)^{n-1}\nabla_x N(z_i,z)\cdot\mu_1+(\rho_i)^n<dz_i,\ D_x^2N(z_i,z)\ \mu_1>\Big]=0$$

for $z\in\Gamma$. Just as was the case with F, the function G has a vanishing normal derivative on $\partial\Omega$ and it is harmonic in Ω except at the points $\{z_i\}$ and $\{z_i'\}$.If

$G(z) = 0,\ z \in \Gamma$, it would now follow from the unique continuation property of harmonic functions that $G(z) \equiv 0$ in Ω. Since the points z_i are mutually different and since the dilatation parameters ρ_i and the vector μ_1 are nonzero, the function G can only identically vanish if $d\rho_i = dz_i = 0,\ i = 1 \dots m$. This, however, would be a contradiction to the fact that $\sum(|d\rho_i| + |dz_i|) = 1$. We therefore conclude that the desired continuous dependence estimate holds.

$\square$

For $n = 2$ we could prove a similar continuous dependence estimate for the Q_i (the angles of rotations) and the "centers" z_i, provided $M(r)$ is *not* isotropic. It is, however, not possible to prove such an estimate for the dilatation parameters, the angles of rotations and the "centers" simultaneously (see [7]). To conclude this section consider the case of two sets of inhomogeneities $\omega_\epsilon = \cup_{i=1}^m (z_i + \epsilon\rho_i B)$ and $\omega'_\epsilon = \cup_{i=1}^m (z_i + \epsilon\rho'_i B)$ with $B =$ the unit ball (centered at the origin).Suppose the inhomogeneity $z_i + \epsilon\rho_i B$ has conductivity k_i and the inhomogeneity $z_i + \epsilon\rho'_i B$ has conductivity k'_i. Notice that the number of inhomogeneities and their centers are the same for ω_ϵ and ω'_ϵ. Suppose $\gamma(x) = \gamma$ (a constant) and $U(x) = x_1$. In this case

$$u_\epsilon(z) - u'_\epsilon(z)$$
$$= \epsilon^n \omega_n \gamma \sum_{i=1}^m \nabla_x N(z_i, z) \cdot e_1 \left[\frac{(\gamma - k_i)\rho_i^n}{(n-1)\gamma + k_i} - \frac{(\gamma - k'_i)(\rho'_i)^n}{(n-1)\gamma + k'_i} \right] + O(\epsilon^{n+\frac{1}{2}}) \ .$$

It is clear from this expression that given a set of dilatation parameters ρ_i and conductivities k_i we can chose any other set of dilatation parameters $\rho'_i > \rho_i$ and then determine conductivities k'_i so that $u_\epsilon - u'_\epsilon$ is of order $\epsilon^{n+\frac{1}{2}}$ on $\partial\Omega$. Similarly given a set of dilatation parameters ρ_i and conductivities k_i we can chose any other set of conductivities k'_i, with k'_i being on the same side of γ as k_i, and then we can determine dilatation parameters ρ'_i so that $u_\epsilon - u'_\epsilon$ is of order $\epsilon^{n+\frac{1}{2}}$ on $\partial\Omega$. In either case $\epsilon^{-n}\|u_\epsilon - u'_\epsilon\|_{L^\infty(\Gamma)} + \epsilon^{1/2}$ is of order $\epsilon^{1/2}$, but $\sum(|\rho_i - \rho'_i| + |k_i - k'_i|)$ is of order 1. Therefore there is no Lipschitz estimate like that in Theorem 3, involving the $\rho's$ and the $k's$. Equivalently, in terms of the inverse problem, it is not possible to determine the $\epsilon\rho'_i s$ and the $k'_i s$ simultaneously from knowledge of the lowest order term in $u_\epsilon - U$.

8 Computational results

In this section we describe some computational experiments concerning the formula (55). These experiments quite clearly demonstrate the viability of this formula as an effective tool for the identification of well separated, internal inhomogeneities inside an otherwise uniform conductor.

The constant background conductivity $\gamma(x) = \gamma$ is in all our actual computations chosen to be 1. We always use the background voltage potential $U(x) = x_1$, corresponding to the boundary current $\psi = \gamma\nu_1$ $(= \nu_1)$. To apply the formula

(55) we need to calculate the tensors M_i. This is in each case done by calculating approximations to the functions ϕ_j (on ∂B) and then employing the formula (58) to calculate the individual entries of M. We also need the boundary data ($u_\epsilon|_{\partial\Omega}$); in practice this data would be measured, but here we generate it numerically (so called synthetic data). Approximations to the functions $\phi_j|_{\partial B}$ and $u_\epsilon|_{\partial\Omega}$ are calculated by discretizations of integral equation formulations. Since the calculation of an approximation to $u_\epsilon|_{\partial\Omega}$ is very similar to the calculation of approximations to the $\phi_j|_{\partial B}$, we describe in some detail how the latter approximations are arrived at, and then we very briefly discuss the main differences concerning u_ϵ. In all our computational work we restrict attention to the two-dimensional case.

Recall, that ϕ_j is the solution to the boundary value problem (52). We note that, as usual, a superscript of $+$ and $-$ will signify the values of the particular function approaching ∂B from the exterior and interior of B respectively. Let Φ denote the fundamental solution

$$\Phi(x,y) = -\frac{1}{2\pi}\log|x-y| \quad ,$$

(Φ is as before, only with $\gamma = 1$). Let $x \in \Re^2 \setminus \bar{B}$ and let S_R be a disk of radius R, with R sufficiently large that $\bar{B} \subset S_R$ and $x \in S_R$. Applying Greens formula for the domain $S_R \backslash \bar{B}$ we obtain

$$\begin{aligned} \phi_j(x) = & \int_{\partial S_R} \frac{\partial \phi_j}{\partial \nu_y}(y)\ \Phi(x,y) d\sigma_y - \int_{\partial S_R} \phi_j(y) \frac{\partial \Phi(x,y)}{\partial \nu_y}\, d\sigma_y \\ & - \int_{\partial B} \frac{\partial \phi_j}{\partial \nu_y}^{+}(y)\ \Phi(x,y) d\sigma_y + \int_{\partial B} \phi_j(y) \frac{\partial \Phi(x,y)}{\partial \nu_y}\, d\sigma_y \quad x \in S_R \backslash \bar{B} \quad . \end{aligned} \tag{74}$$

Here the unit normal, ν, is directed towards the exterior of S_R along ∂S_R, and towards the exterior of B along ∂B. Recall that for $|y| \to \infty$ we have $\phi_j(y) = O(|y|^{-1})$ and $\nabla \phi_j(y) = O(|y|^{-2})$. Thus

$$\lim_{R\to\infty} \int_{\partial S_R} \frac{\partial \phi_j}{\partial \nu_y}(y)\ \Phi(x,y)\, d\sigma_y = 0 \quad ,$$

and

$$\lim_{R\to\infty} \int_{\partial S_R} \phi_j(y) \frac{\partial \Phi(x,y)}{\partial \nu_y}\, d\sigma_y = 0 \quad .$$

As $R \to \infty$ in (74) we now get

$$\phi_j(x) = \int_{\partial B} \phi_j(y) \frac{\partial \Phi(x,y)}{\partial \nu_y}\, d\sigma_y - \int_{\partial B} \frac{\partial \phi_j}{\partial \nu_y}^{+}(y)\ \Phi(x,y)\, d\sigma_y \quad x \in \Re^n \backslash \bar{B} \quad .$$

Using the boundary condition $-\frac{\gamma}{k}\frac{\partial \phi_j}{\partial \nu_y}^{+} + \frac{\partial \phi_j}{\partial \nu_y}^{-} = \nu_j$ on ∂B and the integral identity $\int_{\partial B} \frac{\partial \phi_j}{\partial \nu_y}^{-}(y)\ \Phi(x,y)\, d\sigma_y = \int_{\partial B} \phi_j(y) \frac{\partial \Phi}{\partial \nu_y}(x,y)\, d\sigma_y$, $x \in \Re^2 \setminus \bar{B}$, we calculate

$$\phi_j(x) \quad = \quad \int_{\partial B} \phi_j(y) \frac{\partial \Phi(x,y)}{\partial \nu_y}\, d\sigma_y - \frac{k}{\gamma} \int_{\partial B} \frac{\partial \phi_j}{\partial \nu_y}^{-}(y)\ \Phi(x,y)\, d\sigma_y + \frac{k}{\gamma} \int_{\partial B} \nu_j \Phi(x,y)\, d\sigma_y$$

$$
\begin{aligned}
&= \int_{\partial B} \phi_j(y) \frac{\partial \Phi(x,y)}{\partial \nu_y} \, d\sigma_y - \frac{k}{\gamma} \int_{\partial B} \phi_j(y) \frac{\partial \Phi(x,y)}{\partial \nu_y} \, d\sigma_y + \frac{k}{\gamma} \int_{\partial B} \nu_j \Phi(x,y) \, d\sigma_y \\
&= \frac{\gamma - k}{\gamma} \int_{\partial B} \phi_j(y) \frac{\partial \Phi(x,y)}{\partial \nu_y} \, d\sigma_y + \frac{k}{\gamma} \int_{\partial B} \nu_j \Phi(x,y) \, d\sigma_y \quad .
\end{aligned}
$$

In the limit as $x \to \partial B$, this yields

$$
\phi_j(x) = \frac{\gamma - k}{\gamma} \int_{\partial B} \phi_j(y) \frac{\partial \Phi(x,y)}{\partial \nu_y} \, d\sigma_y + \frac{\gamma - k}{2\gamma} \phi_j(x) + \frac{k}{\gamma} \int_{\partial B} \nu_j \Phi(x,y) \, d\sigma_y \quad ,
$$

where the second term in the right hand side reflects the jump associated with the double layer potential. Rearranging terms and inserting the expression for $\Phi(x,y)$ we finally arrive at the integral equation

$$
\phi_j(x) + \frac{(k-\gamma)}{\pi(k+\gamma)} \int_{\partial B} \phi_j(y) \frac{(x-y) \cdot \nu}{|x-y|^2} \, d\sigma_y = -\frac{k}{\pi(k+\gamma)} \int_{\partial B} \nu_j \log|x-y| \, d\sigma_y \quad ,
$$

for $x \in \partial B$. Equivalently, ϕ_j is the solution to

$$
\phi_j + A\phi_j = f_j \quad , \tag{75}
$$

with

$$
(A\phi_j)(x) = \frac{k-\gamma}{\pi(k+\gamma)} \int_{\partial B} \phi_j(y) \frac{(x-y) \cdot \nu}{|x-y|^2} \, d\sigma_y
$$

and

$$
f_j(x) = -\frac{k}{\pi(k+\gamma)} \int_{\partial B} \nu_j \log|x-y| \, d\sigma_y \quad .
$$

We use a collocation method based on trigonometric interpolation for approximating the solution to (75). That is, we seek an approximate solution for ϕ_j on ∂B from the finite dimensional subspace T_p =span$\{\xi_l\}_{l=1}^p$, generated by the first p functions of the family $\{1, \cos t, \sin t, \cos 2t, \sin 2t, ...\}$. We require that the system of equations be satisfied only at a finite number of collocation points, $\{x_1, x_2, ..., x_p\}$, along ∂B. To be more precise, the approximation to ϕ_j is expressed as a linear combination $\phi_j^{(p)} = \sum_{l=1}^p c_j^{(l)} \xi_l$, where the coefficients $c_j^{(l)}$ are determined so that $\phi_j^{(p)}$ satisfies

$$
\phi_j^{(p)}(x_i) + A\phi_j^{(p)}(x_i) = f_j(x_i) \quad i = 1, ..., p \quad . \tag{76}
$$

The collocation points $x_1, ..., x_p$ are chosen so that the subspace T_p is unisolvent with respect to these points. Inserting $\phi_j^{(p)} = \sum_{l=1}^p c_j^{(l)} \xi_l$ into equation (76) we get

$$
\sum_{l=1}^p c_j^{(l)} \xi_l(x_i) + \sum_{l=1}^p (A\xi_l)(x_i) c_j^{(l)} = f_j(x_i) \quad i = 1, ..., p \quad ,
$$

or equivalently

$$
S\hat{c}_j = \hat{f}_j \quad ,
$$

where $\hat{c}_j$ is the coefficient vector $\hat{c}_j = (c_j^{(1)}, ..., c_j^{(p)})^T$, and the entries in the matrix, $S = \{s_{il}\}$, and the right hand side, $\hat{f}_j = \{\hat{f}_j^{(i)}\}$, are given by

$$s_{il} = \xi_l(x_i) + (A\xi_l)(x_i) \quad ,$$

$$\hat{f}_j^{(i)} = f_j(x_i) \quad .$$

The integral equation for u_ϵ is, as mentioned earlier, derived in a totally similar fashion. There will now be equations corresponding to the boundaries of all the inhomogeneities as well as to the outer boundary. We again use a trigonometric collocation method to find an approximate solution. We take the same number of collocation points for each inhomogeneity, this number being somewhat smaller than the number of collocation points on the outer boundary. Typically we take 25–50 points for each inhomogeneity and 50–100 points on the outer boundary.

Having calculated the rescaled polarization tensors and the synthetic data we use the formula (55) to identify the inhomogeneities. In the two dimensional case, with the expression $-\frac{1}{2\pi\gamma}\log|x-y|$ inserted in place of $\Phi(x,y)$, this formula reads

$$\begin{aligned} u_\epsilon(z) - U(z) + \frac{1}{\pi}\int_{\partial\Omega}(u_\epsilon(x) - U(x))\frac{(z-x)\cdot\nu_x}{|x-z|^2}\,d\sigma_x \\ = \epsilon^2\frac{1}{\pi}\sum_{i=1}^{m}\frac{\gamma-k_i}{k_i}\frac{(z-z_i)\cdot\nu_x}{|z_i-z|^2}\cdot M_i\nabla_x U(z_i) + O(\epsilon^{2+\frac{1}{2}}) \quad . \end{aligned} \tag{77}$$

The left hand side $L(u_\epsilon - U)(z) = u_\epsilon(z) - U(z) + \frac{1}{\pi}\int_{\partial\Omega}(u_\epsilon(x) - U(x))\frac{(z-x)\cdot\nu_x}{|x-z|^2}\,d\sigma_x$ is entirely known on $\partial\Omega$ once $u_\epsilon - U$ is known on $\partial\Omega$. The sum on the right hand side $\Sigma(z) = \epsilon^2\frac{1}{\pi}\sum_{i=1}^{m}\frac{\gamma-k_i}{k_i}\frac{(z-z_i)\cdot\nu_x}{|z_i-z|^2}\cdot M_i\nabla_x U(z_i)$ is a fairly explicit function of the location and certain other features of the inhomogeneities. Our computational identification algorithm consists in minimizing a discreet L^2 norm of the residual $L(u_\epsilon - U) - \Sigma$ on $\partial\Omega$. We choose the boundary expression $L(u_\epsilon - U) - \Sigma = L(u_\epsilon - U - L^{-1}\Sigma)$ from (55) over the essentially equivalent boundary expression $u_\epsilon - U - \tilde{\Sigma}$ from (54) because of its explicitness and simplicity. It is clear that both expressions are based on finding the second term in an asymptotic expansion of u_ϵ in powers of ϵ^n. In the calculations presented here we restrict our attention to inhomogeneities, $z_i + \epsilon\rho_i Q_i B$, which are dilatations, rotations and translations of one common domain, and the conductivities of which are all the same. In this case $M_i = \rho_i^2 Q_i M Q_i^T$ where $\rho_i > 0$ are dilatation parameters, Q_i are rotations and M is the rescaled polarization tensor corresponding to the common domain, B, and the common conductivity ratio γ/k. We take U to be the special solution $U(x) = x_1$, corresponding to the boundary current $\psi = \gamma\nu_1$ $(= \nu_1)$. The domain Ω is taken to be the disk of radius 10, centered at the origin, and we shall here only consider inhomogeneities that are shaped as disks or ellipsoids (even though we have performed experiments with other shapes). We select J equidistant points,

$y_1, ..., y_J$, on $\partial\Omega$ (typically $J = 50$) and we seek the unknown parameters of the inhomogeneities as the solution to the nonlinear least squares problem

$$\min \sum_{j=1}^{J} \Bigg[u_\epsilon(y_j) - U(y_j) + \frac{1}{\pi} \int_{\partial\Omega} (u_\epsilon(x) - U(x)) \frac{(y_j - x) \cdot \nu_x}{|x - y_j|^2} d\sigma_x$$
$$- \frac{1}{\pi} \sum_{i=1}^{m} (\epsilon\rho_i)^2 \; \frac{(\gamma - k)}{k} \frac{y_j - z_i}{|z_i - y_j|^2} \cdot Q_i M_i Q_i^T e_1 \Bigg]^2 . \tag{78}$$

To calculate the inside, $L(u_\epsilon - U) - \Sigma$, of the least squares functional, we use the equivalent formula

$$L(u_\epsilon - U)(z) - \Sigma(z) = u_\epsilon(z) - \frac{1}{\pi} \Bigg(\sum_{i=1}^{m} (\epsilon\rho_i)^2 \; \frac{(\gamma - k)}{k} \frac{z - z_i}{|z_i - z|^2} \cdot Q_i M_i Q_i^T e_1$$
$$- \int_{\partial\Omega} \nu_1 \log|x - z| \, d\sigma_x - \int_{\partial\Omega} u_\epsilon \frac{(z - x) \cdot \nu}{|x - z|^2} \, d\sigma_x \Bigg) \, .$$

We minimize over $\{z_i, \epsilon\rho_i\}$ when the orientations are known or when they are irrelevant, as is the case when the inhomogeneities are all disks, and we minimize over $\{z_i, Q_i\}$ when the dilatation parameters are known. When all the parameters are unknown one can attempt to minimize over $\{z_i, \epsilon\rho_i, Q_i\}$, however, as pointed out earlier, there may be considerable non-uniqueness of the minimizer in that case. The approximate formula $L(u_\epsilon - U) \approx \Sigma$ on which our identification algorithm is based has been verified to hold asymptotically as $\epsilon \to 0$. Before we proceed to describe any of our identification experiments we shall try to assess the practical validity of this formula.

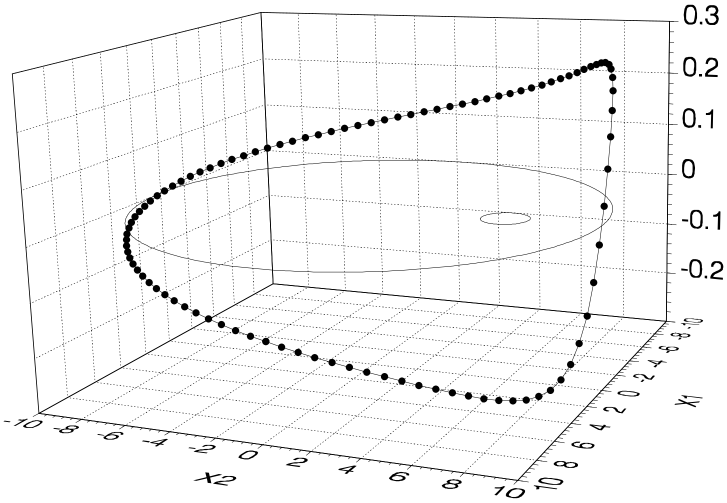
We consider a single inhomogeneity of the form $z_1 + \epsilon\rho_1 B$, having conductivity $k = 10$, and compare the graph of the function

$$u_\epsilon(z) - U(z), \;\; z \in \partial\Omega \tag{79}$$

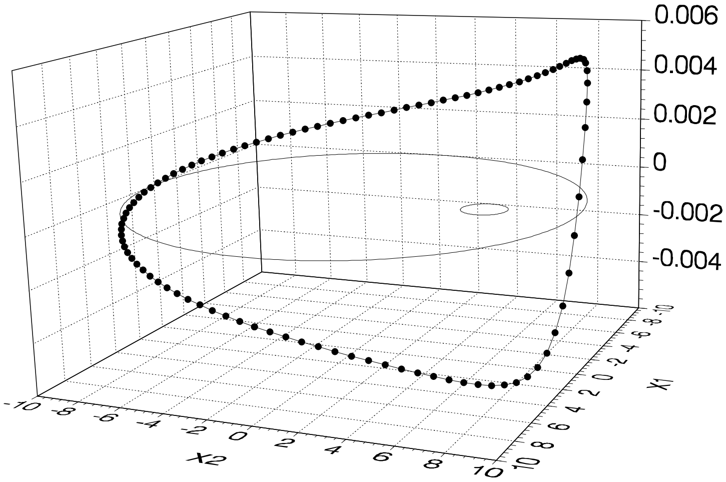
to the graph of

$$L(u_\epsilon - U)(z) - \Sigma(z)$$
$$= u_\epsilon(z) - U(z) + \frac{1}{\pi} \int_{\partial\Omega} (u_\epsilon(x) - U(x)) \frac{(z - x) \cdot \nu_x}{|x - z|^2} \, d\sigma_x$$
$$-(\epsilon\rho_1)^2 \frac{1}{\pi} \frac{\gamma - k}{k} \frac{(z - z_1) \cdot \nu_x}{|z_1 - z|^2} \cdot M \nabla_x U(z_1), \;\; z \in \partial\Omega \;\; . \tag{80}$$

The reference conductivity is, as always, taken to be $\gamma = 1$. Figure 1(a) and (b) displays the graph of these two functions for a circular inclusion with center $z_1 = (0, 6)$ and radius $\epsilon\rho_1 = 1.00$ (in this case B is the unit disk). In figure 2 (a) and (b) we graph the results obtained for an elliptical shaped inhomogeneity having eccentricity ratio 1:2, center of mass $z_1 = (0, 6)$, and scaling factor $\epsilon\rho_1 = 0.50$. In this case B is an ellipse with major axis 4 and minor axis 2; the major

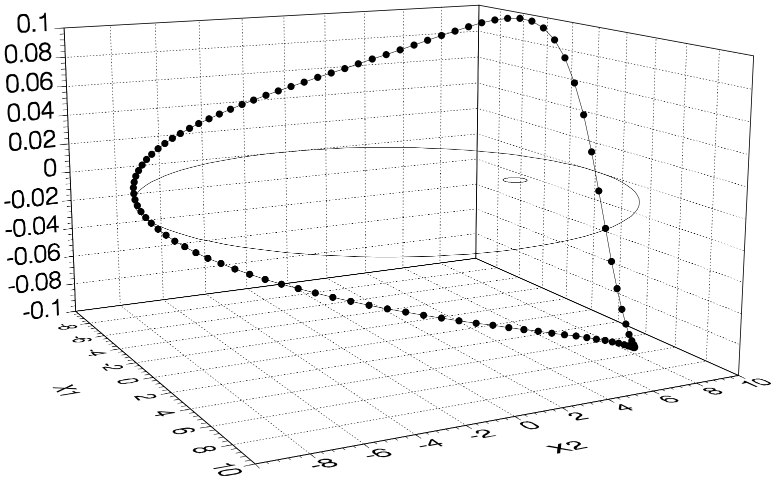


(a) $u_\epsilon - U$

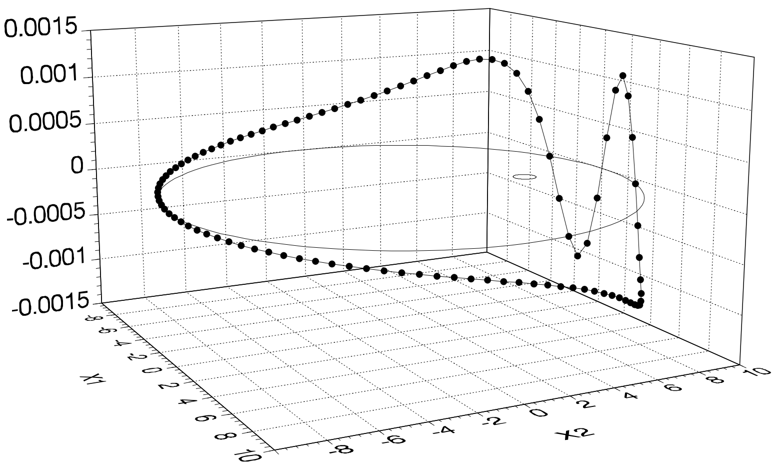


(b) $L(u_\epsilon - U) - \Sigma$

Figure 1: Remainder terms for a circular inhomogeneity



(a) $u_\epsilon - U$



(b) $L(u_\epsilon - U) - \Sigma$

Figure 2: Remainder terms for an elliptical inhomogeneity

axis is parallel to the x_2 axis. The graphs obtained in figures 1 and 2 were based on evaluating (79) and (80) at 100 uniformly spaced points along $\partial\Omega$. As anticipated, the remainder $L(u_\epsilon - U) - \Sigma$ is much smaller than the remainder $u_\epsilon - U$. In both cases the largest values are on that part of the boundary which is nearest to the inhomogeneity. The two graphs are approximately of the same shape for the disk, whereas their shapes differ for the ellipse.

Our theoretical result estimate the term $L(u_\epsilon - U) - \Sigma$ to be of the order $\epsilon^{n+\frac{1}{2}} = \epsilon^{2.5}$, however, this is far too conservative; one should expect it to be of order $\epsilon^{2n} = \epsilon^4$, since it largely represents the "quadratic" term when $L(u_\epsilon - U)$ is expanded in a power series in the volume of the inhomogeneity. To demonstrate this point, we have calculated a discrete L^2 norm of the term $L(u_\epsilon - U) - \Sigma$ correponding to four different cases of a circular inhomogeneity centered at $(0, 6)$ with conductivity $k = 10$. For the l^2 norm we used 25 evenly spaced points along $\partial\Omega$. The four inhomogeneities have the form $(0, 6) + \epsilon\rho_1 B$ with B being the unit disk and the scaling factor $\epsilon\rho_1$ taking the values $\epsilon\rho_1 = .5, 1, 2,$ and 3. The graph of the l^2 norm, E, as a function of the parameter $\epsilon\rho_1$ is displayed on a log-log plot in figure 3(a). Geometric regression analysis shows the correlation coefficient for the data to be 0.919 and produces the "best fit"

$$E \approx 0.0141(\epsilon\rho_1)^{3.95} \quad .$$

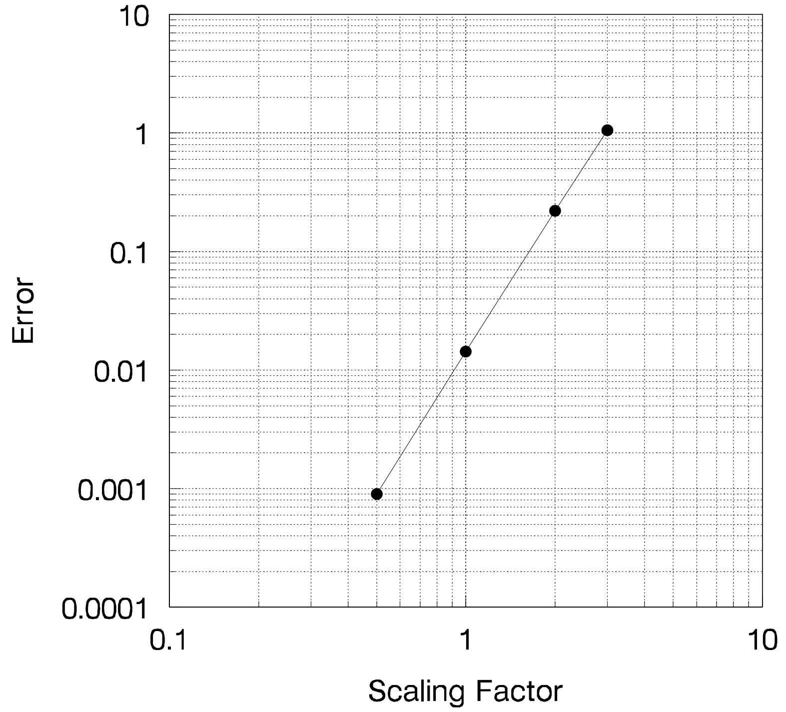
We have repeated this experiment for four different inhomogeneities of the form $(0, 6) + \epsilon\rho_1 B$, where B is the elliptical domain centered at the origin with major axis 4 (parallel to the x_2-axis) and minor axis 2. The l^2 norm of the remainder, based on 25 evenly spaced points along $\partial\Omega$, was calculated for the values $\epsilon\rho_1 = 0.25, 0.50, 1.00,$ and 1.50. The corresponding graph is shown in figure 3(b). The correlation coefficient for this data is calculated to be 0.914 and geometric regression analysis produces the "best fit"

$$E \approx 0.0435(\epsilon\rho_1)^{4.01} \quad .$$

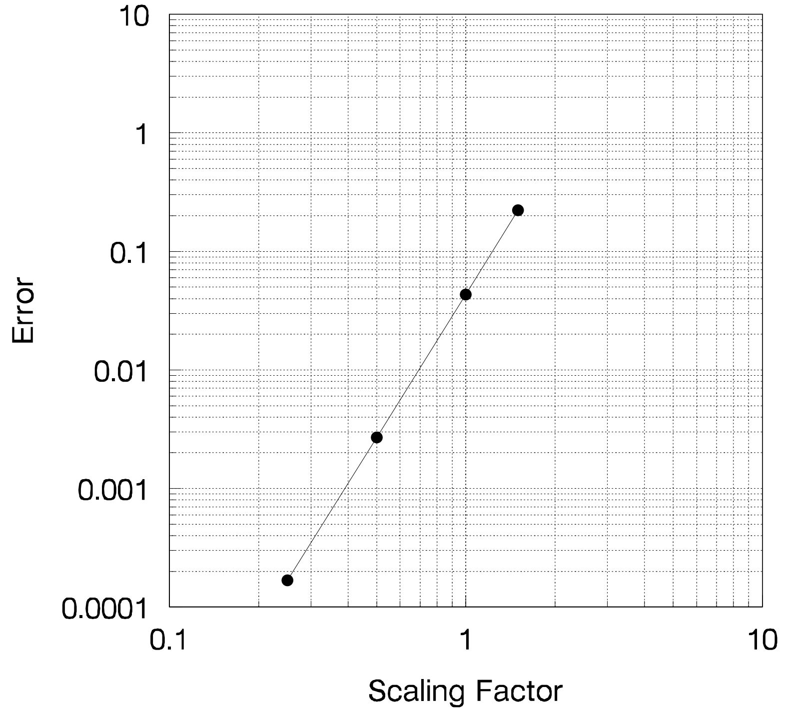
We next study the remainder terms for a conductor containing a single inhomogeneity, as we vary the conductivity, k, of the inhomogeneity. Specifically, we examine the two norms

$$(a) \ \ \|u_\epsilon - U\|_{L^\infty(\partial\Omega)} \quad \text{and} \quad (b) \ \ \|L(u_\epsilon - U) - \Sigma\|_{L^\infty(\partial\Omega)} \quad ,$$

for conductivities satisfying $.00001 < k < 100000$. The inhomogeneity is a unit disk centered at $(0, 6)$. The results are displayed in figure 4. As expected, both norms approach 0 as the conductivity on the inhomogeneity approaches the reference conductivity, $\gamma = 1$. Figure 5 provides a comparison of the two remainders. Here we have plotted $\|L(u_\epsilon - U) - \Sigma\|_{L^\infty(\partial\Omega)}$ against $\|u_\epsilon - U\|_{L^\infty(\partial\Omega)}$ on a log-log scale. In plots 5 (a) and (b) we display the results for conductivity values $k < 1$ and $k > 1$ respectively. Geometric regression analysis shows the correlation

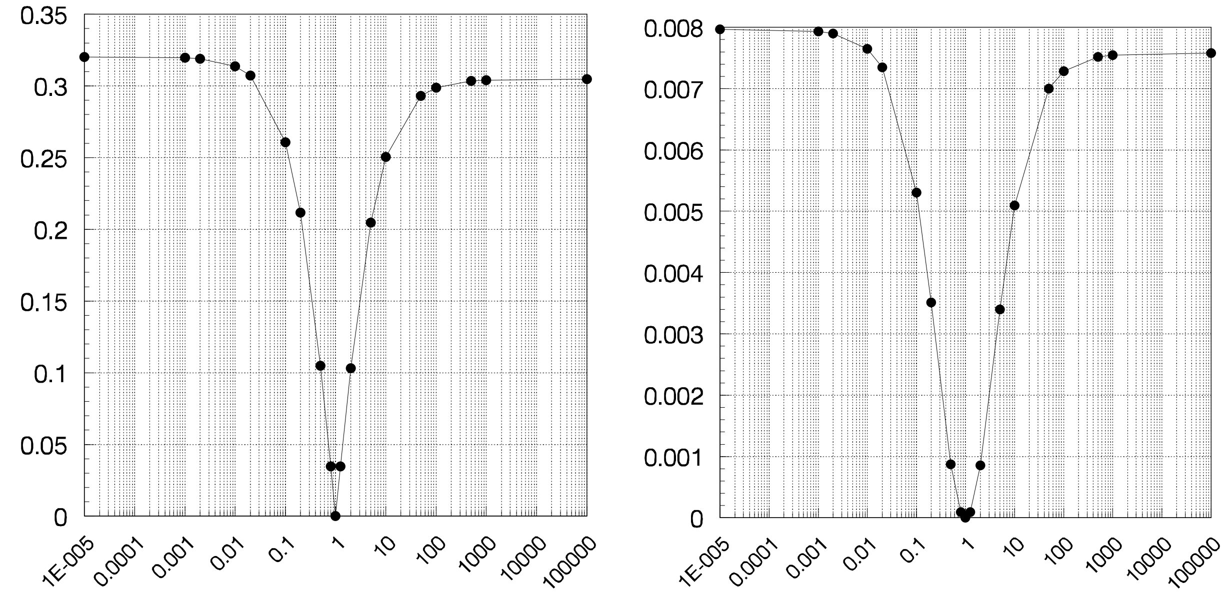


(a) Plot for circular inhomogeneities



(b) Plot for elliptical inhomogeneities

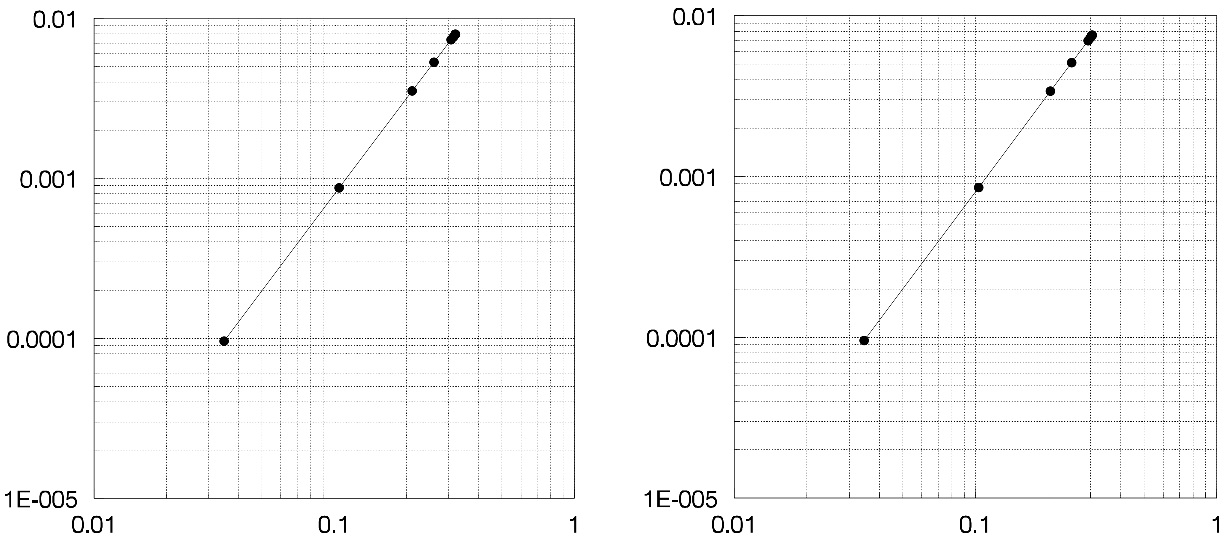
Figure 3: Log-log plots of the remainder (error) vs the scaling factor, $\epsilon\rho_1$.



(a) k vs. $\|u_\epsilon - U\|_\infty$.

(b) k vs. $\|L(u_\epsilon - U) - \Sigma\|_\infty$

Figure 4: The remainder terms as a function of the conductivity, k.



(a) $k < 1$

(b) $k > 1$

Figure 5: Graph of $\|L(u_\epsilon - U) - \Sigma\|_\infty$ vs. $\|u_\epsilon - U\|_\infty$.

$z_1; z_2$	k	$u_\epsilon - U$	$L(u_\epsilon - U) - \Sigma$
(0,6); (0,-6)	10	0.281	8.49E-3
(0,6); (0,3.9)	10	0.360	7.14E-3
(0,6); (0,3.9)	1000	0.425	.100
(6,0); (3.9,0)	10	0.823	.150
(6,0); (3.9,0)	1000	1.075	.255

Table 1: $L^\infty(\partial\Omega)$ norms of remainder terms

coefficient for $k < 1$ to be 0.984, and produces the "best fit", $t = 0.0768s^{1.99}$, where t represents $\|L(u_\epsilon - U) - \Sigma\|_{L^\infty(\partial\Omega)}$ and s represents $\|u_\epsilon - U\|_{L^\infty(\partial\Omega)}$. The correlation coefficient for $k > 1$ is 0.984, and the corresponding "best fit" is $t = 0.0826s^{2.01}$. These results clearly indicate that the remainder $L(u_\epsilon - U) - \Sigma$ is of the order of the square of the remainder $u_\epsilon - U$ itself, uniformly in k. This helps to explain why the polarization tensor term in (54) (or (55) has the "correct" limit as $k \to 0$ and $k \to \infty$.

In our last example to assess the accuracy of our representation formula we study the remainder terms $u_\epsilon - U$ and $L(u_\epsilon - U) - \Sigma$ as we vary the location (closeness) and common conductivity, k, of two circular inhomogeneities. The results are displayed in Table 1. In the first case, the (unit) disks, located at (0,6) and (0,-6), are relatively far apart and the corresponding remainder $L(u_\epsilon - U) - \Sigma$ is quite small. In the remaining four cases the disks are only a distance 0.1 apart. When the two disks are centered on the x_2-axis and k is moderately to very large they have less impact on the electric field (which largely flows parallel to the x_1-axis), than when they are centered along the x_1-axis. This explains why the remainders are smaller in cases 2 and 3 when compaired with cases 4 and 5. When k is 10 the remainder $L(u_\epsilon - U) - \Sigma$ is smaller than when k is 1000 (for the same geometric configuration). In the second case when k is 10 and the disks are close, but centered on the x_2-axis the remainder $L(u_\epsilon - U) - \Sigma$ is small enough (compared to $u_\epsilon - U$) that our approach would very easily locate these two disks, however, as a general rule we must conclude that our approach will be somewhat inaccurate when it comes to locating inhomogeneities that are closely spaced.

8.1 Identification experiments

As mentioned earlier our identification approach consists in solving the least squares problem (78). For the actual minimization we employ Moré's routine, a modified Levenberg-Marquardt algorithm. To be quite specific, we call the general least-squares solver LMDIF1 which is found in MINPACK. Note that we solve the unconstrained problem. That is, in solving (78) we do not place any restrictions on z_i or $\epsilon\rho_i$. Although we could have imposed conditions requiring that for each i, $z_i \in \Omega$, and $\epsilon\rho_i$ be sufficiently small so that the inhomogeneity centered at z_i remains strictly inside Ω, these constraints were unnecessary. When the true number of inhomogeneities is known, these restrictions have always been automatically satisfied in the cases we have run. Moreover, when the number of inhomogeneities is a priori unknown, then the unconstrained problem can in our experience be used to determine the number of inhomogeneities as follows. If the number of inhomogeneities in the initial configuration is less than the actual number present (in the data), then the final l^2 residual produced remains relatively large. Upon increasing the number of initial inhomogeneities, this residual decreases until the number of inhomogeneities corresponds to the actual number present. If the initial number of inhomogeneities is further increased, the additional inhomogeneities were eventually placed outside the domain Ω or shrunk to zero. This provides an excellent indicator that one has exceeded the actual number present.

When it comes to the requirement that the inhomogeneities stay disjoint and well separated our intermediate iterates are frequently in violation. Admitting overlapping inhomogeneities has in practice proven extremely useful in improving the convergence of our procedure and in providing a simple (and quite reasonable) initial estimate for the location and the size. The initial inhomogeneities are all placed at the center of the domain. A "rough" estimate for the "size" of the inhomogeneities is obtained by considering (78) with a prescribed number of inhomogeneities (all "centered" at the origin and of identical "size") and then solving for $\epsilon\rho_i = \epsilon\rho$.

If the common scale (ϵ) of the inhomogeneities is extremely small, then our discretization errors may well dominate the data $u_\epsilon - U|_{\partial\Omega}$, and therefore, one can not expect to locate such inhomogeneities with good accuracy. Consequently, in all our computational work we have considered only those values for $\epsilon\rho_i$ satisfying $\epsilon\rho_i > 0.10, \quad i = 1, ...m$.

Our first experiment involves determining the location, size, and angle of rotation of an elliptical inhomogeneity which is parametrically defined by

$$x_1 = 0 + 0.25\,(\cos(t)\cos(\frac{\pi}{4}) - 2\sin(t)\sin(\frac{\pi}{4}))$$

$$x_2 = 6 + 0.25\,(\cos(t)\sin(\frac{\pi}{4}) + 2\sin(t)\cos(\frac{\pi}{4})),$$

and has conductivity $k = 10$. Figure 6 depicts the convergence of the iterates obtained from the least squares routine. The initial guess and all the iterations have been graphed. After three iterations the location, size, and rotation of the inhomogeneity were acurately determined to three significant figures. The final result corresponds to the solid line ellipse.

We have (understandably) not been very successful in computationally determining the location, size, and angle of rotation for multiple inhomogeneities. However, if the sizes of the inhomogeneities are known then we have been able to determine their locations and their angles of rotation. Our next experiment addresses such a problem. That is, we determine the locations and angles of rotation of the three ellipses which are parametrically given by

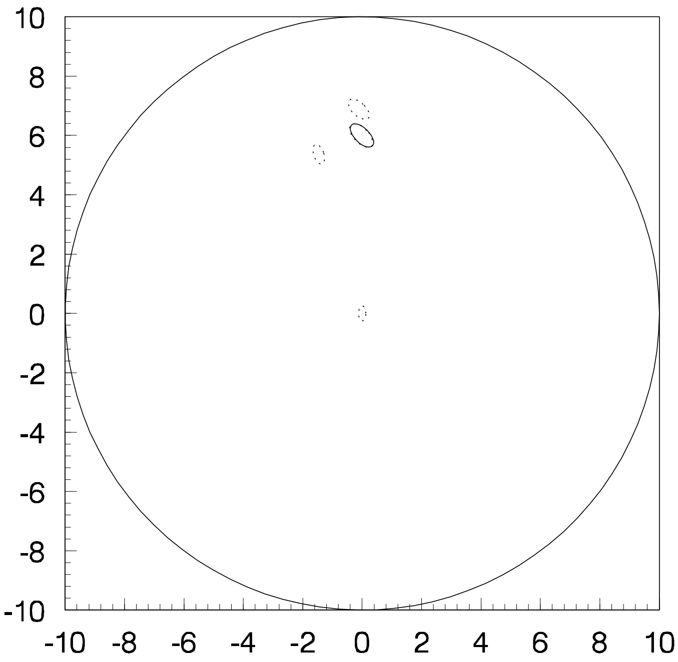


Figure 6: Graph of the iterations for an ellipse.

$$
\begin{array}{llll}
(a) & x_1 = 7 + 0.25\cos(t) & (b) & x_1 = -1 + 0.50\cos(t) \\
& x_2 = 0 + 0.50\sin(t) & & x_2 = -6 + 0.25\sin(t), \\
(c) & x_1 = -2 + 0.25(\cos(t)\cos(\frac{\pi}{4}) - 2\sin(t)\sin(\frac{\pi}{4})) & & \\
& x_2 = \ 4 + 0.25(\cos(t)\sin(\frac{\pi}{4}) + 2\sin(t)\cos(\frac{\pi}{4})). & &
\end{array}
$$

For all three ellipses we utilize the same rescaled polarization tensor which is computed for the ellipse parametrically defined by $x_1 = \cos(t)$, $x_2 = 2\sin(t)$. The program is able to accurately determine the locations, z_i, and the angles of rotation, θ_i, for all the inhomogeneities by the fifteenth iteration. Each of the inhomogeneities has a (fixed) scaling factor of $\epsilon\rho = 0.25$. For our initial data we center all the inhomogeneities at the origin with a zero rotation angle as shown in figure 7(a). The inhomogeneities, determined by the fifth, tenth, and fifteenth iterations of our program, are displayed in figure 7(b)-(d) respectively. After the fifth iteration, two of the inhomogeneities were estimated to have approximately the same location and rotation. As a result, their images nearly coincide and only two inhomogeneities are visible in figure 7(b). Our computations yield the following locations and angles of rotation

$Iteration$	$z - Error$	$\epsilon\rho - Error$	$Residual\ Norm$
0	12.767145	0.385066	4.28148E-3
5	11.786228	0.346988	2.38138E-3
10	7.608637	0.308908	1.92556E-4
15	5.232940	0.180206	1.12363E-4
20	1.517874	0.124015	4.46768E-5
25	0.055479	0.005526	1.45346E-5

Table 2: Error analysis for iterations 0,5,10,15,20 and 25

$$
\begin{aligned}
&(a)\ z_1 = (6.982, -0.005), \quad \theta_1 = 3.134 \\
&(b)\ z_2 = (-0.996, -5.976), \quad \theta_2 = 1.568 \\
&(c)\ z_3 = (-2.026, 4.024), \quad \theta_3 = 0.761 \quad .
\end{aligned}
$$

In our third experiment we successfully locate and determine the sizes of five circular inhomogeneities. Each of the inhomogeneities has conductivity $k = 15$. Figure 8 shows the initial estimate for the location and size of the inhomogeneities, as well as their estimated locations and sizes after $I = 5$, 10, 15, 20, and 25 iterations of the least squares algorithm. Table 2 provides an error analysis for the iterations. More specifically, column 2 represents the l^2 error in the location of the inhomogeneities after the I-th iteration. Column 3 represents the relative l^2 error in the size of the inhomogeneities after the I-th iteration, and column 4 represents the relative l^2 norm of the residual vector, that is,

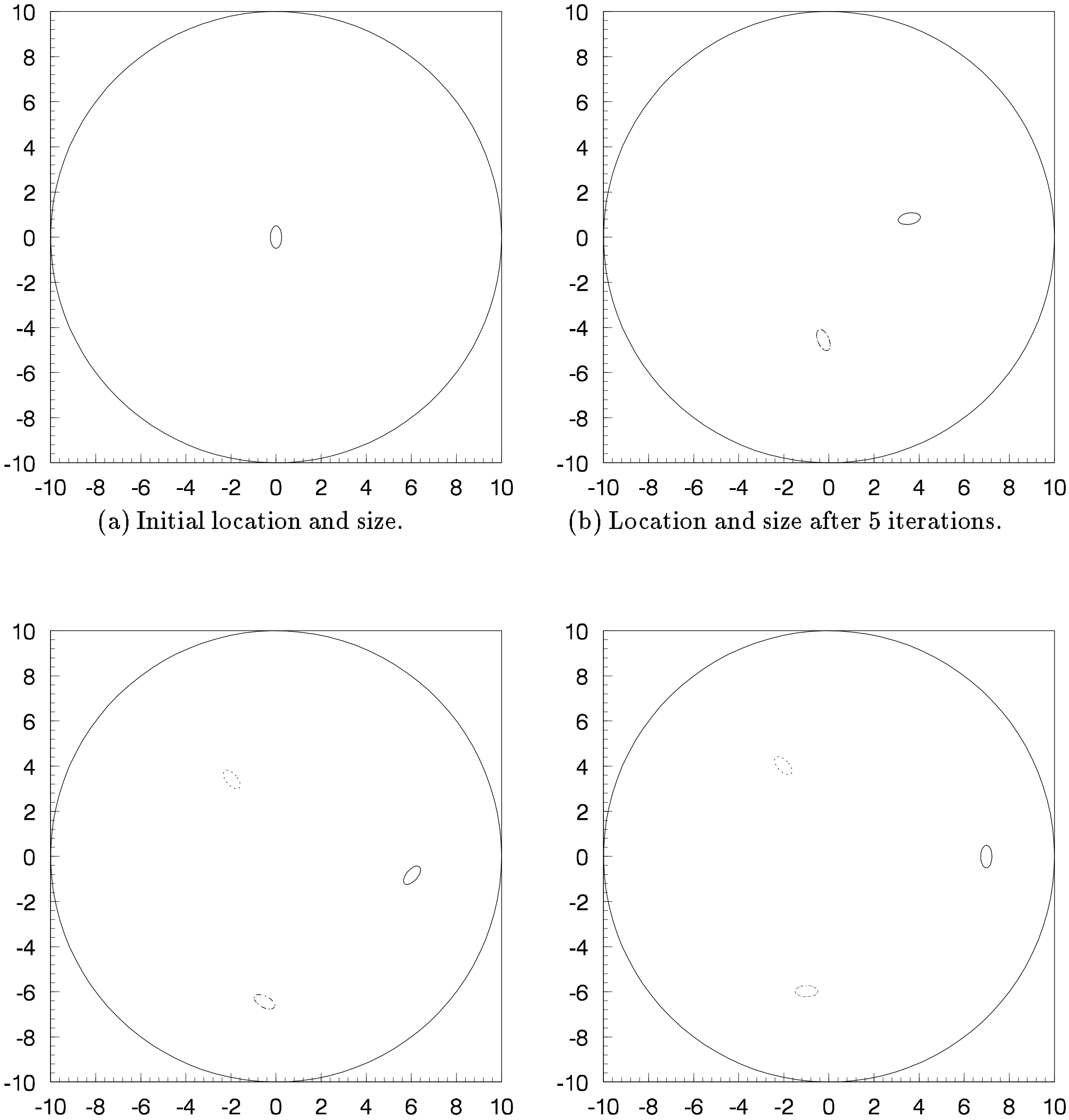
$$
Residual\ Norm = \frac{\sqrt{\sum_{j=1}^{J}(L(u_\epsilon - U)(y_j) - \Sigma_I(y_j))^2}}{\sqrt{\sum_{j=1}^{J}(u_\epsilon(y_j))^2}}.
$$

Here u_ϵ is the (discrete) solution to the conductivity problem coresponding to the the "true" location of the inhomogeneities and Σ_I is the sum from (78) with z_i and $\epsilon\rho_i$ replaced by the approximate locations and scaling factors obtained after the Ith iteration.

For this example we have also calculated the relative l^2 norm of the boundary deviation $u_\epsilon - u_\epsilon^F$, where u_ϵ^F now is the (discrete) solution to the conductivity problem with inhomogeneities the locations and sizes of which are given by the final output from the minimization algorithm. This calculation yields

$$
\frac{\sqrt{\sum_{j=1}^{J}(u_\epsilon(y_j) - u_\epsilon^F(y_j))^2}}{\sqrt{\sum_{j=1}^{J}(u_\epsilon(y_j))^2}} = 7.75533\text{E-}5.
$$

As our last experiment we try to locate two circular inhomogeneities of a fixed radius $\epsilon\rho = .25$ as they approach one another. We consider five different



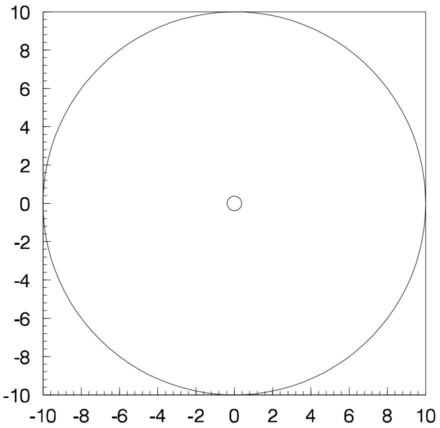
(a) Initial location and size.

(b) Location and size after 5 iterations.

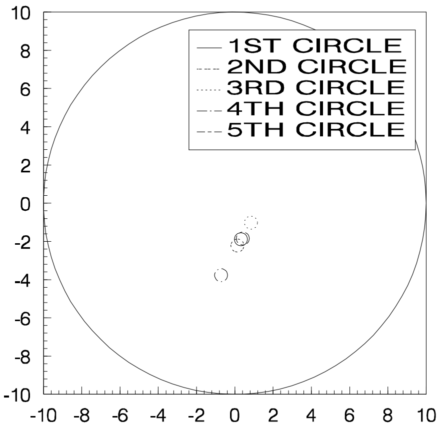
(c) Location and size after 10 iterations.

(d) Location and size after 15 iterations.

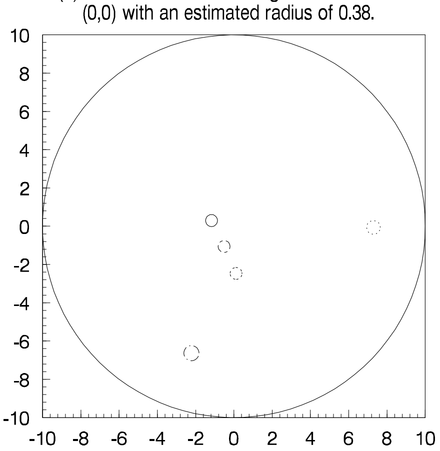
Figure 7: Convergence of iterates for 3 ellipses



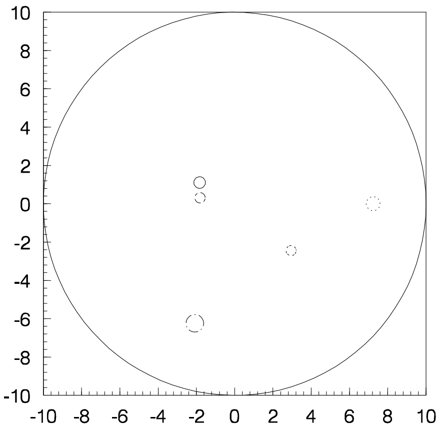
(a) Initial Guess: All inhomogeneities start at (0,0) with an estimated radius of 0.38.



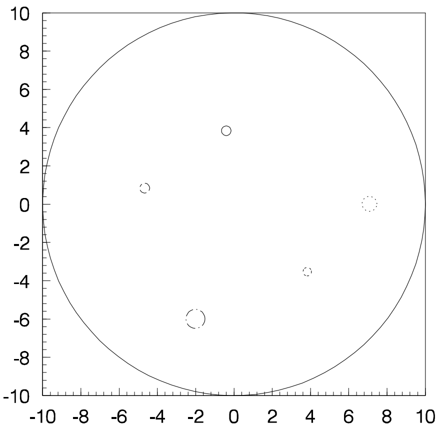
(b) Location and size of the inhomogeneities after 5 iterations.



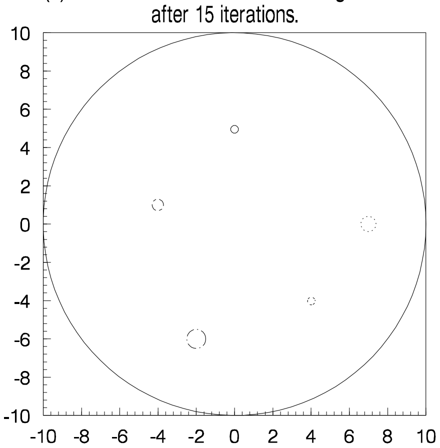
(c) Location and size of the inhomogeneities after 10 iterations.



(d) Location and size of the inhomogeneities after 15 iterations.



(e) Location and size of the inhomogeneities after 20 iterations.



(f) Location and size of the inhomogeneities after 25 iterations.

Figure 8: Convergence of the iterates for 5 circular inhomogeneities.

scenarios: the two circles are centered at $z_1 = (-s, 0)$ and $z_2 = (s, 0)$ with $s = 2, 1, .75, .5$ and $.3$ corresponding to a distance between the two inhomogeneities of $3.5, 1.5, 1, .5$ and $.1$ respectively. For each scenario we run our minimization three times corresponding to three different initial guesses: (i) $z_1^{(init)} = (0, 0)$, $z_2^{(init)} = (0, 5)$, (ii) $z_1^{(init)} = (0, -5)$, $z_2^{(init)} = (0, 5)$, and (iii) $z_1^{(init)} = (-5, 0)$, $z_2^{(init)} = (5, 0)$. In all cases we start with initial scaling factors between 0.14 and 0.16. In the first scenario ($s = 2$) our algorithm always succesfully determined the locations and sizes of the inhomogeneities. In the second scenario ($s = 1$) two of the runs were very succesful – the third got stuck near a local minimum but with a residual sufficiently large that it would indicate a problem (and thus warrant another run). In the third and fourth scenario the ability to identifty the two inhomogeneities accurately depended even more on the initial guess – and quite troubling, the residual for a somewhat "inaccurate" identification was not always significantly larger than for a more "accurate" identification. Among other things this signals that the approximate formula on which our algorithm is based is starting to become "invalid". In the final scenario none of our initial guesses produced a very accurate identification of the inhomogeneities. In the best case the estimated locations and sizes of the inhomogeneities were $z_1 = (-0.280, 0)$, $\epsilon\rho_1 = 0.274$ and $z_2 = (0.292, 0)$, $\epsilon\rho_2 = 0.269$.

We have tested our approach in a variety of other cases including those with multiple inhomogeneities having different shapes and different conductivities. Our experience in these cases is very similar to that described above. But we should note that for such problems the computational cost increases somewhat, since multiple polarization tensors are required.

Acknowledgments

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