Homological Aspects of the Normalization of Algebraic Structures

Wolmer Vasconcelos

Trieste, 2004

Syllabus

- Toolbox of Normalization Processes
- Divisorial Extensions
- Bounds on Generators
- Tracking Numbers
- Ideals and Modules
- The X-Topic

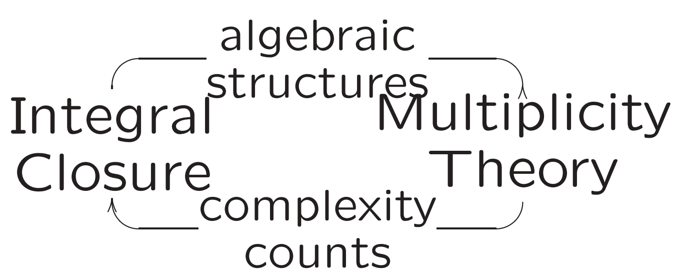
Introduction

Our theme are the structures that arise as solutions of collections of equations of integral dependence in an algebra A,

$$\boxed{z^n + a_1 z^{n-1} + \cdots + a_n = 0.}$$

In such equations, the a_i's and z are required to satisfy set theoretic restrictions of various kinds, with the solutions assembled into algebras, ideals, or modules, each process adding a particular flavor to the subject. The study of these equations–including the search for the equations themselves–is a region of convergence of many interests, with the overall goal being to find the equations defining the assemblages.

What are the motivations? An algebraic structure–a ring, an ideal or even a module–is often susceptible to smoothing processes that enhance their properties. One major process is the *integral closure* of the structure. This often enable them to support new constructions, including analytic ones. In the case of algebras, the divisors acquire a group structure, the cohomology tends to slim down. To make this more viable, multiplicity theory–broadly seen as the assignment of measures of size to an structure–must be built up with the introduction of new families of degree functions suitable for tracking the processes through their complexity costs. The synergy between these two regions is illustrated in the diagram:



The overall goal is to describe the developments leading to this picture, and, hopefully, of setting the stage for further research.

How to start? For one of these structures, $\mathbf{S}$, the analysis and/or construction of its integral closure $\overline{\mathbf{S}}$ usually passes through the study of its so–called reductions $\mathbf{S}_0$:

$$\begin{tikzcd}
 & \mathbf{S} \arrow[dl, dashed] \arrow[dr] & \\
\mathbf{S}_0 \arrow[rr] & & \overline{\mathbf{S}}
\end{tikzcd}$$

The $\mathbf{S}_0$ are structures similar to $\mathbf{S}$, with the same closure as $\mathbf{S}$ but geometrically and computationally simpler than $\mathbf{S}$ and therefore provide for a convenient platform from which to examine $\overline{\mathbf{S}}$. A drawback is that there are many $\mathbf{S}_0$. Rather than a source of frustration, this diversity is a mine of opportunities to examine $\mathbf{S}$, and often it is the springboard to the examination of other properties of $\mathbf{S}$ besides its closure.

There is an obvious organization for the several problems that arise. Without emphasizing relationships one has:

- *Membership Test*: $f \in \overline{\mathbf{S}}$?
- *Completeness Test*: $\mathbf{S} = \overline{\mathbf{S}}$?
- *Construction Task*: $\mathbf{S} \rightsquigarrow \overline{\mathbf{S}}$?
- *Complexity Cost*: $\mathrm{cx}(\mathbf{S} \rightsquigarrow \overline{\mathbf{S}})$?

Toolbox

Normalization: The Setting

A is a Noetherian domain of finite integral closure $\overline{A}$. The background for a comparison between these two rings is:

Theorem: [Krull-Serre]

$$A = \overline{A} \;\Leftrightarrow\; A \text{ satisfies } R_1 \text{ and } S_2.$$

R_1 means that the **conductor ideal** $C = \text{ann}\,(\overline{A}/A)$ has height at least 2

S_2 means that associated primes of principal ideals of A have height 1

Observe that R_1 is about 2, while S_2 is about 1!

The explanation of the theorem is the representation

$$A = \bigcap A_{\mathfrak{p}},$$

where $\mathfrak{p}$ runs over the prime ideals of grade 1.

Two natural questions: (1) How to recognize these conditions, and when missing (2) how to turn them on?

- If A is an affine domain over a field of characteristic 0, R_1 also means that the Jacobian ideal $J(A)$ of A has codimension at least 2: This by the Jacobian criterion. More significant however is one of Emmy Noether last theorems:

$$J(A) \subset C = \mathsf{ann}\,(\overline{A}/A).$$

- S_2 is a module-theoretic condition and can be recognized when A admits a subring R which is normal and over which A is finite [like in a Noether normalization]. Then S_2 means

$$A \simeq A^{**} = \mathsf{Hom}_R(\mathsf{Hom}_R(A, R), R)$$

This show how to turn S_2 on: pass from A to A^{**}.

- Note that

$$A^{**} = \bigcap A_{\mathfrak{p}},$$

where $\mathfrak{p}$ runs over the height 1 primes of A

- A similar representation is

$$\overline{A} = \bigcap \overline{A}_{\mathfrak{p}},$$

where again $\mathfrak{p}$ runs over the height 1 primes of A.

One Step Normalization

Theorem: Let A be a reduced affine algebra over a field of characteristic zero, that satisfies the condition R_1 of Serre (i.e. A is smooth in codimension 1). There exist two elements f and g in the Jacobian ideal J of A such that if

$$L = \{(a,b) \in A \times A, \quad af - bg = 0,\}$$

then

$$\boxed{\overline{A} = \{a/g, \quad (a,b) \in L\}.}$$

The explanation is very clear: Pick f and g so that $\mathrm{height}(f,g) = 2$. Any $\alpha \in \overline{A}$ occurs as one of these ratios, since $f\alpha$ and $g\alpha$ lie in A, by Noether Theorem. Conversely,

$$a/g = b/f \in A_{\mathfrak{p}}$$

for each height one prime. Now use

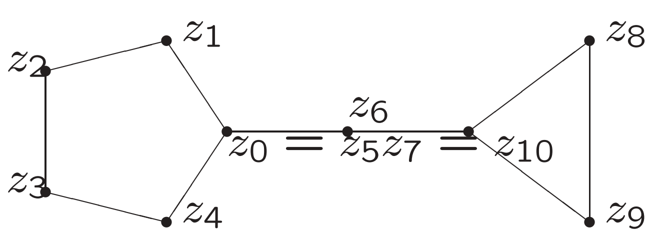
$$\overline{A} = \bigcap A_{\mathfrak{P}},$$

for rings with R_1.

One Explicit Normalization

Very often A is described not as $A = k[x_1, \ldots, x_n]/I$, that is, by generators and relations, but is given as a subring of a ring of polynomials $A = k[f_1, \ldots, f_q] \subset k[x_1, \ldots, x_n]$. Here is a family of examples for which it is possible to describe $\overline{A}$ in the same form.

Consider any graph G with vertex set $V = \{x_1, \ldots, x_n\}$. Let F be the set of all monomials $x_i x_j$ in R, such that $\{x_i x_j\}$ is an edge of G. For simplicity of notation we denote $k[F]$ by $k[G]$. Given the subgraph w of G consisting of two edge disjoint odd cycles



$$Z_1 = \{z_0, z_1, \ldots, z_r = z_0\}$$

and

$$Z_2 = \{z_s, z_{s+1}, \ldots, z_t = z_s\},$$

joined by a path (those subgraphs will be called *bow ties*), we associate the monomial $M_w = z_1 \cdots z_r z_{s+1} \cdots z_t$. We observe that Z_1 and Z_2 are allowed to intersect and that only the variables in the cycles occur in M_w, not those in the path itself.

Theorem:[Simis-Villarreal-V, Hibi-Ohsugi] Let G be a graph and let k be a field. Then the integral closure $\overline{k[G]}$ of $k[G]$ is generated as a k-algebra by the set

$$\mathcal{B} = \{f_1, \ldots, f_q\} \cup \{M_w \mid w \text{ is a bow tie}\},$$

where $f_1, \ldots, f_q$ denote the monomials defining the edges of G.

Bring-Jerrard Extensions

These are extensions defined by an equation

$$z^n + az + b = 0,$$

over a domain R. Recently, S.-L. Tan and D.-Q. Zhang have proposed a description of the integral closure of $A = R[z]$, where R is a factorial domain containing a field (whose char does not divide $n(n-1)$) (one can assume that there is no prime p such that $p^n \mid b$ & $p^{n-1} \mid a$):

$$0 \to \overline{A} \longrightarrow R^{2n-2} \xrightarrow{\varphi} R^{n-2},$$

where φ is a mapping explicitly given.

Localization & Normalization

We treat a role that localization plays in the construction of the normalization of certain rings. The setting will be that of an affine domain A over a field k with a Noether normalization

$$R = k[x_1, \ldots, x_d] \hookrightarrow A.$$

To start, consider the following recast of the Krull-Serre criterion:

Proposition: Let A be a Noetherian domain and let f, g be a regular sequence. If $\overline{A}$ is the integral closure of A, then

$$\overline{A} = \overline{A}_f \cap \overline{A}_g.$$

The assertion holds for arbitrary Noetherian domains although we will only use for affine domains. When f and g are taken as a regular sequence in R, we still have the equality $\overline{A} = \overline{A}_f \cap \overline{A}_g$. Since $\overline{A_f} = \overline{A}_f$ (and similarly for g), there are

R–submodules $C = (c_1, \ldots, c_r)$ and $D = (d_1, \ldots, d_s)$ such that $C_f = \overline{A}_f$ and $D_f = \overline{A}_g$.

Proposition: If f, g is a regular sequence in R, setting $B = (c_1, \ldots, c_r, d_1, \ldots, d_s)$, one has a natural isomorphism

$$\boxed{B^{**} = \mathsf{Hom}_R(\mathsf{Hom}_R(B, R), R) \simeq \overline{A}.}$$

Proof. Consider the inclusion $B \subset \overline{A}$. Since $\overline{A}$ satisfies the condition S_2, it will also satisfy S_2 relative to the subring R. This means that the bidual of B will be contained in $\overline{A}$, $B^{**} \hookrightarrow \overline{A}$. To prove they are equal, it will suffice to show that for each prime ideal $\mathfrak{p} \subset R$ of codimension 1, $B_{\mathfrak{p}}^{**} = \overline{A}_{\mathfrak{p}}$. But from our choices of f and g, $B_{\mathfrak{p}} = \overline{A}_{\mathfrak{p}}$. $\square$

A special case is when f belongs to the conductor of A, since $A_f = \overline{A}_f$ already, so that we simply take $C = A$. For simplicity we denote $B = A(f, g)$, and the special case by $A(1, g)$. As an application, let us consider a reduction technique that converts the problem of finding the integral closure of a standard graded algebra into another involving finding the integral closure of a lower dimension affine domain (but not graded) and the computation of duals.

Proposition: Suppose as above that f lies in the conductor of A and g is a form of degree 1 (in particular, $d \geq 2$)). The integral closure of A is obtained as the R–bidual of a set of generators of the integral closure of $A/(g-1)$.

Proof. The localization A_f has a natural identification with the ring $A_g = S[T, T^{-1}]$, where S is the set of fractions in A_g of degree 0 and T is an indeterminate. Further, as it is well-known, $S \simeq A/(g-1)$. $\square$

What Algorithms Do

The construction $\mathbb{P}$ is often characterized by iterations of a basic procedure $\mathcal{P}$ producing integral, rational extensions of the affine ring A terminating at its integral closure $\overline{A}$

$$A = A_0 \mapsto A_1 \mapsto A_2 \mapsto \cdots \mapsto A_n = \overline{A}$$

- What are these processes like?

- How long these chains might be?

- How long the description of $\overline{A}$ might take in terms of the description of A?

In other words: how to do it, what are the costs, what is there at the end?

Short of a direct description of the integral closure of A by a single operation as above, the construction of $\overline{A}$ is usually achieved by a 'smoothing' procedure: An operation $\mathcal{P}$ on affine rings with the properties:

- $A \subset \mathcal{P}(A) \subset \overline{A}$;
- If $A \neq \overline{A}$ then $A \neq \mathcal{P}(A)$.

The general style of the operation $\mathcal{P}$ of the form:

$$A \rightsquigarrow I(A) \rightsquigarrow \mathcal{P}(\mathcal{A}) = \mathsf{Hom}_{\mathcal{A}}(\mathcal{I}(\mathcal{A}), \mathcal{I}(\mathcal{A})),$$

where $I(A)$ is an ideal somewhat related to the conductor of A.

Two examples of such methods, using Jacobian ideals, are featured in [Va] and [deJong], respectively. Let J be the Jacobian ideal of A. The corresponding smoothing operations are as follows:

(i) $\mathcal{P}_1(A) = \mathrm{Hom}_A(J^{-1}, J^{-1})$

(ii) $\mathcal{P}_2(A) = \mathrm{Hom}_A(\sqrt{J}, \sqrt{J})$

These algorithms have a great advantage in that they incorporate normality criteria:

$$\mathcal{P}(A) = A \quad \text{if and only if } A = \overline{A}$$

They are restricted to affine algebras over fields which are nice. A more abstract method applies to arithmetic algebras (say defined over $\mathbb{Z}$) but lacks this feature. The drawback is that to compute $\mathcal{P}(A)$ one needs a presentation of the algebra A: Thus a fresh batch of new indeterminates is required for each iteration.

Consider an ideal I of A and set

$$\mathcal{P}_3(A) = \bigcup_n \mathrm{Hom}_A(I^n, I^n).$$

When I is an unmixed ideal of codimension 1, its subring

$$\mathrm{Hom}_A(I^e, I^e),$$

for e some multiplicity of A (and therefore fixed) will have the same bidual (S_2–closure) as $\mathcal{P}_3(A)$. This follows because whenever we localize in codimension 1, I has a reduction $J = (a)$ of reduction number at most $e-1$,

$$I^e = aI^{e-1},$$

and therefore $Ia^{-1} \subset \mathrm{Hom}_R(I^e, I^e)$.

When applied to the canonical ideal I of A, this implies that if $\mathcal{P}_3(A) = A$, then A is Gorenstein in codimension 1. Whenever this happens, algorithms involving $\mathcal{P}_3$ would have to be quick-started by other means. When we discuss tracking numbers, we are going to see that this cannot happen often! Of course the great advantage here is that no new variables are required.

Example

Let us illustrate with one example how the two methods differ markedly in the presence of the condition R_1 of Serre. Let A be an affine domain over a field of characteristic zero, and let J be its Jacobian ideal. Suppose $\mathsf{height} J \geq 2$ (the R_1 condition). As we have discussed,

$$\overline{A} = J^{-1} \subset \mathrm{Hom}_A(J^{-1}, J^{-1}) \subset \overline{A}.$$

Consider now the following example. Let n be a positive integer, and set

$$A = \mathbb{R}[x, y] + (x, y)^n \mathbb{C}[x, y],$$

whose integral closure is the ring of polynomials $\mathbb{C}[x, y]$. Note that $A_x = \mathbb{C}[x, y]_x$ and $A_y = \mathbb{C}[x, y]_y$. This means that $\sqrt{J}$ is the maximal ideal

$$M = (x, y)\mathbb{R}[x, y] + (x, y)^n \mathbb{C}[x, y].$$

It is clear that

$$\mathrm{Hom}_A(M, M) = \mathbb{R}[x, y] + (x, y)^{n-1} \mathbb{C}[x, y].$$

It will take precisely n passes of the operation to produce the integral closure. In particular, neither the dimension ($d = 2$), nor the multiplicity ($e = 1$) play any role.

The 'length' or 'order' of the construction is the smallest integer n such that $\mathcal{P}^n(A) = \overline{A}$. The 'cost' of the computation $C(\overline{A})$ however will consist of $\sum_{i=1}^{n} c(i)$, where $c(i)$ is the complexity of the operation $\mathcal{P}$ on the data set represented by $\mathcal{P}^{i-1}(A)$.

Obviously, in a Gröbner basis setting, different iterations of $\mathcal{P}$ may carry non-comparable costs. This holds true particularly if each iteration uses its own local variables.

Almost irresponsibly one could define the astronomical complexity of a smoothing operation by the order of $\mathcal{P}$:

$$\boxed{n = C_{\mathcal{P}}(\overline{A}).}$$

It is not yet clear whether this number carries any significance. The fact however is that $\mathcal{P}$ usually acts not on the full set $B(A)$ of integral birational extensions of A,

$$\mathcal{P} : B(A) \mapsto B(A),$$

but also on much smaller subsets of extensions (containing $\overline{A}$)

$$\mathcal{P} : B_0(A) \mapsto B_0(A).$$

One Main Problem

A very important issue is the development of efficient procedures acting on very sparse sets of extensions:

$$\boxed{\{\mathcal{P}, B_0(A)\}}$$

We are going to consider the extensions $A \subset B \subset \overline{A}$ which satisfy the condition S_2 of Serre, and define smoothing operations on them. The counting will be based on the following general property of a reduced, equidimensional affine algebra A.

Divisorial Extensions

A divisorial extension B of $\mathbf{A}$ is an algebra in $\mathcal{A}$ with the condition S_2 of Serre–e.g. $\overline{\mathbf{A}}$ itself. The subset of such algebras will be denoted $\mathcal{S}_2(\mathbf{A})$.

They can be obtained as follows. Let $\mathbf{S} \subset \mathbf{A}$ be a subalgebra over which of $\mathbf{A}$ is finite and satisfies S_2. For instance, subalgebras such those that occur in the classical Noether normalization process. For any $B \in \mathcal{A}$,

$$B \subset \mathrm{Hom}_{\mathbf{A}}(\mathrm{Hom}_{\mathbf{S}}(B, \mathbf{S}), \mathrm{Hom}_{\mathbf{S}}(B, \mathbf{S})) \in \mathcal{S}_2(\mathcal{A}).$$

In fact, this is a sub construction of the normalization process. Its usefulness already arises in:

Proposition: Suppose that $\mathbf{S}$ has S_2 and is Gorenstein in codimension 1 and $\mathbf{A}$ is rational over it. Then

- $\mathsf{Hom}_{\mathbf{S}}(B, \mathbf{S})$ is a divisorial ideal of $\mathbf{S}$.
- $\mathsf{Hom}_{\mathbf{S}}(\cdot, \mathbf{S})$ is an involution of $S_2(\mathcal{A})$.
- The chains in $S_2(\mathcal{A})$ are bounded.

The proofs are elementary using essentially duality in codimension 1. Since the divisorial ideals of $\mathbf{S}$ satisfy the descending chain condition between two fixed endpoints [in the case of $S_2(\mathcal{A})$: $\mathsf{Hom}_{\mathbf{S}}(\mathbf{A}, \mathbf{S})$ and $\mathsf{Hom}_{\mathbf{S}}(\overline{\mathbf{A}}, \mathbf{S})$] the last assertion follows easily:

Replace A by a hypersurface ring

$$S = k[x_1, \ldots, x_d, x_{d+1}]/(f) \hookrightarrow A, \quad \overline{S} = \overline{A}.$$

By duality there is a one–one correspondence between the extensions $B \in S_2(S)$ and their conductors $\gamma(B) = \mathsf{Hom}_S(B, S)$ (these are divisorial ideals of S). Note that $\gamma(\overline{A}) \subset \gamma(B)$. This sets up an order reversing embedding of partially ordered sets

$$S_2(S) \hookrightarrow \text{ideals of } \mathsf{Quot}(S/\gamma(\overline{A})).$$

But $\mathsf{Quot}(S/\gamma(\overline{A}))$ is an Artinian ring.

Numerical Bounds

We are going to make more precise the preceding discussion. Throughout we will assume that A is a reduced affine ring and $\overline{A}$ is its integral closure.

Definition: An integral extension B of A is *divisorial* if $A \subset B \subset \overline{A}$ and B satisfies the S_2 condition of Serre. The set of divisorial extensions of A will be denoted by $\mathcal{S}_2(A)$. Let $A = k[x_1, \ldots, x_n]/I$ be a reduced equidimensional affine algebra over a field k of characteristic zero, let $R = k[x_1, \ldots, x_d] \subset A$ be a Noether normalization and $S = k[x_1, \ldots, x_d, x_{d+1}]/(f)$ a hypersurface ring such that the extension $S \subset A$ is birational. Denote by J the Jacobian ideal of S, that is the image in S of the ideal generated by the partial derivatives of the polynomial f.

From $S \subset A \subset \overline{S} = \overline{A}$ we have that J is contained in the conductor of $\overline{S}$. To fix the terminology, we denote the annihilator of the S-module A/S by $\mathfrak{c}(A/S)$. Note the identification $\mathfrak{c}(A/S) = \mathrm{Hom}_S(A, S)$.

We want to benefit from the fact that S is a Gorenstein ring, in particular that its divisorial ideals have a rich structure.

Definition: Let I be an ideal containing regular elements of a Noetherian ring S. The *degree* of I is the integer

$$\deg(I) = \sum_{\text{height } \mathfrak{p}=1} \lambda((S/I)_{\mathfrak{p}}).$$

Definition: A *proper operation* (or smoothing) is an order preserving mapping

$$\mathcal{P} : S_2(A) \longrightarrow S_2(A)$$

such that if $B \neq \overline{A}$ then $B \subsetneq \mathcal{P}(A)$.

Theorem: Let S be a reduced hypersurface ring

$$S = k[x_1, \ldots, x_{d+1}]/(f)$$

over a field of characteristic zero and let J be its Jacobian ideal. Then the integral closure of S can be obtained by carrying out at most $\deg(J)$ proper operations on S.

Proof. Denote $S = k[x_1, \ldots, x_{d+1}]/(f)$, where f is a form of degree $e = \deg(A)$. By Euler's formula, $f \in L = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{d+1}}\right)$. Let then g, h be forms of degree $e-1$ in L forming a regular sequence in $T = k[x_1, \ldots, x_{d+1}]$. Clearly we have that $\deg(g,h)S \geq \deg(J)$. On the other hand, we have the following estimation of ordinary multiplicities

$$
\begin{aligned}
(e-1)^2 &= \deg(T/(g,h)) \\
&= \sum_{\text{height } \mathfrak{P}=2} \lambda(T/(g,h)_{\mathfrak{P}}) \deg(T/\mathfrak{P}) \\
&\geq \sum_{\text{height } \mathfrak{P}=2} \lambda(T/(g,h)_{\mathfrak{P}}) \\
&\geq \sum_{\text{height } \mathfrak{P}=2} \lambda(T/(f,g,h)_{\mathfrak{P}}) \\
&= \deg((g,h)S) \\
&\geq \deg(\mathfrak{c}).
\end{aligned}
$$

Corollary: Let A be a reduced equidimensional standard graded algebra over a field of characteristic zero, and set $e = \deg(A)$. Then any sequence

$$A = A_1 \subset A_2 \subset \cdots \subset A_n \subset \overline{A}$$

of finite extensions of A with the property S_2 of Serre has length at most $(e-1)^2$.

Corollary: Let k be a field of characteristic zero and let A be a standard graded domain over k of dimension d and multiplicity e. Let S be a hypersurface subring of A such that $S \subset A$ is finite and birational. Then the integral closure of A can be obtained after $(e-1)^2$ proper operations on S.

Non-Homogeneous Algebras

We will now treat affine algebras which are not homogeneous. Suppose A is a reduced equidimensional algebra over a field of characteristic zero, of dimension d. Let

$$S = k[x_1, \ldots, x_d, x_{d+1}]/(f) \hookrightarrow A$$

be a hypersurface ring over which A is finite and birational. The degree of the polynomial f will play the role of the multiplicity of A. Of course, we may choose f of as small degree as possible.

Our aim is to find estimates for the length of chains of algebras

$$S = A_0 \subset A_1 \subset \cdots \subset A_q = \overline{A}$$

satisfying the condition S_2, between S and its integral closure $\overline{A}$. The argument we used required the length estimates for the length of the total ring of fractions of $S/\mathfrak{c}$, where $\mathfrak{c}$ is the conductor ideal of S, $\mathrm{ann}\,(\overline{A}/S)$. Actually, it only needs estimates for the length of the total ring of fractions of $(S/\mathfrak{c})_{\mathfrak{m}}$, where $\mathfrak{m}$ ranges over the maximal ideals of S.

In the homogeneous case, we found convenient to estimate these lengths in terms of the multiplicities of $(S/\mathfrak{c})_{\mathfrak{m}}$; we will do likewise here.

A first point to be made is the observation that we may replace k by $K \cong S/\mathfrak{m}$ and $\mathfrak{m}$ by a maximal ideal $\mathfrak{M}$ of $K \otimes_k A$ lying over it. In other words, we can replace R by a faithfully flat (local) extension R'. The conditions are all preserved in that $S' = K \otimes_k S$ is reduced, $\overline{S'} = K \otimes_k \overline{A}$, the conductor of S extends to the conductor of S', and chains of extensions with the S_2 conditions give like to likewise extensions of K-algebras. Furthermore the length of the total ring of fractions of $R/\mathfrak{c}$ is bounded by the length of the total ring of fractions of $R'/\mathfrak{c}'$.

What this all means is that we may assume that $\mathfrak{m}$ is a rational point of the hypersurface $f = 0$. We may change the coordinates so that $\mathfrak{m}$ corresponds to the actual origin.

Proposition: Let $A = k[x_1, \ldots, x_d]$ be the ring of polynomials over the infinite field k and let f, g be polynomials in A vanishing at the origin. Suppose f, g is a regular sequence and $\deg f = m \leq n = \deg g$. Then the multiplicity of the local ring $(A/(f,g))_{(x_1,\ldots,x_n)}$ is at most nm^2.

Proof. Write f as the sum of its homogeneous components,

$$f = f_m + f_{m-1} + \cdots + f_r,$$

and similarly for g,

$$g = g_n + g_{n-1} + \cdots + g_s.$$

We first discuss the route the argument will take. Suppose that g_s is not a multiple of f_r. We denote by R the localization of A at the origin, and its maximal ideal by $\mathfrak{m}$. We observe that $A/(f_r)$ is the associated graded ring of $R/(f)$, and the image of g_s is the initial form g. Thus the associated graded ring of $R/(f,g)$ is a homomorphic image of $A/(f_r, g_s)$. If f_r and g_s are relatively prime polynomials, it will follow that the multiplicity of $R/(f,g)$ will be bounded by $r \cdot s$,

$$\deg R/(f,g) \leq r \cdot s.$$

We are going to ensure that these conditions on f and g are realized for f and another element h of the ideal (f, g). After a linear, homogeneous change of variables (as k is infinite), we may assume that each non-vanishing component of f and of g has unit coefficient in the variable x_d. For that end it suffices to use the usual procedure on the product of all nonzero components of f and g. At this point we may assume that f and g are monic.

Rewrite now

$$\begin{aligned} f &= x_d^m + a_{m-1} x_d^{m-1} + \cdots + a_0, \\ g &= x_d^n + b_{n-1} x_d^{n-1} + \cdots + b_0 \end{aligned}$$

with the a_i, b_j in $k[x_1, \ldots, x_{d-1}]$. Consider now the resultant $\mathrm{Res}(f, g)$ of these two polynomials with respect to x_d:

$$\det \begin{bmatrix} 1 & a_{m-1} & a_{m-2} & \ldots & a_0 & & & \\ & 1 & a_{m-1} & \ldots & a_1 & a_0 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & 1 & a_{m-1} & a_{m-2} & \ldots & a_0 \\ 1 & b_{n-1} & b_{n-2} & \ldots & b_0 & & & \\ & 1 & b_{n-1} & \ldots & b_1 & b_0 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & 1 & b_{n-1} & b_{n-2} & \ldots & a_0 \end{bmatrix}.$$

We recall that $h = \mathsf{Res}(f, g)$ lies in the ideal (f, g). Scanning the rows of the matrix above (n rows of entries of degree at most m, m rows of entries of degree at most n), it follows that $\deg h \leq 2mn$. A closer examination of the distribution of the degrees shows that $\deg h \leq mn$. If h_p is the initial form of h, then clearly h_p and f_r are relatively prime since the latter is monic in x_d, while h_p lacks any term with x_d.

Assembling the estimates, one has

$$\deg R/(f, g) \leq \deg R/(f, h) \leq r{\cdot}p \leq m{\cdot}mn = nm^2.$$

Corollary: Let $S = k[x_1, \ldots, x_{d+1}]/(f)$ be a reduced hypersurface ring over a field of characteristic zero, with $\deg f = e$. Then any chain of algebras between S and its integral closure, satisfying the condition S_2, has length at most $e(e-1)^2$.

AUs: Astronomical Units

Theorem: Let k be a field of characteristic zero and consider the reduced hypersurface ring

$$S = k[x_1, \ldots, x_d, x_{d+1}]/(f), \quad \deg f = e.$$

Then

$$\ell(\mathrm{Quot}(S/\gamma(\overline{S}))) \leq \begin{cases} (e-1)^2 & f \text{ is homogeneous} \\ e(e-1)^2 & \text{otherwise} \end{cases}$$

Arithmetic Noether Normalization

Let R be a normal domain, and suppose $A = R[x_1, \ldots, x_n]$ is an integral domain of dimension d. To use the approach above to build divisorial extensions between A and $\overline{A}$ requires the presence of a Gorenstein subalgebra

$$S \subset A$$

over which A is finite (or rational).

These approaches may not be always possible. Here is a positive case:

Theorem: Suppose that R is a ring of algebraic integers.

- (Shimura) If A is graded there are elements $y_1, \ldots, x_r$, $r = \dim A - 1$, such that $R[y_1, \ldots, y_r]$ is a Noether normalization of A.

- In general there are elements $y_1, \ldots, y_r$, $r = \dim A$, such A is finite over the hypersurface ring $R[y_1, \ldots, y_r]$.

Embedding Dimensions and Degrees

Let A be a reduced, equidimensional, affine ring over field k, given by generators and relations, $A = k[x_1, \ldots, x_n]/I$. The smallest n is the **embedding dimension of** A (over k). Let $\overline{A}$ be its integral closure.

$$\boxed{A = k[x_1, \ldots, x_n]/I \hookrightarrow \overline{A} = k[y_1, \ldots, y_m]/J}$$

- What is $\mathsf{emb}(\overline{A})$ like?

- If A is graded, are there bounds for the degrees of generating sets of $\overline{A}$?

Some Conjectures

Conjecture: There exist elementary functions $\beta(d,e)$, $\delta(d,e)$, polynomial in e for fixed d, such that for any standard graded integral domain A of dimension $d = \mathsf{dim}\, A$ and multiplicity $e = \mathsf{deg}(A)$,

$$\mathsf{emb}(\overline{A}) \leq \beta(\mathsf{dim}\, A, \mathsf{deg}(A)),$$

$$\mathsf{embd}(\overline{A}) \leq \delta(\mathsf{dim}\, A, \mathsf{deg}(A)).$$

We will describe joint work with Bernd Ulrich on these two questions.

Cohen-Macaulay $\overline{A}$

Proposition: Let A be a reduced equidimensional affine ring. If $\overline{A}$ is Cohen–Macaulay, then

$$\mathrm{emb}(\overline{A}) \leq d + e - 1 \quad (2d + e - 1 \text{ in the global case}).$$

Proof. If $R \subset A$ is a Noether normalization. $\overline{A}$ is a free R–module of rank e.

Dimension 3^{++}

Theorem: Let k be a field of characteristic zero and let A be a reduced and equidimensional k–algebra of dimension d and multiplicity $e > 1$.

- If $A \subset B$ is a finite and birational extension of graded rings $\operatorname{depth}_A B \geq d-1$ then

$$\nu_A(B) \leq (e-1)^2$$

$$\operatorname{emb}(B) \leq (e-1)^2 + d + 1.$$

- If $A \subset B$ is a finite and birational extension of affine rings and for each maximal ideal $\mathfrak{P}$ of A, $\operatorname{depth}_{A_{\mathfrak{P}}} B_{\mathfrak{P}} \geq d-1$, then

$$\nu_A(B) \leq e(e-1)^2 + d + 1$$

$$\operatorname{emb}(B) \leq e(e-1)^2 + 2d + 1.$$

Corollary: Let k be a field of characteristic zero and let A be a reduced and equidimensional k–algebra of dimension 3 and multiplicity e. The integral closure $B = \overline{A}$ satisfies

$$\mathrm{emb}(B) \leq \begin{cases} (e-1)^2 + 4 & \text{if } A \text{ is homogeneous} \\ e(e-1)^2 + 7 & \text{if } A \text{ is non homogeneous.} \end{cases}$$

How are these bounds obtained? A key observation is to use an exact sequence

$$0 \to S \longrightarrow B \longrightarrow C \to 0$$

(Let k be a field of characteristic zero) S is a reduced and equidimensional Noetherian standard graded k–algebra of dimension d and multiplicity $e > 1$ of the form

$$k[x_1, \ldots, x_{d+1}]/(f).$$

If $S \subset B$ is a finite and birational extension of rings, then the S–module B/S satisfies (in the graded case)

$$\deg(B/S) \leq e(e-2).$$

Proof. We use Noether normalization and the theorem of the primitive element to find a hypersurface subring

$$S = k[x_1, \ldots, x_d, t]/(f) \subset B = \overline{A}$$

over which B birational. With the given hypotheses, the S–module $C = B/S$ (may assume $\neq 0$) is Cohen–Macaulay. What is needed is to estimate the multiplicity of C and for that we are going to use the fact that S is Gorenstein and some properties of the Jacobian ideal of (f). We may assume that $S \neq B$. As S satisfies S_2 and the extension $S \subset B$ is finite and birational, it follows that the S–module B/S is of pure codimension one. Thus since S is Gorenstein,

$$\deg(B/S) = \deg(\mathrm{Ext}^1_S(B/S, S)).$$

Applying $\mathrm{Hom}_S(\cdot, S)$ to the exact sequence

$$0 \to S \longrightarrow B \longrightarrow B/S \to 0$$

yields an exact sequence

$$0 \to S/S :_S B \to \mathrm{Ext}^1_S(B/S, S) \to \mathrm{Ext}^1_S(B, S) \to 0.$$

Since B has the property S_2 over the Gorenstein ring S, the S–module $\mathrm{Ext}^1_S(B,S)$ has codimension at least 3. Thus

$$\deg(\mathrm{Ext}^1_S(B/S,S)) = \deg(S/S:_S B).$$

On the other hand $J(S) \subset S :_S \overline{S}$ by [Noether] (see also [Kunzbook] or for more general results, [Lipman–Sathaye]). Hence $J(S) \subset S :_S B$ and then, as both ideals have height one,

$$\deg(S/S:_S B) = \deg(S/J(S)).$$

Finally, write $S = R/(f)$ with $R = k[x_1,\ldots,x_n]$ a polynomial ring and f a form of degree e. Set $J = R\left(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right)$. One has $f \in J$ since characteristic $k = 0$ and hence $S/J(S) = R/J$. Note that J is an ideal of height 2 generated by forms of degree $e-1$. Let g,h be forms of degree $e-1$ in J generating a regular sequence.

Consider the induced exact sequence

$$0 \to L \to R/(g,h) \to \mathrm{Ext}^1_S(C,S) \to \mathrm{Ext}^1_S(B,S) \to 0.$$

Note that we cannot have $L = 0$ as $R/(f,g)$ is Cohen–Macaulay of dimension $d-1$ and therefore $\mathrm{Ext}^1_S(B,S)$ would have codimension at most 2. Thus L must be nonzero, of dimension $d-1$, and therefore

$$\begin{aligned} \deg(S/J(S)) &< \deg(R/(g,h)) - 1 \\ &\leq (e-1)^2 - 1 = e(e-2). \end{aligned}$$

Proof of Theorem, first part. Let S be a hypersurface ring of multiplicity e, $S \subset A \subset B$, as in the introduction. If $S \neq B$, the module B/S is Cohen–Macaulay of dimension $d - 1$ and multiplicity bounded by $e(e-2)$ by the Lemma. By [Bruns-Herzog, Chapter 4], $\nu_S(B/S) \leq \deg(B/S)$, which will prove the first assertion. The other estimate will follow.

The other part is another kind of calculation, but still straightforward.

Small Singularities

Let A be a reduced equidimensional affine algebra over a perfect field k, let J be its Jacobian ideal and B its integral closure. In this section we consider the case of $\dim A = d \geq 4$ and assume that the singular locus of A is suitably small.

Theorem: Let k be a field of characteristic zero and let A be a reduced and equidimensional standard graded algebra of dimension $d \geq 4$ and multiplicity e. Write $B = \overline{A}$ and let t be an integer with $2 \leq t \leq \min\{d-2, \operatorname{depth}_A B\}$. If A satisfies R_{d-t-1} then

$$\nu_A(B) \leq (e(e-1))^{2^{d-t-1}} - (2e(e-1))^{2^{d-t-2}} + 2.$$

If in addition k is algebraically closed and A is a domain then

$$\operatorname{emb}(B) \leq (e(e-1))^{2^{d-t-1}} - (2e(e-1))^{2^{d-t-2}} + t + 3.$$

In particular, this applies if A has an isolated singularity.

Bounding Degrees

Conjecture: [Eisenbud-Goto] Let A be a homogeneous integral domain over an integrally closed field of characteristic zero. Then the Castelnuovo–Mumford regularity of A is bounded by

$$\mathrm{reg}(A) \leq \deg(A) - \mathrm{codim}\,(A) + 1.$$

Theorem: Let k be a field of characteristic zero and let A be a standard graded domain over k of dimension d and multiplicity e. Write $B = \overline{A}$ and suppose that A satisfies the condition R_1. If [Eisenbud-Goto] holds in dimension $\leq d-1$ then

$$\mathrm{embd}(B) \leq (e-1)^2.$$

Theorem: Let k be a perfect field, let A be a reduced and equidimensional standard graded k–algebra of dimension d and multiplicity $e \geq 2$, and let $A \subset B$ be a finite and birational extension of graded rings. If A satisfies R_1 and $\mathrm{depth}\,_A B \geq d - 1$, then the A–module B is generated in degrees at most $3e - 4$.

Module Operations

Some of the constructions discussed involve the passage from an R–module E to its bidual $\mathsf{Hom}_R(\mathsf{Hom}_R(E,R),R)$ or the ring of endomorphisms $\mathsf{Hom}_R(E,E)$. More precisely, we consider an R–algebra A and some of the A–ideals. It is of interest to be able to make predictions about the number/type of generators of both of these modules from data on E. If using numerical data we have found convenient to bring in an extended notion of multiplicity.

Cohomological Degrees

Definition A *cohomological degree*, or *extended multiplicity*, is a function on an appropriate category of modules over a Noetherian local (or graded) ring R

$$\mathsf{Deg}(\cdot) : \mathcal{M}(R) \mapsto \mathbb{N},$$

that satisfies

(i) If $L = \Gamma_{\mathfrak{m}}(A)$ is the submodule of elements of A which are annihilated by a power of the maximal ideal and $A' = A/L$, then

$$\mathsf{Deg}(A) = \mathsf{Deg}(A') + \lambda(L),$$

where $\lambda(\cdot)$ is the ordinary length function.

(ii) (Bertini's rule) If A has positive depth and h is an appropriate generic element, then

$$\mathsf{Deg}(A) \geq \mathsf{Deg}(A/hA).$$

(iii) (The calibration rule) If A is a Cohen–Macaulay module, then

$$\mathsf{Deg}(A) = \mathsf{deg}(A),$$

where $\mathsf{deg}(A)$ is the ordinary multiplicity of the module A.

These numbers tend to be very big, bounding in a manner of speaking, both width and height:

- If $(R, \mathfrak{m}, k)$ is a Gorenstein local ring, for any $\mathsf{Deg}(\cdot)$,

$$\beta_i(A) \leq \beta(k) \cdot \mathsf{Deg}(A),$$

where $\beta_i(A)$ are the Betti numbers of the R–module A.

- [U. Nagel] If $R = k[x_1, \ldots, x_d]$, then for any f.g. graded R–module A, generated in degrees $\leq r$,

$$\mathsf{reg}(A) \leq \mathsf{Deg}(A) + r,$$

where $\mathsf{reg}(A)$ is the Castelnuovo-Mumford regularity of A.

Homological Degree

Definition: Let M be a finitely generated graded module over the graded algebra A and let S be a Gorenstein graded algebra mapping onto A, with maximal graded ideal $\mathfrak{m}$. Assume that $\dim S = r$, $\dim M = d$. The *homological degree* of M is the integer

$$\begin{aligned} \mathrm{hdeg}(M) &= \deg(M) + \\ &\sum_{i=r-d+1}^{r} \binom{d-1}{i-r+d-1} \cdot \mathrm{hdeg}(\mathrm{Ext}^i_S(M,S)). \end{aligned}$$

This expression becomes more compact when $\dim M = \dim S = d > 0$:

$$\begin{aligned} \mathrm{hdeg}(M) &= \deg(M) + \\ &\sum_{i=1}^{d} \binom{d-1}{i-1} \cdot \mathrm{hdeg}(\mathrm{Ext}^i_S(M,S)). \end{aligned}$$

Its recursive character is a negative aspect but it is fairly effective and useful to refine other cohomological degrees, such as the construction by Gunston of a degree $\mathsf{bdeg}(\cdot)$ in which the appropriate Bertini's property

$$\mathsf{bdeg}(A) = \mathsf{bdeg}(A/hA)$$

will hold. In addition, $\mathsf{hdeg}(\cdot)$ can be extended to a degree that uses Samuel's multiplicities.

When A is a Buchsbaum module,

$$\mathsf{hdeg}(A) = \mathsf{deg}(A) + I(A),$$

where $I(A)$ is the Buchsbaum invariant in Stückrad-Vogel theory.

Going back to biduals and modules of endomorphisms. [Restricted to torsionfree modules over an affine domain R admitting a Gorenstein subring S over which R is finite and rational–as the setting of Noether normalization in nice characteristics.]

The HomAB Problem

Conjecture: For any finitely generated torsionfree S–modules A and B,

$$\nu(\mathrm{Hom}_S(A,B)) \leq c(R)\cdot \mathrm{hdeg}(A)\cdot \mathrm{hdeg}(B),$$

where $c(S)$ is a constant determined from the dimension and local Betti numbers of S.

This problem is studied by Dalili in his thesis and special cases have been settled. It will hold for $\mathrm{Hom}_S(A,S)$,

$$\nu(\mathrm{Hom}_S(A,S)) \leq c(R)\cdot \mathrm{hdeg}(A),$$

in the proof does not allow keeping track of homological information on A^*, that we could use to bound the bidual A^{**}. For that one turns to Auslander duals:

Definition: Let E be a finitely generated R–module with a projective presentation

$$F_1 \xrightarrow{\varphi} F_0 \longrightarrow E \to 0.$$

The *Auslander dual* of E is the module $D(E) = \text{coker}(\varphi^t)$,

$$0 \to E^* \longrightarrow F_0^* \xrightarrow{\varphi^t} F_1^* \longrightarrow D(E) \to 0.$$

The module $D(E)$ depends on the chosen presentation but it is unique up to projective summands. In particular the values of the functors $\text{Ext}^i_R(D(E), \cdot)$ and $\text{Tor}^R_i(D(E), \cdot)$, for $i \geq 1$, are independent of the presentation. Its uses here lies in the following:

Proposition: Let R be a Noetherian ring and let E be a finitely generated R-module. There are two exact sequences of functors:

$$0 \to \text{Ext}^1_R(D(E), \cdot) \longrightarrow E \otimes_R \cdot \longrightarrow \text{Hom}_R(E^*, \cdot) \longrightarrow \text{Ext}^2_R(D(E), \cdot) \to 0$$

$$0 \to \text{Tor}^R_2(D(E), \cdot) \longrightarrow E^* \otimes_R \cdot \longrightarrow \text{Hom}_R(E, \cdot) \longrightarrow \text{Tor}^R_1(D(E), \cdot) \to 0.$$

From these sequences, for any extended degree $\text{Deg}(\cdot)$, one has

$$\begin{aligned}
\nu(E^{**}) &\leq \nu(E) + \nu(\text{Ext}^2_R(D(E), R)) \\
&\leq \nu(E) + \text{Deg}(\text{Ext}^2_R(D(E), R)) \\
\nu(\text{Hom}_R(E, E)) &\leq \nu(E^* \otimes E) + \nu(\text{Tor}^R_1(D(E), E)) \\
&\leq \nu(E^*) \cdot \nu(E) + \text{Deg}(\text{Tor}^R_1(D(E), E)).
\end{aligned}$$

This requires information about E^* that can be tracked all the way up to E and a great deal of control over $D(E)$. This is possible in $\dim R \leq 4$. For instance:

Proposition: Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d, and let E be a torsionfree R–module and let F be its bidual. If $\dim F/E \leq 2$, then

$$\mathrm{hdeg}(F) \leq \begin{cases} \mathrm{hdeg}(E) & \text{if} \quad d \geq 6, \\ 2 \cdot \mathrm{hdeg}(E) & \text{if} \quad d = 4, 5. \end{cases}$$

This is relevant to the issue of the number of generators of the integral closure of an affine algebra A (with a Noether normalization R) if A is non-singular in codimension $\leq \dim A - 3$.

Tracking Numbers

There is another approach to the tagging of the elements of $S_2(\mathcal{A})$ by integers instead of by divisorial ideals of a ring. This is developed by Kia Dalili and myself in [DV]. It has advantages of generality–as it applies to all [or none] characteristics and even for modules–but it is restricted to graded objects. It is also helpful in analyzing what often does not happen in divisorial chains.

A tracking number of an algebraic structure–algebra, ideal, or module–is a numerical tag or index that can be used for the purpose of comparing two such structures. A way to introduce them is to attach to an structure $\mathbf{S}$ (consider the case of modules over a ring R) an element of an abelian group $\mathbf{G}$,

$$\mathsf{det}(\mathbf{S}) \in \mathbf{G},$$

and follow it up with a linear functional

$$\mathbf{h} : \mathbf{G} \longrightarrow \mathbb{Z}.$$

Such construction–if it comes with useful functorial properties–can be useful to study chains

$$\mathbf{S}_1 \longrightarrow \mathbf{S}_2 \longrightarrow \cdots \longrightarrow \mathbf{S}_n$$

and to develop techniques to bound their lengths.

We will introduce one such function that has a natural place in the study of the integral closure of algebras, and hopefully of ideals and modules.

Its definition [here limited to graded structures] uses the combinatorial data encoded in the Hilbert function and also geometric information on the multiplicities of some of its components.

- Determinant of a module or algebra

- Tracking number of an algebra

- Calculation rules

- Initial ideals

- Simplicial complexes

- Degree bounds

- Lengths of Chains

- Questions

Determinant of a Module

Let R be a Noetherian normal domain and let E be a finitely generated R–module of rank r. The determinant of E is the reflexive module of rank one

$$\mathsf{det}(E) = (\wedge^r E)^{**}.$$

It is very useful in the comparison of certain modules. Consider the following elementary observation:

Proposition: If $E \subset F$ are two reflexive R–modules of the same rank, then $E = F$ iff $\mathsf{det}(E) = \mathsf{det}(F)$.

Definition: Suppose $R = k[x_1, \ldots, x_d]$ is a ring of polynomials over a field, with the standard grading. For a graded R–module E, of dimension d, its torsionfree rank is its multiplicity $e_0(E)$. Its determinant

$$\mathsf{det}(E) \simeq R[-a].$$

(This is a graded isomorphism!) The integer a is called the tracking number of E: $\mathsf{tn}_R(E) = a$.

When A is a graded algebra that admits a Noether normalization such as R, we set

$$\mathsf{tn}(A) = \mathsf{tn}_R(A).$$

We will see that $\mathsf{tn}_R(A)$ is independent of R.

Proposition: If $E \subset F$ are two reflexive R–modules of the same rank, then $E = F$ iff $\mathsf{tn}(E) = \mathsf{tn}(F)$. In particular, any chain of distinct reflexive modules between E and F has length at most $\mathsf{tn}(E) - \mathsf{tn}(F)$.

Some experience shows that in the case of algebras, the chains may be shorter by a factor than $\mathsf{tn}(E) - \mathsf{tn}(F)$. It makes also one wonder whether using Cohen–Macaulay algebras only [when they exist at all] may further shorten the chains.

Calculation Rules

The basic rules are the following (restricted to graded modules)

- If E_0 is the torsion submodule of E,

$$0 \to E_0 \longrightarrow E \longrightarrow E' \to 0,$$

$$\mathrm{tn}(E) = \mathrm{tn}(E')$$

- If the complex (of graded modules and homogeneous mappings)

$$0 \to E_1 \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_n \to 0,$$

is an acyclic complex of free modules in codimension 1, then

$$\sum_{i=1}^{n} (-1)^i \mathrm{tn}(E_i) = 0$$

Proposition: Let E be a finitely generated graded module over the polynomial ring $R = k[x_1, \ldots, x_d]$. If E is torsionfree over R, $\mathsf{tn}(E) = e_1(E)$, the first Chern number of E.

Proof. Let

$$0 \to \oplus_j R[-\beta_{d,j}] \to \cdots \to \oplus_j R[-\beta_{0,j}] \to E \to 0$$

be a (graded) free resolution of E. The integer

$$e_1(E) = \sum_{i,j} (-1)^i \beta_{i,j}$$

is the next to the leading Hilbert coefficient of E. It is also the integer that one gets by taking the alternating product of the determinants in the free graded resolution. $\square$

In general, the connection between the tracking number and the first Hilbert coefficient has to be 'adjusted' in the following manner.

Proposition: Let E be a finitely generated graded module over $R = k[x_1, \ldots, x_d]$. If $\dim E = d$ and E_0 is its torsion submodule,

$$0 \to E_0 \longrightarrow E \longrightarrow E' \to 0,$$

then

$$\mathsf{tn}(E) = \mathsf{tn}(E') = e_1(E') = e_1(E) + \widehat{e}_0(E_0),$$

where $\widehat{e}_0(E)$ is the multiplicity of E_0 if $\dim E_0 = d-1$, or 0 otherwise.

Proof. Denote by $H_A(t)$ the Hilbert series of an R-module A and write

$$H_A(t) = \frac{h_A(t)}{(1-t)^d},$$

if $\dim A = d$. For the exact sequence defining E', we have

$$h_E(t) = h_{E'}(t) + (1-t)^r h_{E_0}(t),$$

where $r = 1$ if $\dim E_0 = d-1$, or $r \geq 2$ otherwise. Since

$$e_1(E) = h'_E(1) = h'_{E'}(1) + r(1-t)^{r-1}|_{t=1} h_{E_0}(1),$$

the assertion follows. □

Remark: This suggests a reformulation of the notion of tracking number. By using exclusively the Hilbert function, the definition could be extended to all finite modules over a graded algebra.

Corollary: Let E and F be graded R-modules of dimension d. Then

$$\mathsf{tn}(E \otimes_R F) = \deg(E) \cdot \mathsf{tn}(F) + \deg(F) \cdot \mathsf{tn}(E).$$

Proof. By a corollary above, we may assume that E and F are torsionfree modules. Let $\mathbb{P}$ and $\mathbb{Q}$ be minimal projective resolutions of E and F, respectively. The complex $\mathbb{P} \otimes_R \mathbb{Q}$ is acyclic in codimension 1, by the assumption on E and F. We can then use a corollary above,

$$\mathsf{tn}(E \otimes_R F) = \sum_{k \geq 0} (-1)^k \mathsf{tn}(\oplus_{i+j=k} \mathbb{P}_i \otimes_R \mathbb{Q}_j).$$

Expanding gives the desired formula. $\square$

Key Remark: The Proposition gives another way to define **tracking numbers** in a way that does not depend of any Noether normalization. It is helpful in assignment tracking numbers to an associated graded ring $\mathsf{gr}_I(R)$, when there is possibly no characteristic to be used.

Let $A = k[x_1, \ldots, x_n]/I$ be a standard graded algebra. Let $>$ be a monomial order and set $I' = \mathsf{in}_>(I)$ and $A' = k[x_1, \ldots, x_n]/I'$. A comparison result gives:

Theorem: $\mathsf{tn}(A') \geq \mathsf{tn}(A)$.

Proof. Let J be the component of I of maximal dimension and consider the exact sequence

$$0 \to J/I \longrightarrow S/I \longrightarrow S/J \to 0.$$

$\dim J/I < \dim A$ and therefore $\mathsf{tn}(A) = \mathsf{tn}(S/J) = e_1(S/J)$. Denote by J' the corresponding initial ideal of J, and consider the sequence

$$0 \to J'/I' \longrightarrow S/I' \longrightarrow S/J' \to 0.$$

Noting that S/I and S/J have the same multiplicity, and so do S/I' and S/J' by Macaulay's theorem, $\dim J'/I' < \dim A$. This means that

$$\begin{aligned}\mathsf{tn}(S/I') = \mathsf{tn}(S/J') &= e_1(S/J') + \widehat{e}_0(J'/I') \\ &= e_1(S/J) + \widehat{e}_0(J'/I') \\ &= \mathsf{tn}(A) + \widehat{e}_0(J'/I').\end{aligned}$$

Example: $A = k[x, y, z, w]/(x^3 - yzw, x^2y - zw^2)$:

$$\mathsf{tn}(A) = e_1(A) = 18.$$

$A' = k[x, y, z, w]/I'$, where I' is the initial of I for the Deglex order. A calculation with *Macaulay2* gives

$$I' = (x^2y, x^3, xzw^2, xy^3zw, y^5zw).$$

$$\mathsf{tn}(B) = 18 + 5 = 23.$$

The example shows that $\mathsf{tn}(\cdot)$ is independent of the Hilbert function of the algebra. As a piece of philosophy, we tend to view $\mathsf{tn}(\cdot)$ as a 'nonlinear' invariant of the algebra.

Simplicial Complexes

One of our goals is to determine bounds for $\mathsf{tn}(A)$, in particular whether these numbers are non-negative. There are also some explicit formulas:

Theorem: If Δ is a simplicial complex and $k[\Delta]$ is the corresponding face ring,

$$\mathsf{tn}(k[\Delta]) = df_{d-1} - f_{d-2} + f'_{d-2},$$

where f'_{d-1} is the number of maximal faces of dimension $d-2$.

That $\mathsf{tn}(k[\Delta])$ are non-negative also results from general results for graded rings which are reduced.

General Bounds

There are various kinds of bounds for $\mathsf{tn}(A)$, but some require lots of data. One that is very general is the following:

Theorem: Let E be a graded module generated in non-negative degrees. Then

$$\mathsf{tn}(E) \leq \mathsf{deg}(E) \cdot \mathsf{reg}(E),$$

where $\mathsf{reg}(E)$ is the Castelnuovo-Mumford regularity of E.

This follows from standard properties of reg, deg and tn under generic hyperplane sections.

Positivity Results

Let A be a positively generated graded K-algebra, finite over a (standard) polynomial ring. While $\mathsf{tn}(A)$ may be negative [common in the case of modules] one has:

Theorem: If A is reduced, $\mathsf{tn}(A) \geq 0$.

Its proof and of the following intermingle:

Theorem: If A is a domain, $\mathsf{tn}(\overline{A}) \geq 0$.

Sketch of Proof: We have $A \subset A^{**} \subset \overline{A}$ where A^{**} is the bidual $\mathsf{Hom}_R(\mathsf{Hom}_R(A, R), R))$. A^{**} is an algebra and $\mathsf{tn}(A) = \mathsf{tn}(A^{**})$. We thus may assume $\mathsf{dim}\, A > 2$ and A has the S_2-condition of Serre.

We replace the base field K by $K(x)$ and pick a 1–form $h \in R$ such that h is a prime element in $\overline{A}$ and $\mathsf{tn}(A) = \mathsf{tn}(A/hA)$ and $\mathsf{tn}(\overline{A}) = \mathsf{tn}(\overline{A}/h\overline{A})$. Start now with the latter and continue until dimension drops to 2.

Multiplicity-Based Bounds

Theorem: If A is a standard graded domain of multiplicity e,

$$0 \leq \mathsf{tn}(\overline{A}) \leq \mathsf{tn}(A) \leq \binom{e}{2}.$$

In particular, any chain of distinct algebras

$$A = A_0 \subset A_1 \subset \cdots \subset A_n = \overline{A}$$

satisfying the condition S_2 of Serre has length at most $\binom{e}{2}$.

Proof: If K is a field of characteristic zero, A contains a hypersurface ring

$$S = R[t]/(f), \quad \deg f = e.$$

Since $\mathsf{tn}(S) = \binom{e}{2}$, the theorems apply.

To complete the proof in other characteristics we resort to the following construction [a module-theoretic ersatz for the theorem of the primitive element]:

Proposition: Let A be an integral domain finite over the subring R, $A = R[y_1, \ldots, y_n]$, and let $\mathsf{rank}_R(A) = e$. Let K be the field of fractions of R and define the field extensions

$$F_0 = K, \quad F_i = K[y_1, \ldots, y_i], \quad i = 1 \ldots n.$$

Then the module

$$E = \sum R y_1^{j_1} \cdots y_n^{j_n}, \quad 0 \leq j_i < r_i = [F_i : F_{i-1}]$$

is R–free of rank e. If A is a standard graded algebra, with $\mathsf{deg}(y_i) = 1$, $\mathsf{tn}(E) \leq \binom{e}{2}$.

Proof. As the rank satisfies the equality $e = \prod_{i=1}^n r_i$, e is the number of 'monomials' $y_1^{j_1} \cdots y_n^{j_n}$. Their linear independence over R is a simple verification. Note also that there are monomials of all degrees between 0 and $(r_1 - 1, \ldots, r_n - 1)$. Thus according to an earlier observation,

$$\mathsf{tn}(E) \geq \mathsf{tn}(A),$$

and the bound for $\mathsf{tn}(E)$ is obvious, with equality holding only when E is a hypersurface ring over R.

Exercise: Classify the algebras with $\mathsf{tn}(A) \leq 1$.

Lengths of Chains

We now indicate how tracking numbers may throw further light on the lengths of the form:

$$A = A_0 \subset A_1 \subset \cdots \subset A_n = \overline{A}$$

Let us outline a method for generating such chains. Let S be a hypersurface ring of integral closure $\overline{S}$ and A an extension

$$S \subset A \subset \overline{S}$$

If $A \neq \overline{S}$, one seeks a larger extension $A \subset B$. In char 0 a Jacobian ideal will do–but it will add many new variables. A naive approach is to consider

$$C = \mathrm{ann}(A/S),$$

and take the algebra

$$B_0 = C^e : C^e,$$

$e = \mathsf{deg}(S)$. The bidual of B_{0},

$$B = \mathsf{Hom}_S(\mathsf{Hom}_S(B_{\mathsf{0}}, S), S))$$

will often work but not if A is quasi-Gorenstein—i.e. $C \simeq A$.

Proposition: Let $S \subset A \subset \overline{S}$ (graded case). If A is quasi-Gorenstein

$$\mathsf{tn}(A) = \mathsf{deg}(A) \cdot \mathsf{half\text{-}integer.}$$

In particular if $A \subset B$ are two such algebras,

$$\mathsf{tn}(A) - \mathsf{tn}(B) \geq \begin{cases} \mathsf{deg}(A), \mathsf{if}\ \mathsf{deg}(A)\ \mathsf{is\ odd} \\ 1/2 \cdot \mathsf{deg}(A), \mathsf{otherwise.} \end{cases}$$

Proof. The canonical module of A is $\mathsf{Hom}_R(A, R)(-a)$ so if A is quasi-Gorenstein

$$\mathsf{Hom}_R(A, R) \simeq A(-b).$$

In this case,

$$\mathsf{tn}(A^*) = -\mathsf{tn}(A) = \mathsf{tn}(A) + \mathsf{deg}(A) \cdot b.$$

Corollary: Quasi-Gorenstein extensions cannot occur 'too' often in divisorial chains between S and its integral closure.

The issue is more complicated:

$$I^e : I^e = A$$

means that A is Gorenstein in codimension one [pre-quasi-Gorenstein ...].

Conjecture: If A is a graded algebra as above,

$$\mathsf{tn}(A) = \mathsf{deg}(A) \cdot \mathsf{half\text{-}integer}.$$

Note that it would place a lower bound on $\mathsf{tn}(\overline{A})$.

Abstract Tracking Numbers

If R is an integrally closed local ring, we don't know how to construct tracking numbers for its R-modules. While the determinant of an R–module E ($\dim E = \dim R$) can be formed

$$\det(E) = (\wedge^e E)^{**},$$

there doesn't seem to be a natural way to attach a degree to it.

In the case of a R–algebra A of finite integral closure $\overline{A}$, there is an *ad hoc* solution for the set of submodules of $\overline{A}$ of rank $r = \mathrm{rank}_R(A)$. The construction proceeds as follows:

Let F be a normalizing free R–module

$$F = R^r = Re_1 \oplus \cdots \oplus Re_r \subset A \subset \overline{A}.$$

There exist $0 \neq f \in R$ such that $f \cdot \overline{A} \subset F$. For the R–submodule E of $\overline{A}$,

$$\mathsf{det}(fE) \subset R(e_1 \wedge \cdots \wedge e_r) = R\epsilon.$$

This means that

$$\mathsf{det}(fE) = I.R\epsilon$$

where I is a divisorial ideal of R, so If^{-r} is also a divisorial ideal with a primary decomposition

$$If^{-r} = \bigcap p_i^{(r_i)}.$$

Definition: The tracking number of E (offset by F) is the integer

$$\mathsf{tn}(E) = \sum_i r_i \cdot \mathsf{deg}(R/p_i) + \mathsf{tn}(R\epsilon).$$

The value $\mathsf{tn}(E)$ is defined up to an offset but it is independent of f. It will have several of the properties of the tracking number defined for graded modules and can play the same role in the comparison of the lengths of chains of subalgebras lying between A and $\overline{A}$.

Questions

- If $(R, \mathfrak{m})$ is a local ring and I is an $\mathfrak{m}$-primary ideal, what is $\mathsf{tn}(G = \mathsf{gr}_I(R))$ like? If R is Cohen-Macaulay $\mathsf{tn}(G) \geq 0$ and can be bounded by other degree functions. (This suggests that one should define the tracking number of I as $\mathsf{tn}(G)$, not $\mathsf{tn}(I)$.) For instance, a guess for a bound is

$$\mathsf{tn}(G) \leq e_1(I) + e_2(I).$$

- When R is a regular local ring of dimension d and $\mathbf{G}$ is the associated graded ring of the integral closure filtration,

$$\mathsf{tn}(\mathbf{G}) \leq (d-1)e(I).$$

  [Even sharper, according to Claudia's talk.] Are there such bounds for singular rings? (This bound also holds for F–regular rings.)

- It is possible to define tracking numbers for some non-graded algebras and they are interesting. The construction is ad hoc and does not extend for modules just for affine domains but in any characteristic–ah, to paraphrase a Purdue poet, it just requires a long determinant. (Extending the notion to more general graded structures is straightforward.)

- For an associated graded ring $G = \mathsf{gr}_I(R))$, with special fiber $\mathcal{F}(I) = G \otimes (R/\mathfrak{m})$, can $\mathsf{tn}(\mathcal{F}(I))$ be connected to the Hilbert coefficients of I in the manner that the multiplicity of $\mathcal{F}(I)$ does?

- For an algebra A (finite over a standard Noether normalization), an interesting puzzle is the possible relationship between $\mathsf{red}(A)$ and $\mathsf{tn}(A)$. The formulas or bounds that relate these integers to $\mathsf{deg}(A)$ or $\mathsf{reg}(A)$ suggest:

  Conjecture: There exists a function $c(d)$ (dependent on the dimension d of A) such that

$$\mathsf{red}(A) \leq c(d)\sqrt{\mathsf{tn}(A)}.$$

Ideals and Modules

Like rings, there are also notions of integral closure and normalization for ideals and modules. One setting they can be discussed is that of *Rees Algebras*. There are several definitions from which we will pick a relative (and more classical) one. Let R be a ring and E an R–module with an embedding

$$f : E \longrightarrow R^r.$$

The Rees algebra of E is the image of homomorphism of the symmetric algebras

$$S(f) : S(E) \longrightarrow S(R^r) = R[T_1, \ldots, T_r],$$

$$\mathcal{R}(E) = \text{image } S(f).$$

This definition seems to imply that the Rees algebra depends on f, so that the proper notation for it should be $\mathcal{R}_f(E)$.

This is indeed the case according to an example in [EHU]. They have proposed for an absolute notion of Rees algebra,

$$\mathcal{R}(E) = S(E)/\bigcap_f(\ker S(f)).$$

This gives an algebra that is tighter than the standard definition

$$\mathcal{R}(E) = S(E)/(\text{modulo } R\text{–torsion})$$

When R is an integral domain, the 3 definitions coincide. In any event, we will always take a fixed embedding

$$f : E \longrightarrow R^r$$

In fact, we assume R domain and $r = \operatorname{rank}(E)$. The mapping

$$\wedge^r(f) : \wedge^r E \longrightarrow \wedge^r R^r \simeq R$$

define the ideal

$$\det(E) = \operatorname{image}(\wedge^r f).$$

While we are on the subject, we offer yet another definition of Rees algebras of modules. Given an R–module E, let

$$f : E \longrightarrow E_0$$

be the embedding into its *injective envelope*. The the image of

$$S(f) : S(E) \longrightarrow S(E_0),$$

$$\mathcal{R}(E) = \text{image } S(f).$$

If R is Noetherian and E is finitely generated, this definition has obvious functorial properties that includes commuting with localization.

$$\mathcal{R}(E) = R \oplus E \oplus E^2 \oplus \cdots \subset S = R[T_1, \ldots, T_r]$$

One can then extend to $\mathcal{R}(E)$ the notion of integrality etc. (But still with care since the embedding matters–but not if R is normal.) The key notion is that of *reduction*:

Definition: The submodule $U \subset E$ is a reduction of E if $\mathcal{R}(E)$ is integral over $\mathcal{R}(U)$.

This amounts saying that

$$E^{r+1} = U \cdot E^r,$$

for some integer r, with the smallest called the *reduction number* of E relative to U.

Definition: The integral closure of E in R^r is the largest submodule $\overline{E}$ of R^r admitting E as a reduction. If $E = \overline{E}$, E is said to be integrally closed (or complete) (in R^r), with the caveats dropping when R is normal. In such case,

$$\overline{E} = \bigcap_V VE,$$

here V runs over all the valuations of R. It gives rise to

A test of

$$\overline{U} = \overline{E}$$

is

$$\overline{\det(U)} = \overline{\det(E)}$$

From now on R is normal. Then $\mathcal{R}(E)$ is normal when all E^n are integrally closed.

Another notion and a construction are useful. If $(R, \mathfrak{m})$ is a local ring,

$$\mathcal{F}(E) = \mathcal{R}(E) \otimes (R/\mathfrak{m})$$

is its *special fiber*. Its dimension $\ell(E)$ is the *analytic spread* of E (at $\mathfrak{m}$). When $R/\mathfrak{m}$ is infinite, there are reductions generated by $\ell(E)$ elements (they are minimal). Among all such the smallest reduction number is called the reduction number of E.

One can associate to E an ideal of $S = R[T_1, \ldots, T_r]$,

$$(E) = ES,$$

or its Rees algebra (not to be confused with $\mathcal{R}(E)$)

$$S + ESt + E^2St^2 + \cdots$$

Playing one structure against another is useful:

Proposition: Let M be a graded ideal of the polynomial ring $S = R[x_1, \ldots, x_d]$, $M = \oplus_{n \geq 0} M_n$. If M is an integrally closed S-ideal then each component M_n is an integrally closed R-module.

Proof. M_n is a R-module of the module S_n freely R-generated by the monomials T_α in the x_i of degree n. Denote by $\mathcal{R}(M_n) \subset \mathcal{R}(S_n)$ the corresponding Rees algebras. Let $u \in S_n$ be integral over M_n; there is an equation in $\mathcal{R}(S_n)$ of the form

$$u^m + a_1 u^{m-1} + \cdots + a_m = 0, \quad a_i \in M_n^i.$$

Map this equation using the natural homomorphism $\mathcal{R}(S_n) \to S$, that sends the variable T_α into the corresponding monomial of S. The equation converts into an equation of integrality of $u \in S$ over the ideal M. Since $M = \overline{M}$, $u \in M_n$. $\square$

Proposition: Let $S = \mathcal{R}(R^r) = R[x_1, \ldots, x_r]$ and denote by (E) the ideal of S generated by the forms in E. For any positive integer n,

$$\overline{E_n} = \overline{(E^n)}_n.$$

Proof. It suffices to verify the equality of these two integrally closed modules at the valuations of R. At some valuation V, $VS = V[z_1, \ldots, z_r]$ and VE is generated by the forms $a_1 z_1, \ldots, a_r z_r$, with $a_i \in V$. We must show that the ideal (VE) is normal. We may assume that a_1 divides all a_i, $a_i = a_1 b_i$, so the ideal $(VE) = a_1(z_1, b_2 z_2, \ldots, b_r z_r)$, and it will be normal if and only if it is the case for $(z_1, b_2 z_2, \ldots, b_r z_r)$. Obviously we can drop the indeterminate z_1 and iterate. $\square$

Normality of Algebras of Linear Type

An important class of Rees algebras are those of *linear type*:

$$\mathcal{R}(E) = S(E).$$

In this case if

$$R^m \xrightarrow{\varphi} R^n \longrightarrow E \to 0$$

is a free presentation of E, then

$$\mathcal{R}(E) = R[T, \ldots, T_n]/(f_1, \ldots, f_m)$$

where

$$[f_1, \ldots, f_m] = [T_1, \ldots, T_n] \cdot \varphi.$$

It makes the algebras more amenable.

Theorem: Let R be a regular integral domain and let E be a torsionfree module of rank r, with a free presentation

$$R^p \xrightarrow{\psi} R^m \xrightarrow{\varphi} R^n \longrightarrow E \to 0.$$

Suppose $S(E)$ is a domain. E is normal if and only if the following conditions hold:

(i) The ideal $I_c(\psi)S_R(E)$, $c = m + r - n$, is principal at all localizations of $S_R(E)$ of depth 1.

(ii) The modules $S_s(E)$ are complete, for $s = 1, \ldots, n - r$.

$$I_c(\psi)S_R(E) \Leftrightarrow (\text{Jacobian ideal})S_R(E)$$

Let R be a Cohen–Macaulay ring and let E be a complete intersection module. This means that there exists a mapping of rank r

$$\varphi : R^n \longrightarrow R^r,$$

with $E = \mathsf{image}\,\varphi$, with $n = r + c - 1$, for $c \geq 2$, and the ideal $I = \mathsf{det}_0(E)$ of maximal minors of φ has codimension c. These modules meet our assumptions that $S_R(E)$ be an integral domain.

Theorem: Let R be a Cohen–Macaulay integrally closed domain and let E be a complete intersection module. The following conditions are equivalent:

(a) E is integrally closed.

(b) E is normal (i.e. $S_R(E)$ is normal).

(c) $\mathsf{det}_0(E)$ is an integrally closed generic complete intersection.

To prove normality for $S_R(E)$ one must verify the conditions (i) S_2, and (ii) R_1 of Serre. Fortunately for the modules above $S_R(E)$ is Cohen-Macaulay (by [Katz-Naudé, Simis-Ulrich-V]). It will allow the reduction to a result of [Goto], valid for ideals.

For a module defined by a mapping

$$R^n \xrightarrow{\varphi} R^r, \quad I = I_r(\varphi),$$

it converts into:

Theorem: Let I be a generic complete intersection ideal of codimension c. I is integrally closed if and only if the following conditions hold:

(i) height ann $\wedge^{c+1} \sqrt{I} \geq c+1$;

(ii) height ann $\wedge^2 (\sqrt{I}/I) \geq c+1$.

The Hard Problems

One could drop the task onto the *algebra* lap, but the dimension of $\mathcal{R}(E)$ is too large and there may be some opportunities that we may not want to miss.

- Integral Closure of Modules

A non direct construction of the integral closure of an ideal or module can be sketched as follows. Let R be a normal domain and let E be a torsionfree R-module. $\overline{E}$ is the degree 1 component of the integral closure of the Rees algebra of E:

$$E \rightsquigarrow \overline{\mathcal{R}(E)} = R + \boxed{\overline{E}} + \overline{E^2} + \cdots \rightsquigarrow \overline{E}$$

This begs the issue since the construction of $\overline{\mathcal{R}(E)}$, for arbitrary modules, may verge on the impossibility. It takes place in a much larger setting (that of a presentation $\mathcal{R}(E) = R[T_1, \ldots, T_n]/L$). By a *direct* construction $E \rightsquigarrow \overline{E}$ we mean an algorithm whose steps take place entirely in R or R^r. These are lacking in the literature.

A significant difference between the construction of the integral closure of an affine algebra A and that of $\overline{I}$ for an ideal I say lies in the ready existence of *conductors*: Given A by generators and relations (at least in characteristic zero) the Jacobian ideal J of A has the property

$$J \cdot \overline{A} \subset A,$$

in other words, $\overline{A} \subset A : J$. This fact lies at the root of all current algorithms to build $\overline{A}$. There is no known corresponding *annihilator* for $\overline{I}/I$. In several cases, one can cheat by borrowing part of the Jacobian ideal of $R[It]$ by proceeding as follows. Let (this will be part of the cheat) $R[It] = R[T_1, \ldots, T_n]/L$ be a presentation of the Rees algebra of I. The Jacobian ideal is a graded ideal

$$J = J_0 + J_1 t + J_2 t^2 + \cdots,$$

with the components obtained by taking selected minors of the Jacobian matrix. This means that to obtain some of the generators of J_i we do not need to consider all the generators of L. Since J annihilates $\overline{R[It]}/R[It]$, we have that for each i

$$J_i \cdot \overline{I} \subset I^{i+1},$$

and therefore

$$\overline{I} \subset \bigcap_{i \geq 0} I^{i+1} : J_i.$$

Of course when using subideals $J_i' \subset J_i$, or further when only a few J_i' are used, the comparison gets overstated.

Despite these obstacles, in a number of important cases, one is able to understand relatively well the process of integral closure and normalization of ideals. These include monomial ideals and ideals of finite colength in regular local rings. Even here the full panoply of techniques of commutative algebra must be brought to play.

Ad hoc techniques

There are some *annihilator* methods that often produce elements in $\overline{I}$. One introduced in [Corso,Huneke,Katz,..] is grounded on annihilation of homology. Let $I = (f_1, \ldots, f_n)$ be an unmixed ideal of grade g. Let $\mathbb{K}$ be the Koszul complex on the f_i. For $0 \leq j \leq n - g$, Corso Conjecture-Question: Is

$$\mathsf{ann}\, H_j(\mathbb{K}) \subset \overline{I}$$

Lots of experiments lend support to it. Also theoretical results such if $I = \overline{I}$ and $j = 1$, or $j = n - g$. However it may happen that $\mathsf{ann}\,(H_j(\mathbb{K}) = I$ for all $j \leq n - g$.

In the case of a module $E \subset R^r$, for any ideal L not contained in the associated primes of $\mathsf{det}_0(E)$,

$$E :_{R^r} L \subset \overline{E}$$

[Note that for ideals, when $I = \mathsf{det}_0(I)$, this adds nothing.]

Another approach is that of *m-fullness*. All modules have the property that for any prime $\mathfrak{p}$ (assume R contains an infinite field) there is an element $x \in \mathfrak{p}$ such that

$$xE :_{R^r} \mathfrak{p} \subset \overline{E}$$

Numerical Measures

Definition: Let R be a quasi-unmixed normal domain and let I be an ideal.

(i) The *normalization index* of I is the smallest integer $s = s(I)$ such that

$$\overline{I^{n+1}} = I \cdot \overline{I^n} \quad n \geq s.$$

(ii) The *generation index* of I is the smallest integer $s_0 = s_0(I)$ such that

$$\sum_{n \geq 0} \overline{I^n} t^n = R[\overline{I}t, \ldots, \overline{I^{s_0}} t^{s_0}].$$

For example, if $R = k[x_1, \ldots, x_d]$ and $I = (x_1^d, \ldots, x_d^d)$, then $I_1 = \overline{I} = (x_1, \ldots, x_d)^d$. It follows that $s_0(I) = 1$, while $s(I) = \mathrm{red}_I(I_1) = d - 1$.

For primary ideals and some other equimultiple ideals there are relations between the two indices of normalization.

Proposition: Let $(R, \mathfrak{m})$ be an integrally closed, local Cohen-Macaulay domain such that the maximal ideal $\mathfrak{m}$ is normal. Let I be $\mathfrak{m}$-primary ideal of indices of normalization $s(I)$ and $s_0(I)$. Then

$$s(I) \leq e(I)((s_0(I)+1)^d - 1) - s_0(I)(2d-1)+1),$$

where $e(I)$ is the multiplicity of I.

Reduction Number of Good Filtrations

Let R be a quasi unmixed integral domain and let I be a nonzero ideal. Denote by

$$A = \sum_{n \geq 0} \overline{I^n} t^n,$$

the Rees algebra attached to $\{\overline{I^n}\}$. Noting that while A may not be a standard graded algebra, it is finitely generated over $R[It]$. We may thus apply the theory of Castelnuovo-Mumford regularity to A in order to deduce results about the reduction number of the filtration. We state one of these extensions.

We assume that $(R, \mathfrak{m})$ is a local ring and set $\ell(A) = \dim A/\mathfrak{m}A$ for the analytic spread of A (which is equal to the analytic spread of I). If J is a minimal reduction of I, setting $B = R[Jt]$, we can apply the results of [Johnston-Katz] to the pair (B, A):

Theorem: Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and let $\{I_n, \neq 0, I_0 = R\}$ be a multiplicative filtration such that the Rees algebra $A = \sum_{n\geq 0} I_n t^n$ is Cohen-Macaulay and finite over $R[I_1 t]$. Suppose that $\text{height} I_1 \geq 0$ and let J be a minimal reduction of I_1. Then

$$I_{n+1} = JI_n = I_1 I_n, \quad n \geq \ell(I_1) - 1,$$

and in particular, A is generated over $R[It]$ by forms of degrees at most $\ell - 1 = \ell(I_1) - 1$,

$$\sum_{n\geq 0} I_n t^n = R[It, \ldots, I_{\ell-1} t^{\ell-1}].$$

Corollary: If I is a monomial ideal of $k[x_1, \ldots, x_d]$ then

$$\overline{I^n} = I \cdot \overline{I^{n-1}} \quad n \geq d-1.$$

For modules one has:

Theorem: Let R be a Cohen-Macaulay normal domain of dimension d and let E be a torsionfree R–module. If $\overline{\mathcal{R}(E)}$ is Cohen-Macaulay, then it is generated by $\overline{E^n}$ for $n < d$.

Note that the rank of the module does not matter. It is obtained from the ideal case using the technique of Bourbaki ideals and deformation.

Divisorial Extensions

An ordinary transplant of the theory of divisorial extensions to Rees algebras, that is to the chains of subalgebras

$$\mathcal{R}(E) \subset A \subset \overline{\mathcal{R}(E)},$$

would not be very useful since the ordinary multiplicities of these algebras are much too high. Fortunately one can do much better using the multiplicities (Buchsbaum multiplicities in the case of modules) ordinarily associated to ideals.

This is a subject that will be treated at greater depth in Polini's talk in the workshop so we set the stage and discuss only general aspects.

Primary Ideals in Regular Rings

We discuss the role of Briançon-Skoda type theorems in determining some relationships between the coefficients $e_0(I)$ and $e_1(I)$ of the Hilbert polynomial of an ideal. We consider here the case of a normal local ring $(R, \mathfrak{m})$ of dimension d and of an $\mathfrak{m}$-primary ideal I. Set $A = R[It]$ and $B = \overline{R[It]}$; we assume that B is a finite A-module. From the exact sequence

$$0 \to \overline{I^n}/I^n \longrightarrow R/I^n \longrightarrow R/\overline{I^n} \to 0,$$

we obtain as above the relationship

$$\overline{e}_1(I) = e_1(I) + \widehat{e}_0(I),$$

where $\widehat{e}_0(I)$ is the multiplicity of the module of components $\overline{I^n}/I^n$, if $\dim L = d$; otherwise it is set to zero.

Theorem: Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of infinite residue field. Suppose the Briançon-Skoda number of R is $c(R)$. Then for any $\mathfrak{m}$-primary ideal I,

$$\overline{e}_1(I) \leq c(R) \cdot e_0(I).$$

In particular, $e_1(I) \leq c(R) \cdot e_0(I)$.

Proof. The definition of $c = c(I)$: For any ideal L of R

$$\overline{L^{n+c}} \subset L^n, \quad \forall n.$$

To apply this notion to our setting, let J be a minimal reduction of I. Assume $\overline{I^{n+c}} \subset J^n$ for all n. To estimate the multiplicity of the module of components $\overline{I^{n+c}}/J^{n+c}$-which is the same as that of the module of components $\overline{I^n}/J^n$-note that $\overline{I^{n+c}} \subset J^n$, and that J^n admits a filtration

$$J^n \supset J^{n+1} \supset \cdots \supset J^{n+c},$$

whose factors all have multiplicity $e_0(J)$. More precisely, for each positive integer k,

$$\lambda(J^{n+k-1}/J^{n+k}) = e_0(J)\binom{n+k-1+d-1}{d-1} = \frac{e_0(J)}{(d-1)!}n^{d-1} + \text{lower}$$

As a consequence we obtain

$$\overline{e}_1(I) \leq e_1(J) + c(R) \cdot e_0(J) = c(R) \cdot e_0(I),$$

since $e_1(J) = 0$. The other inequality, $e_1(I) \leq c(R) \cdot e_0(I)$ is easy.

Corollary: Let $(R, \mathfrak{m})$ be a Japanese regular local ring of dimension d. Then for any $\mathfrak{m}$-primary ideal I,

$$\overline{e}_1(I) \leq (d-1)e_0(I), \quad e_1(I) \leq (d-1)e_0(I).$$

Proof. In this case, the classical Briançon-Skoda theorem asserts that $c(R) = d - 1$.

Normalization of Rees Algebras

The computation (and of its control) of the integral closure of a standard graded algebra over a field benefits greatly from Noether normalizations and of the structures built upon them. If $A = R[It]$ is the Rees algebra of the ideal I of an integral domain R, it does not allow for many such constructions.

We would still like to develop some tracking of the complexity of the task required to build $\overline{A}$ (assumed A-finite) through sequences of extensions

$$A = A_0 \to A_1 \to A_2 \to \cdots \to A_n = \overline{A}$$

where A_{i+1} is obtained from an specific procedure $\mathcal{P}$ applied to A_i. At a minimum, we would want to bound the length of such chains.

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local of dimension d, integrally closed, of Briançon-Skoda number $c(R)$, and let I be an $\mathfrak{m}$-primary ideal of multiplicity $e_0(I)$. Let A and B be distinct algebras satisfying the S_2 condition of Serre, and such that

$$R[It] \subset A \subset B \subset \overline{R[It]}.$$

For any algebra D such as these, we set $\lambda(R/D_n)$ for its Hilbert function; for $n \gg 0$, one has the Hilbert polynomial

$$\lambda(R/D_n) = e_0(D)\binom{n+d-1}{d} - e_1(D)\binom{n+d-2}{d-1} + \text{lower}$$

The Hilbert coefficients satisfy $e_0(D) = e_0(I)$, and according to where $0 \leq e_1(D) \leq c(R)e_0$.

Theorem: For any two algebras A and B as above,

$$c(R)e_0(I) > e_1(B) > e_1(A) \geq 0,$$

in particular any chain of such algebras has length bounded by $c(R)e_0(I)$.

Proof. Set $C = B/A$. Since A has Krull dimension $d+1$ and satisfies S_2, it follows easily that C is an A-module of Krull dimension d. From the exact sequence,

$$0 \to C_n \longrightarrow R/A_n \longrightarrow R/B_n \to 0,$$

one gets that the multiplicity $e_0(C)$ of C is $e_1(B) - e_1(A)$. As $e_0(C) > 0$, we have all the assertions.

Corollary: If $(R, \mathfrak{m})$ is a regular local ring of dimension d and I is an $\mathfrak{m}$-primary ideal, then $(d-1)e_0(I)$ bounds the lengths of the divisorial chains between $R[It]$ and $\overline{R[It]}$.

Non-Primary Ideals

Few cases are known, an exception being:

Theorem: If I is an equimultiple ideal of codimension g of a regular local $(R, \mathfrak{m})$ then any chain of S_2 graded algebras between $R[It]$ and $\overline{R[It]}$ has length at most

$$(g-1)\,\mathrm{deg}(R/I).$$

If the proof of this result is any indication, this is a more general phenomenon and should be approachable using some extended multiplicity.