THE TRACKING NUMBER OF AN ALGEBRA

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ABSTRACT. We introduce the technique of tracking numbers of graded algebras and modules. It is a modified version of the first Chern class of its free resolution relative to any of its standard Noether normalizations. Several estimations are obtained which are used to bound the length of chains of algebras occurring in the construction of the integral closure of a graded domain. Noteworthy is a quadratic bound on the multiplicity for all chains of algebras that satisfy the condition S_2 of Serre.

1. INTRODUCTION

A **tracking code** of an algebraic structure–algebra, ideal, or module–is a tag or index that can be used for the purpose of comparing two such structures. A way to introduce them is to attach to an structure $\mathbf{S}$ (consider the case of modules over a ring R) an element of an abelian group $\mathbf{G}$,

$$\det(\mathbf{S}) \in \mathbf{G},$$

and follow it up with a linear functional

$$\mathbf{h} : \mathbf{G} \longrightarrow \mathbb{Z}.$$

Such construction–if it comes with interesting functorial properties–can be useful to study chains

$$\mathbf{S}_1 \longrightarrow \mathbf{S}_2 \longrightarrow \cdots \longrightarrow \mathbf{S}_n$$

and to develop techniques to bound their lengths. As an application of an index introduced here, we bound lengths in *all* methods for construction of integral closures of standard graded domains that uses extensions with the property S_2 of Serre, in *any* characteristic.

The function we treat, $\mathrm{tn}(E)$, is mimicked on the first Chern class $c_1(E)$ of the graded module E over the polynomial ring $R = k[x_1, \ldots, x_d]$:

$$\det(E) \simeq R[-\delta].$$

The integer δ will be called the *tracking number* of E: $\delta = \mathrm{tn}(E)$. In particular, unlike $c_1(E)$ the number $\mathrm{tn}(E)$ is independent of the Hilbert function of E. Its usefulness comes from two contrasting rules for its computation–the definition via determinants of complexes and its Hilbert polynomials.

We now describe the contents of this paper. For a positively generated algebra A, finite over a standard Noether normalization R, $\mathrm{tn}_R(A)$ is independent of R. For such algebras, a stream of results emerges quickly:

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(1) (**Theorem 3.6**:) $\mathrm{tn}(A)$ is not defined entirely by the Hilbert function. In general $\mathrm{tn}(A') \geq \mathrm{tn}(A)$, where A' is the algebra defined by the corresponding initial ideal (strict inequality can occur).
(2) (**Theorem 4.2**:) $\mathrm{tn}(A) \leq \deg(A) \cdot \mathrm{reg}(A)$, where $\deg(A)$ is the multiplicity and $\mathrm{reg}(A)$ the Castelnuovo-Mumford regularity of A.
(3) (**Theorem 5.1**:) If A is a reduced standard graded algebra, of integral closure $\overline{A}$,

$$0 \leq \mathrm{tn}(\overline{A}) \leq \mathrm{tn}(A) \leq \binom{\deg(A)}{2}.$$

This surprising positivity statement is valid in all characteristics and allows for defining *a priori* bounds in general processes of integral closure.

2. Determinant of an algebra or a module

Throughout we use basic results and terminology from [1] and/or [2]. In particular, for an R–module E, we refer to $E^* = \mathrm{Hom}_R(E, R)$ as its *dual*, and to $E^{**} = \mathrm{Hom}_R(\mathrm{Hom}_R(E, R), R)$ as its *bidual* (sometimes with the natural mapping $E \rightarrow E^{**}$). Polynomial rings, $k[x_1, \ldots, x_d]$, will have the standard grading, $\deg x_i = 1$.

Let E be a finitely generated graded module over the polynomial ring $R = k[x_1, \ldots, x_d]$. If $\dim E = d$, denote by $\det_R(E)$ the determinantal divisor of E: If E has multiplicity e,

$$\det(E) = (\wedge^e E)^{**} \simeq R[-\delta].$$

Definition 2.1. The integer δ will be called the *tracking number* of E: $\delta = \mathrm{tn}(E)$.

The terminology *tracking number* (or *twist*) refers to the use of integers as locators, or tags, for modules and algebras in partially ordered sets. A forerunner of this use was made in [5], when divisorial ideals were employed to bound chains of algebras with the property S_2 of Serre.

Example 2.2. If A is a homogeneous domain over a field k, R is a homogeneous Noether normalization and S is a hypersurface ring over which A is birational,

$$R \subset S = R[t]/(f(t)) \subset A$$

($f(t)$ is a homogeneous polynomial of degree e), we have $(\wedge^e S)^{**} = R[-\binom{e}{2}]$. As a consequence $\mathrm{tn}(A) \leq \binom{e}{2}$.

Another illustrative example, is that of the fractionary ideal $I = (x^2/y, y)$. I is positively generated but a simple calculation shows $\mathrm{tn}(I) = -1$. Additionally, the algebra obtained by forming the trivial extension of R by I, $A = R \propto I$, has $\mathrm{tn}(A) = -1$.

The following observation shows the use of tracking numbers to locate the members of certain chains of modules.

Proposition 2.3. *If $E \subset F$ are graded modules with the same multiplicity that satisfy the S_2 condition of Serre, then $\mathrm{tn}(E) \geq \mathrm{tn}(F)$ with equality only if $E = F$.*

Corollary 2.4. *If the distinct modules in the chain*

$$E_0 \subset E_1 \subset \cdots \subset E_n$$

have the same multiplicity and satisfy the condition S_2 of Serre, then $n \leq \mathrm{tn}(E_0) - \mathrm{tn}(E_n)$.

We list some of the more elementary properties of the function $\mathrm{tn}(\cdot)$.

Proposition 2.5. *If the complex of finitely generated graded R–modules*

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

is an exact sequence of free modules in each localization $R_{\mathfrak{p}}$ at height one primes, then $\mathrm{tn}(B) = \mathrm{tn}(A) + \mathrm{tn}(C)$.

Proof. We break up the complex into simpler exact complexes:

$$0 \to \ker(\varphi) \longrightarrow A \longrightarrow A' = \mathrm{image}(\varphi) \to 0$$

$$0 \to A' \longrightarrow \ker(\psi) \longrightarrow \ker(\psi)/A' \to 0$$

$$0 \to B' = \mathrm{image}(\psi) \longrightarrow C \longrightarrow C/B' \to 0$$

and

$$0 \to \ker(\psi) \longrightarrow B \longrightarrow B' \to 0.$$

We note that by hypothesis, codim $\ker(\varphi) \geq 1$, codim $C/B' \geq 2$, codim $\ker(\psi)/A' \geq 2$, so that we have the equality of determinantal divisors: $\det(A) = \det(A') = \det(\ker(\psi))$, and $\det(C) = \det(B')$. What this all means is that we may assume the given complex is exact.

The rest of the proof is well-known but it is a short argument that is given for completeness. Suppose $r = \mathrm{rank}(A)$ and $\mathrm{rank}(C) = s$ and set $n = r + s$. Consider the pair $\wedge^r A$, $\wedge^s C$. For $v_1, \ldots, v_r \in A$, $u_1, \ldots, u_s \in C$, pick w_i in B such that $\psi(w_i) = u_i$ and consider

$$v_1 \wedge \cdots \wedge v_r \wedge w_1 \wedge \cdots \wedge w_s \in \wedge^n B.$$

Different choices for w_i would produce elements in $\wedge^n B$ that differ from the above by terms that would contain at least $r+1$ factors of the form

$$v_1 \wedge \cdots \wedge v_r \wedge v_{r+1} \wedge \cdots ,$$

with $v_i \in A$. Such products are torsion elements in $\wedge^n B$. This implies that modulo torsion we have a well defined pairing

$$[\wedge^r A/\mathrm{torsion}] \otimes_R [\wedge^s C/\mathrm{torsion}] \longrightarrow [\wedge^n B/\mathrm{torsion}].$$

When localized at primes $\mathfrak{p}$ of codimension at most 1 the complex becomes an exact complex of projective $R_{\mathfrak{p}}$–modules and the pairing is an isomorphism. Upon taking biduals and the $\circ$ divisorial composition, we obtain the asserted isomorphism. $\square$

Corollary 2.6. *Let*

$$0 \to A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_n \to 0$$

be a complex of graded R–modules and homogeneous homomorphisms which is an exact complex of free modules in codimension 1*. Then*

$$\sum_{i=1}^{n}(-1)^i \mathrm{tn}(A_i) = 0.$$

Proposition 2.7. *Let $R = k[x_1, \ldots, x_d]$ and let*

$$0 \to A \longrightarrow B \longrightarrow C \longrightarrow D \to 0,$$

be an exact sequence of graded R–modules and homogeneous homomorphisms. If $\dim B = \dim C = d$*,* $\operatorname{codim} A \geq 1$ *and* $\operatorname{codim} D \geq 2$*, then* $\mathrm{tn}(B) = \mathrm{tn}(C)$*.*

Corollary 2.8. *If E is a graded R–module of dimension d, then* $\mathrm{tn}(E) = \mathrm{tn}(E/\text{mod torsion}) = \mathrm{tn}(E^{**})$*.*

3. Calculation rules for algebras

Proposition 3.1. *Let E be a finitely generated graded module over the polynomial ring $R = k[x_1, \ldots, x_d]$. If E is torsionfree over R,* $\mathrm{tn}(E) = e_1(E)$ *the first Chern number of E.*

Proof. Let

$$0 \to \oplus_j R[-\beta_{d,j}] \to \cdots \to \oplus_j R[-\beta_{1,j}] \to \oplus_j R[-\beta_{0,j}] \to E \to 0$$

be a minimal (graded) free resolution of E. The integer

$$e_1(E) = \sum_{i,j}(-1)^i \beta_{i,j}$$

is (see [1, Proposition 4.1.9]) the next to the leading Hilbert coefficient of E. It is also the integer that one gets by taking the alternating product of the determinants in the free graded resolution (see Corollary 2.6). □

In general, the connection between the tracking number and the first Hilbert coefficient has to be 'adjusted' in the following manner.

Proposition 3.2. *Let E be a finitely generated graded module over $R = k[x_1, \ldots, x_d]$. If* $\dim E = d$ *and E_0 is its torsion submodule,*

$$0 \to E_0 \longrightarrow E \longrightarrow E' \to 0,$$

then

$$\mathrm{tn}(E) = \mathrm{tn}(E') = e_1(E') = e_1(E) + \hat{e}_0(E_0),$$

where $\hat{e}_0(E)$ is the multiplicity of E_0 if $\dim E_0 = d - 1$*, or* 0 *otherwise.*

Proof. Denote by $H_A(t)$ the Hilbert series of an R–module A (see [1, Chap. 4]) and write

$$H_A(t) = \frac{h_A(t)}{(1-t)^d},$$

if $\dim A = d$. For the exact sequence defining E', we have

$$h_E(t) = h_{E'}(t) + (1-t)^r h_{E_0}(t),$$

where $r = 1$ if $\dim E_0 = d-1$, or $r \geq 2$ otherwise. Since

$$e_1(E) = h'_E(1) = h'_{E'}(1) + r(1-t)^{r-1}|_{t=1} h_{E_0}(1),$$

the assertion follows. □

Remark 3.3. Let A be a homogeneous algebra defined over a field k that admits a Noether normalization $R = k[x_1, \ldots, x_d]$, then clearly $\mathrm{tn}_R(A) = \mathrm{tn}_{R'}(A')$, where K is a field extension of k, $R' = K \otimes_k R$ and $A' = K \otimes_k A$. Partly for this reason, we can always define the tracking number of an algebra by first enlarging the ground field. Having done that and chosen a Noether normalization R that is a standard graded algebra, it will follow that $\mathrm{tn}_R(A)$ is independent of R.

Proposition 3.2 suggests a reformulation of the notion of tracking number. By using exclusively the Hilbert function, the definition could be extended to all finite modules over a graded algebra (positively graded but not necessarily standard).

Corollary 3.4. *Let E and F be graded R–modules of dimension d. Then*

$$\mathrm{tn}(E \otimes_R F) = \deg(E)\mathrm{tn}(F) + \deg(F)\mathrm{tn}(E).$$

Proof. We may assume that E and F are torsionfree modules. Let $\mathbb{P}$ and $\mathbb{Q}$ be minimal projective resolutions of E and F, respectively. The complex $\mathbb{P} \otimes_R \mathbb{Q}$ is acyclic in codimension 1, by the assumption on E and F. We can then use Corollary 2.6,

$$\mathrm{tn}(E \otimes_R F) = \sum_{k \geq 0} (-1)^k \mathrm{tn}(\oplus_{i+j=k} \mathbb{P}_i \otimes_R \mathbb{Q}_j).$$

Expanding gives the desired formula. □

Theorem 3.5. *Let Δ be a simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$, and denote by $k[\Delta]$ the corresponding Stanley–Reisner ring. If $\dim k[\Delta] = d$,*

$$\mathrm{tn}(k[\Delta]) = d f_{d-1} - f_{d-2} + f'_{d-2},$$

where f_i denotes the number of faces of dimension i, and f'_{d-2} denotes the number of maximal faces of dimension $d-2$.

Proof. Set $k[\Delta] = S/I_\Delta$, and decompose $I_\Delta = I_1 \cap I_2$, where I_1 is the intersection of the primary components of dimension d and I_2 of the remaining components. The exact sequence

$$0 \to I_1/I_\Delta \longrightarrow S/I_\Delta \longrightarrow S/I_1 \to 0,$$

gives, according to Proposition 3.2,

$$\mathrm{tn}(k[\Delta]) = e_1(k[\Delta]) + \hat{e}_0(I_1/I_\Delta).$$

From the Hilbert function of $k[\Delta]$ ([1, Lemma 5.1.8]), we have that $e_1 = df_{d-1} - f_{d-2}$, while if I_1/I_Δ is a module of dimension $d-1$, its multiplicity is the number of maximal faces of dimension $d-2$. □

Theorem 3.6. *Let $S = k[x_1, \ldots, x_n]$ be a ring of polynomials, and $A = S/I$ a graded algebra. For a monomial ordering $>$, denote by $I' = in_>(I)$ the initial ideal associated to I and set $B = S/I'$. Then* $\operatorname{tn}(B) \geq \operatorname{tn}(A)$.

Proof. Let J be the component of I of maximal dimension and consider the exact sequence

$$0 \to J/I \longrightarrow S/I \longrightarrow S/J \to 0.$$

$\dim J/I < \dim A$ and therefore $\operatorname{tn}(A) = \operatorname{tn}(S/J) = e_1(S/J)$. Denote by J' the corresponding initial ideal of J, and consider the sequence

$$0 \to J'/I' \longrightarrow S/I' \longrightarrow S/J' \to 0.$$

Noting that S/I and S/J have the same multiplicity, and so do S/I' and S/J' by Macaulay's theorem, $\dim J'/I' < \dim A$. This means that

$$\operatorname{tn}(S/I') = \operatorname{tn}(S/J') = e_1(S/J') + \hat{e}_0(J'/I') = e_1(S/J) + \hat{e}_0(J'/I') = \operatorname{tn}(A) + \hat{e}_0(J'/I').$$

Example 3.7. Let $A = k[x, y, z, w]/(x^3 - yzw, x^2y - zw^2)$. The Hilbert series of this (Cohen–Macaulay) algebra is

$$H_A(t) = \frac{h_A(t)}{(1-t)^2} = \frac{(1+t+t^2)^2}{(1-t)^2},$$

so that

$$\operatorname{tn}(A) = e_1(A) = h'_A(1) = 18.$$

Consider now the algebra $B = k[x, y, z, w]/J$, where J is the initial ideal of I for the Deglex order. A calculation with *Macaulay2* gives

$$J = (x^2y, x^3, xzw^2, xy^3zw, y^5zw).$$

By Macaulay's Theorem, B has the same Hilbert function as A. An examination of the components of B gives the exact sequence

$$0 \to B_0 \longrightarrow B \longrightarrow B' \to 0,$$

where B_0 is the ideal of elements with support in codimension 1. By Proposition 3.2,

$$\operatorname{tn}(B) = \operatorname{tn}(B') = e_1(B').$$

At same time, one has the equality of h–polynomials,

$$h_B(t) = h_{B'}(t) + (1-t)h_{B_0}(t),$$

and therefore

$$e_1(B') = e_1(B) + e_0(B_0).$$

A final calculation of multiplicities gives $e_0(B_0) = 5$, and

$$\operatorname{tn}(B) = 18 + 5 = 23.$$

The example shows that $\operatorname{tn}(A)$ is independent of the Hilbert function of the algebra.

4. General bounds

We now describe how the technique of generic hyperplane sections leads to bounds of various kinds. We are going to assume that the algebras are defined over infinite fields.

One of the important properties of the tracking number is that it will not change under hyperplane sections as long as the dimension of the ring is at least 3. So one can answer questions about the tracking number just by studying the 2 dimensional case. The idea here is that tracking number is more or less the same material as e_1 and hence cutting by a superficial element will not change it unless the dimension is to drop below 2.

Proposition 4.1. *Let $R = k[x_1, \ldots, x_d]$ be a ring of polynomials over the infinite field k, $\deg(x_i) = 1$, $d > 2$, and let E be a finitely generated graded R–module of dimension d. Then for a general element $h \in R$ of degree one $R' = R/(h)$ is also a polynomial ring, and $\mathrm{tn}_R(E) = \mathrm{tn}_{R'}(E')$, where $E' = E/hE$.*

Proof. First we will prove the statement for a torsion free module E. Consider the exact sequence

$$0 \to E \longrightarrow E^{**} \longrightarrow C \to 0.$$

Note that C has codimension at least 2 since after localization at any height 1 prime E and E^{**} are equal. Now for a linear form h in R that is a superficial element for C, we can tensor the above exact sequence with $R/(h)$ to get the complex

$$\mathrm{Tor}_1(C, R/(h)) \longrightarrow E/hE \longrightarrow E^{**}/hE^{**} \longrightarrow C/hC \to 0.$$

Now as an R–module C/hC has codimension at least 3, so as an $R' = R/(h)$ module it has codimension at least 2. Also as $\mathrm{Tor}_1(C, R/(h))$ has codimension at least 2 as an R–module, so it is a torsion $R/(h)$–module. Hence we have $\mathrm{tn}_{R'}(E/hE) = \mathrm{tn}_{R'}(E^{**}/hE^{**})$. But E^{**} is a torsion free $R/(h)$–module, so

$$\mathrm{tn}_{R'}(E/hE) = e_1(E^{**}/hE^{**}) = e_1(E^{**}) = \mathrm{tn}_R(E^{**}) = \mathrm{tn}_R(E).$$

To prove the statement for a general R module E, we consider the short exact sequence

$$0 \to E_0 \longrightarrow E \longrightarrow E' \to 0,$$

where E_0 is the torsion submodule of E. E' is torsion free, so by the first case we know that for a general linear element h of R, $\mathrm{tn}_{R/(h)}(E'/hE') = \mathrm{tn}_R(E')$. Now if in addition we restrict ourselves to those h that are superficial for E and E_0, we can tensor the above exact sequence with $R/(h)$ and get

$$0 = \mathrm{Tor}_1(E', R/(h)) \longrightarrow E_0/hE_0 \longrightarrow E/hE \longrightarrow E'/hE' \to 0,$$

but since E_0/hE_0 is a torsion $R' = R/(h)$–module, $\mathrm{tn}_{R'}(E/hE) = \mathrm{tn}_{R'}(E'/hE') = \mathrm{tn}_R(E') = \mathrm{tn}_R(E)$. ∎

We shall now derive the first of our general bounds for $\mathrm{tn}(E)$ in terms of the Castelnuovo–Mumford regularity $\mathrm{reg}(E)$ of the module. For terminology and basic properties of the $\mathrm{reg}(\cdot)$ function, we shall use [2, Section 20.5].

Theorem 4.2. *Let $R = k[x_1, \ldots, x_d]$ and E a generated graded R–module of dimension d. Then*

$$\mathrm{tn}(E) \leq \deg(E) \cdot \mathrm{reg}(E).$$

Proof. The assertion is clear if $d = 0$. For $d \geq 1$, if E_0 denotes the submodule of E of the elements with finite support, $\deg(E) = \deg(E/E_0)$, $\mathrm{tn}(E) = \mathrm{tn}(E/E_0)$ and $\mathrm{reg}(E/E_0) \leq \mathrm{reg}(E)$, the latter according to [2, Corollary 20.19(d)]. From this reduction, the assertion is also clear if $d = 1$.

If $d \geq 3$, we use a general hyperplane section h so that $\mathrm{tn}_R(E) = \mathrm{tn}_{R/(h)}(E/hE)$, according to Proposition 4.1, and $\mathrm{reg}(E/hE) \leq \mathrm{reg}(E)$, according to [2, Proposition 20.20]. (Of course, $\deg(E) = \deg(E/hE)$.)

With these reductions, we may assume that $d = 2$ and consider the natural exact sequence

$$E \longrightarrow E^{**} \longrightarrow C \to 0,$$

where C is a torsion module. One has $\mathrm{tn}(E) = \mathrm{tn}(E^{**})$. Since E^{**} is R–free, $\mathrm{tn}(E^{**}) \leq \deg(E^{**})\mathrm{reg}(E^{**})$, where $\deg(E^{**}) = \deg(E)$. We claim that $\mathrm{reg}(E^{**}) \leq \mathrm{reg}(E)$. If E is generated by elements of degree $< \mathrm{reg}(E^{**})$, its image in E^{**} would be a module of rank $< \deg(E^{**})$, forcing $\deg(C) > 0$. □¡

5. Positivity results

We shall now prove our main result, the somewhat surprising fact that for a reduced homogeneous algebra A, $\mathrm{tn}(A) \geq 0$. Since such algebras already admit a general upper bound for $\mathrm{tn}(A)$ in terms of its multiplicity, together these statements are useful in the construction of integral closures by all algorithms that use intermediate extensions that satisfy the condition S_2 of Serre.

Theorem 5.1. *Let A be a reduced, non-negatively graded algebra that is finite over a standard graded Noether normalization R. Then $\mathrm{tn}(A) \geq 0$.*

Proof. Let $A = S/I$, $S = k[x_1, \ldots, x_n]$, be a graded presentation of A. From our earlier discussion, we may assume that I is height unmixed (as otherwise the lower dimensional components gives rise to the torsion part of A, which is dropped in the calculation of $\mathrm{tn}(E)$ anyway).

Let $I = P_1 \cap \cdots \cap P_r$ be the primary decomposition of I, and define the natural exact sequence

$$0 \to S/I \longrightarrow S/P_1 \times \cdots \times S/P_r \longrightarrow C \to 0,$$

from which a calculation with Hilbert coefficients gives

$$\mathrm{tn}(A) = \sum_{i=1}^{r} \mathrm{tn}(S/P_i) + \hat{e}_0(C).$$

This shows that it suffices to assume that A is a domain.

Let $\overline{A}$ denote the integral closure of A. Note that $\overline{A}$ is a non-negatively graded algebra and that the same Noether normalization R can be used. Since $\operatorname{tn}(A) \geq \operatorname{tn}(\overline{A})$, we may assume that A is integrally closed.

Since the cases $\dim A \leq 1$ are trivial, we may assume $\dim A = d \geq 2$. The case $d = 2$ is also clear since A is then Cohen–Macaulay. Assume then $d > 2$. We are going to change the base field using rational extensions of the form $k(t)$, which do not affect the integral closure condition. (Of course we may assume that the base field is infinite.)

If h_1 and h_2 are linearly independent hyperplane sections in R, they define a regular sequence in A, since the algebra being normal satisfies the S_2 condition of Serre. Effecting a change of ring of the type $k \to k(t)$ gives a hyperplane section $h_1 - t \cdot h_2 \in R(t)$, which is a prime element in A, according to Nagata's trick ([3, Lemma 14.1]). Clearly we can choose h_1 and h_2 so that $h_1 - t \cdot h_2$ is a generic hyperplane section for the purpose of applying Proposition 4.1 to A. This completes the reduction to domains in dimension $d - 1$. $\square$

One application is to the study of constructions of the integral closure of an affine domain (see [5] where details are given and the characteristic zero case is exploited via resultants).

Theorem 5.2. *Let A be a standard graded domain over a field k and let $\overline{A}$ be its integral closure. Then any chain of distinct subalgebras satisfying the condition S_2 of Serre,*

$$A \subset A_1 \subset \cdots \subset A_n = \overline{A},$$

has length at most $\binom{e}{2}$, where $e = \deg(A)$.

Proof. It will suffice, according to Corollary 2.4, to show that $0 \leq \operatorname{tn}(A) \leq \binom{e}{2}$. The positivity having been established in Theorem 5.1, we now prove the upper bound.

If k is a field of characteristic zero, by the theorem of the primitive element, A contains a hypersurface ring $S = R[t]/(f(t))$, where R is a ring of polynomials $R = k[z_1, \ldots, z_d]$, $\deg(z_i) = 1$, and $f(t)$ is a homogeneous polynomial in t of degree e. As $\operatorname{tn}(A) \leq \operatorname{tn}(S) = \binom{e}{2}$, the assertion would hold in this case.

To complete the proof in other characteristics we resort to the following construction:

Proposition 5.3. *Let A be an integral domain finite over the subring R, $A = R[y_1, \ldots, y_n]$, and let $\operatorname{rank}_R(A) = e$. Let K be the field of fractions of R and define the field extensions*

$$F_0 = K, \quad F_i = K[y_1, \ldots, y_i], \quad i = 1 \ldots n.$$

Then the module

$$E = \sum R y_1^{j_1} \cdots y_n^{j_n}, \quad 0 \leq j_i < r_i = [F_i : F_{i-1}]$$

is R–free of rank e. If A is a standard graded algebra, with $\deg(y_i) = 1$, $\operatorname{tn}(E) \leq \binom{e}{2}$.

Proof. As the rank satisfies the equality $e = \prod_{i=1}^{n} r_i$, e is the number of 'monomials' $y_1^{j_1} \cdots y_n^{j_n}$. Their linear independence over R is a simple verification. Note also that there are monomials of all degrees between 0 and $\sum_{i=1}^{n}(r_i - 1)$. Thus according to Proposition 3.1, the bound for $\operatorname{tn}(E)$ is obvious, with equality holding only when E is a hypersurface ring over R. (The precise value for $\operatorname{tn}(E)$ could be derived from Corollary 3.4.) $\square$

References

[1] W. Bruns and J. Herzog, *Cohen–Macaulay Rings*, Cambridge University Press, 1993.

[2] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer-Verlag, New York, 1995.

[3] R. Fossum, *The Divisor Class Group of a Krull Domain*, Springer-Verlag, New York, 1973.

[4] D. Grayson and M. Stillman, *Macaulay2*, 1996. Available via anonymous ftp from `math.uiuc.edu`.

[5] W. V. Vasconcelos, Divisorial extensions and the computation of integral closures, *J. Symbolic Computation* **30** (2000), 595–604.

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