# Integrally Closed Modules and their Divisors

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**Abstract**

There is a beautiful theory of integral closure of ideals in regular local rings of dimension two, due to Zariski, several aspects of which were later extended to modules. Our goal is to study integral closures of modules over normal domains by attaching divisors/determinantal ideals to them. They will be of two kinds: the ordinary Fitting ideal and its divisor, and another 'determinantal' ideal obtained through Noether normalization. They are useful to describe the integral closure of some class of modules and to study the completeness of the modules of Kähler differentials.

## 1 Introduction

The theory of normality of ideals has been extended to modules. The development of tools to detect completeness of modules serves a useful purpose. In this paper, we study determinantal ideals associated to modules and we investigate how they play a role in establishing conditions for modules and ideals to be integrally closed. Throughout this paper, let $R$ be a commutative Noetherian ring with total ring of fractions $K$. An $R$–module $E$ is said to be of rank $r$ if $K \otimes_R E \simeq K^r$. Let $E$ be a finitely generated torsionfree

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$R$–module of rank $r$. Then there is an (non-canonical) embedding

$$E \hookrightarrow R^r,$$

which allows us to define the Rees algebra $\mathcal{R}(E)$ of $E$ as the subalgebra of the polynomial ring $R[T_1, \ldots, T_r]$ generated by all linear forms

$$a_1T_1 + \cdots + a_rT_r \in E$$

(we are taking $T_1, \ldots, T_r$ as a basis of $R^r$). While the Rees algebra $\mathcal{R}(E)$ is defined using a given embedding of $E$ into a free $R$–module, the Rees algebra $\mathcal{R}(E)$ is isomorphic to the symmetric algebra $S(E)$ of $E$ modulo its $R$–torsion $T(E)$. Therefore the Rees algebra $\mathcal{R}(E)$ is independent of the embedding. The degree 1 component of $\mathcal{R}(E)$ is $E$ and we denote the other components by $E^n$. The integral closure of $\mathcal{R}(E)$ in $R[T_1, \ldots, T_r]$ is a graded algebra, which we denote by

$$\overline{\mathcal{R}(E)} = \sum_{n \geq 0} \overline{E^n}.$$

An $R$–module $E$ is said to be *integrally closed or complete* if $E$ is equal to the integral closure $\overline{E}$. An $R$–module $E$ is said to be *normal* if all $E^n$ are integrally closed. There arise several issues of interest: What are the module theoretic properties of integrally closed modules? How to decide whether a given module is integrally closed? What are interesting classes of integrally closed modules? If a torsionfree module $E$ is not integrally closed, what are possible routes to the integral closure of $E$ which do not involve the computation of the integral closure of the Rees algebra $\mathcal{R}(E)$? What are potential applications? For two-dimensional regular local rings some of these aims were fully realized in [7] (see also [6]).

One of our interests is to use the theory of integral closure of modules to study the integral closure of ideals in affine rings. Specifically, if $A$ is a normal affine ring and $R = k[z_1, \ldots, z_d]$ is one of Noether normalizations of $A$, we would like to examine when the $R$–module structure of $A$ helps in studying the integral closure of an $A$–ideal $I$.

Our approach to general problems about the integral closures of modules and ideals is through the introduction of various determinantal ideals and their associated divisors. In the case of a module $E$ embedded into a free module $R^r$, we relate the integral closure of $E$ to the properties of the Fitting ideal $\det_0(E)$ (see the definition of $\det_0(E)$ in Section 2). For an ideal $I$ of an affine ring $A$, we use a Noether normalization $R$ of $A$, and the norm mapping $\mathrm{N} : A \to R$ and we attach an $R$–ideal $\mathrm{N}(I)$ to $I$ from which we infer properties of the integral closure of $I$. We observe that the norm mapping has a determinantal character.

Let us describe the contents of this paper. In Section 2, we examine relationships between the integral closure of a module $E$ and the associated prime ideals of either $\det_0(E)$ or the module $R^r/E$. When these sets of associated prime ideals coincide, the analysis is simpler (but not complete). One of its highlights is the question on (in a normal domain) whether if $\det_0(E)$ is a prime ideal then $E$ is integrally closed. In Section 3, for a given Noether normalization $R \subset A$, we consider a norm mapping which attaches an $R$–ideal $\mathrm{N}(I)$ to an $A$–ideal $I$. It has the property that $\overline{I} \subset \overline{J}$ if and only if $\overline{\mathrm{N}(I)} \subset \overline{\mathrm{N}(J)}$ which converts some questions from $A$ to $R$. In Section 4, we consider the role of the divisor of $\det_0(E)$ in the completeness of $E$. The applications are to instances of the conormal module or of the module of Kähler differentials.

## 2 Integral closure and associated prime ideals

Throughout this section, unless explicitly asserted, we assume that $R$ is a Noetherian normal domain. We consider the determinantal ideal associated to an embedding $E \hookrightarrow R^r$ and the role it plays in the analysis of the completeness of $E$. Some of the definitions below apply to more general rings and modules, but we will leave these adjustments to the reader.

**Definition 2.1** Let $E$ be a finitely generated torsionfree $R$–module of rank $r$. The *order determinant* of the embedding $E \stackrel{\varphi}{\longrightarrow} R^r$ is the ideal defined by the image of the mapping $\wedge^r \varphi$,

$$\text{image}(\wedge^r \varphi) = I \cdot \wedge^r (R^r).$$

When the embedding is clear, we denote the order determinant of $E$ by $\det_0(E)$.

Since the order determinant depends on embeddings, a more appropriate notation would have been $\det_\varphi(E)$. In any event, for any embedding one has

$$\det_0(E) \simeq \wedge^r E/(\text{torsion}).$$

Let $E$ be a submodule of a free module $R^r$. We deal with the associated prime ideals of the order determinant $\det_0(E)$ and those of the modules in the diagram

$$E \hookrightarrow R^r \to R^r/E \hookleftarrow \overline{E}/E.$$

For instance, the associated prime ideals of $R^r/E$ can be used in the following general observation.

**Proposition 2.2** *Let $R$ be a Noetherian normal domain with field of fractions $K$. Then a finitely generated torsionfree $R$–module $E$ of rank $r$ is integrally closed if and only if $E_{\mathfrak{p}}$ is integrally closed for each associated prime ideal $\mathfrak{p}$ of $R^r/E$.*

We first recall how the order determinant $\det_0(E)$ mirrors the integral closure $\overline{E}$ of $E$ (see [8], [10, Chapter 8]).

**Proposition 2.3** *Let $R$ be a Noetherian normal domain with field of fractions $K$. Let $E \subset F$ be finitely generated torsionfree $R$–modules embedded into $R^r$. Then $F \subset \overline{E}$ if and only if*

$$\det_0(F) \subset \overline{\det_0(E)}.$$

**Proof.** Let $V$ be an arbitrary discrete valuation ring containing $R$ in $K$. We denote the images of $E \otimes_R V$ and $F \otimes_R V$ in $R^r \otimes_R V$ by $VE$ and $VF$ respectively. Then $F \subset \overline{E}$ if and only if $VF = VE$ for any discrete valuation ring $V$ containing $R$ in $K$. Note that $\det_0(VE) = \det_0(E)V$. Suppose that $F \subset \overline{E}$. Then $\det_0(E)V = \det_0(F)V$ and hence $\det_0(F) \subset \overline{\det_0(E)}$. For the converse, suppose that for any discrete valuation ring $V$ containing $R$ in $K$, we have $\det_0(E)V = \det_0(F)V$. From the following embeddings of free modules

$$VE \xrightarrow{\alpha} VF \xrightarrow{\beta} V^r,$$

we obtain that $\det_0(VF)$ and $\det_0(VE)$ are the ideals generated by $\det(\beta)$ and $\det(\beta \circ \alpha)$. In other words,

$$\begin{array}{ccl} \det_0(F)V & = & \det_0(VF) = (\det(\beta)) \\ \| & & \\ \det_0(E)V & = & \det_0(VE) = (\det(\beta) \cdot \det(\alpha)) \end{array}$$

It follows that $\det(\alpha)$ is a unit in $V$ and hence $VF = VE$. □

This fact can be used to describe elements in the integral closure $\overline{E}$ of $E$.

**Proposition 2.4** *Let $R$ be a Noetherian normal domain and $E$ a finitely generated torsionfree $R$–module of rank $r$. Let ${}_UE$ be the submodule of $R^r$ defined as*

$${}_UE = \{v \in R^r \mid u \cdot v \in E, \quad u \text{ regular mod } \det_0(E)\}.$$

*Then ${}_UE$ is contained in the integral closure $\overline{E}$ of $E$.*

**Proof.** Let $v_1, \ldots, v_r$ be elements of ${}_UE$ with corresponding conductors $u_1, \ldots, u_r$, as in the definition of ${}_UE$. Observe that by setting $u = u_1 \cdot \cdots \cdot u_r$, we have

$$u \cdot \det_0({}_UE) \subset \det_0(E).$$

Since $u$ is regular mod $\det_0(E)$, it follows that $\det_0({}_UE) = \det_0(E)$. By Proposition 2.3, we conclude that ${}_UE$ is contained in $\overline{E}$. □

Let us denote the bidual $\mathrm{Hom}_R(\mathrm{Hom}_R(E, R), R)$ of an $R$–module $E$ by $E^{**}$.

**Corollary 2.5** *Let $R$ be a Noetherian normal domain and $E$ a finitely generated torsionfree $R$–module having a rank. If the order determinant $\det_0(E)$ is divisorial, then the integral closure $\overline{E}$ of $E$ is equal to the bidual $E^{**}$.*

**Proof.** Since $R$ is a Krull domain, the bidual $E^{**}$ is equal to $\bigcap E_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over all prime ideals of $R$ of height 1. In particular $E^{**}$ is integrally closed and hence $\overline{E} \subset E^{**}$. Note that height of the annihilator $\mathrm{ann}(E^{**}/E)$ is at least 2. Since $\det_0(E)$ is divisorial (i.e. all of its primary components are of height 1), it forces $E^{**} \subset {}_UE$. It follows from Proposition 2.4 that $E^{**}$ is equal to $\overline{E}$. □

This suggests a more general description of the integral closure $\overline{E}$.

**Corollary 2.6** *Let $R$ be a Noetherian normal domain and $E$ a finitely generated torsion-free $R$–module having a rank. Suppose that for any associated prime ideal $\mathfrak{p}$ of the order determinant* $\det_0(E)$*, the module $E_\mathfrak{p}$ is integrally closed. Then ${}_UE$ is equal to the integral closure $\overline{E}$ of $E$.*

**Proof.** Since $\overline{E}_\mathfrak{p} = E_\mathfrak{p}$ for any associated prime ideal $\mathfrak{p}$ of $\det_0(E)$, none of the associated prime ideals of $\overline{E}/E$ is contained in any of the associated prime ideals of $\det_0(E)$. Therefore $\overline{E} \subset {}_UE$. Since the converse holds by Proposition 2.4, the assertion is proved. □

There are some unresolved questions on what the structure of the order determinant $\det_0(E)$ might imply on the integral closure $\overline{E}$. One of these is the following.

**Conjecture 2.7** Let $R$ be a Cohen–Macaulay Noetherian normal domain and let $E$ be a torsionfree $R$–module having a rank. If the order determinant $\det_0(E)$ is a prime ideal, then $E$ is integrally closed.

If $E$ has finite projective dimension less than height of $\det_0(E)$, then the assertion is not difficult to establish.

**Proposition 2.8** *Let $R$ be a Cohen-Macaulay Noetherian normal domain and let $E$ be a finitely generated torsionfree $R$–module of rank $r$. Suppose that the order determinant* $\det_0(E)$ *is a prime ideal of height $s$. If $E$ has projective dimension less than $s$, then $E$ is integrally closed. In particular,* $\det_0(E)$ *is generically a complete intersection.*

**Proof.** Let us denote $\det_0(E)$ by $\mathfrak{p}$ and the minimal number of generators of $E$ by $n$. Let $\varphi : R^n \to R^r$ be the mapping such that image$(\varphi)$ is isomorphic to $E$. Then $\mathfrak{p}$ is the ideal $I_r(\varphi)$ generated by $r \times r$ minors of $\varphi$. Localizing at $\mathfrak{p}$, by Nakayama Lemma, we have

$$E_\mathfrak{p} \simeq R_\mathfrak{p}^{r-1} \oplus \mathfrak{p}R_\mathfrak{p}.$$

In particular $E_\mathfrak{p}$ is integrally closed. Now consider a prime ideal $\mathfrak{m}$ associated to $R^r/E$. It is enough to show that $\mathfrak{m} = \mathfrak{p}$, according to Proposition 2.2. Suppose that $\mathfrak{p} = I_r(\varphi)$ is not contained in $\mathfrak{m}$. Then $E_\mathfrak{m} = \text{image}(\varphi)_\mathfrak{m}$ contains $(R^r)_\mathfrak{m}$ and hence $(R^r/E)_\mathfrak{m} = 0$. Therefore $\mathfrak{p}$ is contained in $\mathfrak{m}$. Now suppose that $\mathfrak{m}$ contains properly $\mathfrak{p}$. Then depth $R_\mathfrak{m} \geq s+1$ and by Auslander-Buchsbaum equality, $(R^r/E)_\mathfrak{m}$ has depth at least 1, which is a contradiction. Therefore $E$ is integrally closed by Proposition 2.2. Furthermore the order determinant $\det_0(E)$ is generically a complete intersection by Theorem 4.8. □

## 3 Norms of ideals

Another determinantal ideal arises when we consider the integral closure of ideals in an affine domain over a field. An affine domain $A$ over a field $k$ and its Noether normalization $R \hookrightarrow A$ already provide a setting to which we can apply the integral closure techniques of modules.

We briefly recall the notion of the determinant of an endomorphism of a module. Let $R$ be a domain with field of fractions $K$ and let $\varphi$ be an endomorphism of a finitely generated torsionfree $R$–module $E$. Let $\varphi'$ be the endomorphism of $K \otimes_R E$ extended from $\varphi$, that is,

$$\varphi' = K \otimes \varphi : \; K \otimes_R E \longrightarrow K \otimes_R E.$$

We denote the determinant of $\varphi'$ by $\det \varphi$.

**Proposition 3.1** *Let $R$ be a domain with field of fractions $K$ and let $\varphi$ be an endomorphism of a finitely generated torsionfree $R$–module $E$. If $R$ is integrally closed, then $\det \varphi \in R$.*

**Proof.** Let $\varphi'$ be the endomorphism of $K \otimes_R E$ extended from $\varphi$. Since $R$ is an integrally closed domain, it suffices to show that for each prime ideal $\mathfrak{p}$ of height 1, the determinant $\det \varphi' \in R_{\mathfrak{p}}$. Since $R_{\mathfrak{p}}$ is a discrete valuation ring, the torsionfree $R_{\mathfrak{p}}$–module $E_{\mathfrak{p}}$ is free. Moreover the restriction of $\varphi'$ to $E_{\mathfrak{p}}$, denoted by $\varphi'|_{E_{\mathfrak{p}}}$, defines an endomorphism. Thus the determinant $\det(\varphi'|_{E_{\mathfrak{p}}})$ can be computed using a basis of $E_{\mathfrak{p}}$, since it is also a basis (over $K$) for $E \otimes K$. □

If $R$ is integrally closed, we introduce the notion of a norm function from $E$ to $R$ as follows.

**Definition 3.2** Let $R$ be an integrally closed domain with field of fractions $K$ and $E$ a finitely generated torsionfree $R$–module. Let $b \in E$ and let $f_b$ be the $R$–endomorphism of $E$ defined by

$$f_b : E \mapsto E, \quad a \mapsto ba.$$

The $R$–element $\det f_b$ is defined to be *norm* of $b$, denoted by $\mathrm{N}_R(b)$.

If $E$ has further a $R$-algebra structure, then we define norm of an $E$–ideal as follows.

**Definition 3.3** Let $R$ be an integrally closed domain with field of fractions $K$ and let $A$ be a finite $R$–algebra integral over $R$. We define *norm* $\mathrm{N}_R(I)$ of an $A$–ideal $I$ as the $R$–ideal generated by all $\mathrm{N}_R(b)$, $b \in I$.

If there is no ambiguity, we denote the norm function $\mathrm{N}_R(\cdot)$ of an $R$–module over $R$ simply by $N(\cdot)$.

**Remark 3.4** The norm function can also be defined if $A$ is integral over $R$ and has finite rank over $R$. To define $N(b)$ for $b \in A$, let $G$ be a finitely generated $R$–submodule of $A$ such that $A \otimes_R K \simeq G \otimes_R K (\simeq K^r)$. Let $E = R[G, b]$ and $f_b$ be the endomorphism of $E$ defined by multiplication by $b$. For $b \in A$, $N(b)$ can then be defined to be $\det f_b$ which is independent of $E$.

Note that the norm function $\mathrm{N}(\cdot)$ is a non-additive mapping and some properties of the norm function can be easily proved.

**Remark 3.5** Let $R$ be an integrally closed domain and $A$ be a finite $R$–algebra. For ideals $I$ and $J$ of $A$ and elements $a, b$ of $A$, the followings hold true:

(i) $\mathrm{N}(a)\mathrm{N}(b) = \mathrm{N}(ab)$.

(ii) $\mathrm{N}(Aa) = R\mathrm{N}(a)$.

(iii) An $A$–element $b$ is a unit if and only if $\mathrm{N}(b)$ is a unit in $R$.

(iv) $\mathrm{N}(I)N(J) \subseteq \mathrm{N}(IJ)$.

(v) $I \cap R \supset N(I)$, $I \supset N(I)A$.

In the rest of this section, let $A$ be an affine domain over a field $k$ and let $R \hookrightarrow A$ be a Noether normalization. If $A$ is not a ring of polynomials–when computation is more amenable–we can use a Noether normalization $R \hookrightarrow A$ in order to study issues of integral closures of $A$–ideals while using their $R$–module structures.

We can think of $A$ as embedded in a free $R$–module $R^r$, where $r$ is the rank of $A$ over $R$ (which is the degree of the field of fractions of $A$ over the field of fractions of $R$). Note that a nonzero $A$–ideal $I$ is also a torsionfree $R$–module of rank $r$. Now we associate two integral closures with an $A$–ideal $I$. Let $F$ and $K$ be fields of fractions of $A$ and $R$ respectively. For a valuation ring $V$ containing $R$ in $K$, we denote the image of $I \otimes V$ in $V^r$ by $IV$.

$$\begin{aligned} \overline{I}^A &= \bigcap_U IU \cap A, \quad U \text{ valuation ring containing } A \text{ in } F, \\ \overline{I}^R &= \bigcap_V IV \cap R^r, \quad V \text{ valuation ring containing } R \text{ in } K, \end{aligned}$$

that is, $\overline{I}^A$ denotes the integral closure of $I$ as an $A$–ideal and $\overline{I}^R$ denotes that of $I$ as an $R$–module. The following collects some elementary observations. For simplicity we assume that $A$ is integrally closed.

**Remark 3.6** *Let $A$ be an integrally closed affine domain over a field $k$ and let $R \hookrightarrow A$ be a Noether normalization. Let $I$ and $J$ be $A$–ideals.*

(i) $\overline{I}^R = \bigcap_V IV \cap R^r$, *where $V$ runs over the set of Rees valuations of the order determinant* $\det_0(I)$ *of $I$ as an $R$–module.*

(ii) $\overline{\det_0(I)} = \overline{\det_0(J)}$ *if and only if* $\overline{I}^R = \overline{J}^R$ *as $R$–modules.*

**Proof.** For the proof we refer to [10, Chapter 8]. □

We emphasize the fact that the integral closure of $I$ as an $A$–ideal is not always equal to the integral closure of $I$ as an $R$–module. It follows from the fact that for each valuation ring

$U$ of $A$, its restriction to field of fractions of $R$ yields a valuation ring $V$ of $R$. Furthermore for any such $V$ there are only finitely many $U$'s, say $U_1, \ldots, U_s$, and

$$IV = \bigcap_{i=1}^{s} I(U_i \cap K).$$

**Proposition 3.7** *Let $A$ be an integrally closed affine domain over a field $k$ and let $R \hookrightarrow A$ be a Noether normalization. Let $I$ and $J$ be $A$–ideals. If $\overline{\det_0(I)} = \overline{\det_0(J)}$, then $\overline{I}^A = \overline{J}^A$ as $A$–ideals.*

**Proof.** Let $F$ and $K$ be fields of fractions of $A$ and $R$ respectively. For any discrete valuation ring $U$ containing $A$ in $F$, let $V = U \cap K$. By Remark 3.6–(ii), we have $IV = JV$. It follows that $IU = IVU = JVU = JU$ and hence $\overline{I}^A = \overline{J}^A$ as $A$-ideals. □

The converse of Proposition 3.7 may not be true as in the following example.

**Example 3.8** Let $R = k[x^2]$ and $A = k[x^2, x^3] = R \oplus Rx^3$. Let $I = (x^3, x^4) \supset J = (x^3)$ be $A$–ideals. Then, $\overline{I}^A = \overline{J}^A$ as $A$–ideals. As $R$-modules, $\det_0(J) = Rx^6$ and $\det_0(I) = Rx^4$. We claim that $x^4 \in \overline{\det_0(I)} \setminus \overline{\det_0(J)}$. Suppose $x^4$ is integral over $\det_0(J)$. Then, it satisfies a monic equation

$$(x^4)^n + f_1(x^4)^{n-1} + \cdots + f_{n-1}x^4 + f_n = 0,$$

for $n \geq 2$ and $f_i \in (x^6)^i$ for all $i$ in $R$. It then implies that

$$x^{4n}(1 + a_1x^2 + a_2x^4 + \cdots + a_nx^{2n}) = 0,$$

for $a_i \in R$ which lead to a contradiction since $x^{4n} \neq 0$ in $R$.

For $A$–ideals $J \subset I$, the integral closures of order determinants $\det_0(I)$ and $\det_0(J)$ detect whether $\overline{J}^R = \overline{I}^R$ as $R$–modules while they cannot capture the whole relation of $I$ over $J$ as $A$–ideals. There is a companion observation that uses norms but it requires modifications. Note its relative strength.

**Proposition 3.9** *Let $A$ be an integrally closed affine domain over a field $k$ and let $R \hookrightarrow A$ be a Noether normalization. Let $J \subset I$ be $A$–ideals. Then, $\overline{I}^A = \overline{J}^A$ as $A$–ideals if and only if $\overline{\mathrm{N}(I)} = \overline{\mathrm{N}(J)}$ as $R$–ideals.*

**Proof.** Let $F$ and $K$ be fields of fractions of $A$ and $R$ respectively such that $r = [F : K]$. Fix an embedding $\varphi : A \hookrightarrow R^r$ and denote the norm function of $A$ over $R$ by $\mathrm{N}(\cdot)$.

We first assume that $\overline{\mathrm{N}(I)} = \overline{\mathrm{N}(J)}$ in $R$. Let $U$ be an arbitrary discrete valuation ring containing $A$ in $F$ and let $V$ be $U \cap K$. Let $B$ be the image of $A \otimes V$ under $\varphi \otimes V$ and let $\overline{B}$ be the integral closure of $B$ in $F$. Since $V$ is a discrete valuation ring and $F$ is finite algebraic over $K$, by Krull-Akizuki, $\overline{B}$ is a Noetherian domain of dimension 1. Furthermore,

$\overline{B}$ is a Dedekind domain with only finitely many maximal ideals and hence $\overline{B}$ is a principal ideal domain. For $\overline{B}$–ideals $J\overline{B}$ and $I\overline{B}$, there exist $h \in J$ and $g \in I$ such that

$$\overline{B}h = J\overline{B} \subseteq I\overline{B} = \overline{B}g,$$

and $h = gb$ for some $b \in \overline{B}$.

Note that $\overline{B}$ is a free $V$–submodule of $V^r$ which is also integral over $V$ with finite rank since $F$ is finite algebraic over $K$. We can then apply the norm function $N_V(\cdot)$ from $\overline{B}$ to $V$. Since $\overline{\mathrm{N}(I)} = \overline{\mathrm{N}(J)}$ in $R$, we have $\mathrm{N}(I)V = \mathrm{N}(J)V$ and hence

$$\begin{array}{ccccc} \mathrm{N}(I)V & = & \mathrm{N}_V(I\overline{B}) & = & \mathrm{N}(g)V \\ \| & & & & \\ \mathrm{N}(J)V & = & \mathrm{N}_V(J\overline{B}) & = & \mathrm{N}(h)V. \end{array}$$

Let $v$ be the valuation associated with $V$. Then $v(\mathrm{N}(g)) = v(\mathrm{N}(h))$. On the other hand, since $h = gb$,

$$\mathrm{N}(h) = \mathrm{N}(gb) = \mathrm{N}(g)\mathrm{N}(b).$$

It follows that

$$v(\mathrm{N}(g)) = v(\mathrm{N}(g)) + v(\mathrm{N}(b)),$$

and hence $v(\mathrm{N}(b)) = 0$. Now $\mathrm{N}(b)$ is a unit in $V$ and it means that $b$ is a unit in $\overline{B}$ by Remark 3.5–(iii). Therefore $J\overline{B} = I\overline{B}$ and furthermore, $JU = IU$, which proves that $\overline{I} = \overline{J}$ as $A$–ideals.

Conversely let us assume that $\overline{I} = \overline{J}$ as $A$–ideals. Let $V$ be an arbitrary discrete valuation ring containing $R$ in $K$. Let $B$ be the image of $A \otimes V$ under $\varphi \otimes V$ and let $\overline{B}$ be the integral closure of $B$ in $F$. Then $\overline{B}$ is a Dedekind domain with only finitely many maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$, and we have a norm function $\mathrm{N}_V(\cdot)$ of $\overline{B}$ over $V$. Now let $U_i$ be $\overline{B}_{\mathfrak{p}_i}$ for each $i$. Then each $U_i$ is a discrete valuation ring and $U_i \cap K = V$. Since $IU_i = JU_i$ for each $i$ and $\overline{B} = \bigcap_i U_i$, it follows that $I\overline{B} = J\overline{B}$. Therefore $\mathrm{N}(I)V = \mathrm{N}_V(I\overline{B}) = \mathrm{N}_V(J\overline{B}) = \mathrm{N}(J)V$, and hence $\overline{\mathrm{N}(I)} = \overline{\mathrm{N}(J)}$. □

Using Proposition 3.9, we see that the norm function $\mathrm{N}(\cdot)$ tells the reduction relation of two ideals of $A$ as well.

**Corollary 3.10** *Let $A$ be an integrally closed affine domain over a field $k$ and let $R \hookrightarrow A$ be a Noether normalization. Let $J \subset I$ be $A$–ideals. Then, $J$ is a reduction of $I$ if and only if* $\mathrm{N}(J)$ *is a reduction of* $\mathrm{N}(I)$.

Given an $A$–ideal $I$, however there is a real puzzle here on how to determine generators of $\mathrm{N}(I)$. It may not be enough to take a set of generators of $I$, either as an $A$–ideal or even as an $R$–module. Interestingly enough, the situation for the integral closure $\overline{\mathrm{N}(I)}$ is more clear in several examples. We are going to show that there is a finite set of elements of generators of $I$ so that $\overline{\mathrm{N}(I)}$ is the integral closure of the ideal generated by their norms.

Let us assume that $R$ contains an infinite field $k$. Let $I$ be an $A$–ideal generated by $a_1, \ldots, a_n$. For a discrete valuation ring $V$ containing $R$, let $B = AV$ and $\overline{B}$ the integral closure of $B$ in the field of fractions of $A$. Then $\overline{B}$ is finitely generated over $V$ and it is a Dedekind domain with only finitely many maximal ideals, in particular $\overline{B}$ is a principal ideal domain. Now the ideal $I\overline{B}$ is principal and in any localization of $\overline{B}$ one of the $a_i$'s generates $I\overline{B}$. We may assume that there is $t \in k$ such that a linear combination $b = \sum_{i=1}^{n} t^i a_i$ is the generator of the principal ideal $I\overline{B}$. This argument shows that kind of elements are needed to define the integral closure $\overline{\mathrm{N}(I)}$ of the norm ideal. The difficulty is that we do not know how many such combinations to take.

Consider the Noether normalization $R[x] \hookrightarrow A[x]$, where $x$ is an indeterminate. We observe that the corresponding norm mapping from $A[x]$ to $R[x]$ is the extension of the norm mapping from $A$ to $R$. Let $I$ be an $A$–ideal generated by $a_1, \ldots, a_n$. For an $I[x]$–element $b$ of the form

$$b = f_1(x)a_1 + \cdots + f_n(x)a_n, \quad \text{where } f_i(x) \in R[x] \text{ for each } i,$$

and for $u \in k$, we have

$$N(b)(u) = N(b(u)),$$

where $\mathrm{N}(b)(u)$ is the evaluation of $\mathrm{N}(b) \in R[x]$ by setting $x = u$ and $b(u) = \sum_{i=1}^{n} f_i(u)a_i$.

**Proposition 3.11** *Let $A$ be an integrally closed affine domain over an infinite field $k$ and let $R \hookrightarrow A$ be a Noether normalization. Let $I$ be an $A$–ideal generated by $a_1, \ldots, a_n$ and set*

$$\alpha = \sum_{i=1}^{n} x^i a_i$$

*be the generic element of $I[x]$. Denote by $L$ the $R$–ideal generated by the coefficients of the polynomial $\mathrm{N}(\alpha) \in R[x]$. Then*

$$\overline{\mathrm{N}(I)} = \overline{L}.$$

**Proof.** We first show that $L$ is contained in $\mathrm{N}(I)$, which will imply that the integral closure $\overline{L}$ is contained in $\overline{\mathrm{N}(I)}$. Write

$$\mathrm{N}(\alpha) = \sum_{j=0}^{m} b_j x^j \in R[x], \quad \text{and} \quad L = (b_0, \ldots, b_m).$$

We note that for each evaluation $x \to u \in k$,

$$\sum_{j=0}^{m} b_j u^j = \mathrm{N}(\alpha)(u) = \mathrm{N}\left(\sum_{i=1}^{n} u^i a_i\right) \in \mathrm{N}(I).$$

It follows that the polynomial $\mathrm{N}(\alpha)$ has coefficients in $\mathrm{N}(I)$ for each such evaluation. Since $k$ is infinite, by Vandermonde it follows that each $b_j \in \mathrm{N}(I)$, and therefore $L \subset \mathrm{N}(I)$.

For the converse, it is enough to show that $LV = \mathrm{N}(I)V$ for any discrete valuation ring $V$ containing $R$. Let $B$ be $AV$. For some $u \in k$, there exists $b = \sum_i u^i a_i$ such that $I\overline{B} = b\overline{B}$. Note that $b = \alpha(u)$. Then

$$\mathrm{N}(I)V = \mathrm{N}_V(I\overline{B}) = \mathrm{N}_V(b\overline{B}) = \mathrm{N}(b)V = \mathrm{N}(\alpha(u))V = \mathrm{N}(\alpha)(u)V \subseteq LV.$$

□

## 4 Divisors and integral closures

We will recall the usual notion of divisors of modules and use it to study the completeness of some classes of modules. For an $R$–module $E$, we denote $\mathrm{Hom}_R(E, R)$ by $E^*$. We will have several occasions to use the notions of determinant (or determinant class) of a module. We will limit our definition to modules having rank.

**Definition 4.1** Let $R$ be a Noetherian normal domain and let $E$ be a finitely generated torsionfree $R$–module having rank $r$. The *determinant* or *determinant divisor* of $E$ is the fractionary ideal

$$\det(E) = \det_0(E)^{-1-1}.$$

We denote the isomorphism class of $\det(E)$ in the divisor class group of $R$ by $[\det(E)]$ or by $\mathrm{div}(E)$.

Note that if $R$ is factorial, then $\det(E) \simeq R$ for every module $E$ of positive rank, while if $E$ is a projective module of positive rank over a ring $R$, then $\det(E)$ is an invertible ideal. We observe the following two rules of computation ([10, Chapter 8]).

**Lemma 4.2** *Let $R$ be a Noetherian normal domain.*

(i) *If*

$$0 \to E \longrightarrow F \longrightarrow G \to 0$$

*is a complex of finitely generated modules such that in each localization $R_{\mathfrak{p}}$ at a codimension one prime is an exact sequence of free modules, then*

$$\det(F) = \det(E) \circ \det(G) = (\det(E)\det(G))^{-1-1},$$

*where $\circ$ is divisorial composition.*

(ii) *If $\varphi : E \longrightarrow F$ is a homomorphism of finitely generated $R$–modules whose kernel is a torsion module and whose cokernel is a torsion module of codimension at least two, then*

$$\mathrm{div}(E) = \mathrm{div}(F).$$

We can sum up these observations as follows ([10, Chapter 8]).

**Proposition 4.3** *Let $R$ be a Noetherian normal domain and let*

$$0 \to E_n \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0 \to 0$$

*be a complex of finitely generated $R$–modules. If the complex is exact at each localization $R_{\mathfrak{p}}$ of dimension at most 1, then the divisor classes of their determinants satisfy*

$$\sum_{i=0}^{n} (-1)^i \mathrm{div}(E_i) = 0.$$

For the proofs of Lemma 4.2 and Proposition 4.3 we refer to [10, Chapter 8]. Those who are comfortable in algebraic K-theory will recognize the divisor class group of $R$ as the $K_0$ group of the category of torsionfree modules with isomorphisms up to torsion, and Proposition 4.3 as its fundamental property. Of course, one key application of this result is to show the factoriality of regular local rings.

A well-known calculation of determinants is that in the case of affine domains(see [5]).

**Proposition 4.4** *Let $R = k[x_1, \ldots, x_n]/\mathfrak{p}$ be a normal affine domain over field $k$ of characteristic zero. Let $E$ be the conormal $R$–module $\mathfrak{p}/\mathfrak{p}^2$ and $\Omega_k(R)$ the module of Kähler $k$–differentials of $R$. Then*

$$\operatorname{div}(\Omega_k(R)) = -\operatorname{div}(E) = \operatorname{div}(\omega_R),$$

*where $\omega_R$ is the canonical module of $R$.*

**Proof.** This follows from Proposition 4.3 together with the Jacobian criterion, and the chain complexes

$$0 \to \mathfrak{p}/\mathfrak{p}^2 \longrightarrow R^n \longrightarrow \Omega_k(R) \to 0,$$

$$0 \to H_1 \longrightarrow R^m \longrightarrow \mathfrak{p}/\mathfrak{p}^2 \to 0,$$

where $H_1$ is the 1–dimensional Koszul homology module on a set of generators of $\mathfrak{p}$ with $m$ elements. The multiplication in the Koszul homology algebra readily implies that $\operatorname{div}(H_1)$ is equal to $\operatorname{div}(\omega_R)$. $\square$

Another application of the divisors is to the conormal module of an almost complete intersection ideal(see also [5]). At this point we need the notion of $\mathfrak{m}$–full modules. Let $(R, \mathfrak{m})$ be a Noetherian local ring. A torsionfree $R$–module $E$ of rank $r$ is called an *$\mathfrak{m}$–full module* if there is an element $x \in \mathfrak{m}$ such that $\mathfrak{m}E :_{R^r} x = E$. Integrally closed modules are $\mathfrak{m}$–full modules ([1, Proposition 2.6]). For a submodule $M$ of $R^r$ containing an $\mathfrak{m}$–full module $E$ with $\lambda(M/E) < \infty$, the minimal number of generators $\nu(M)$ of $M$ is less than or equal to $\nu(E)$([1, Corollary 2.7]).

**Theorem 4.5** *Let $R = k[x_1, \ldots, x_n]/\mathfrak{p}$ be a Cohen–Macaulay normal domain. If $\mathfrak{p}$ is locally an almost complete intersection of height $g$ (this includes the possibility that some localization is a complete intersection), then the conormal module $E = \mathfrak{p}/\mathfrak{p}^2$ is integrally closed if and only if it is reflexive.*

**Proof.** There is a presentation (we may assume that $R = A/\mathfrak{p}$ where $A$ is a regular local ring and $\mathfrak{p}$ is generated by $1 + \text{height } \mathfrak{p} = 1 + g$ elements)

$$0 \to H_1 \longrightarrow R^{g+1} \longrightarrow E \to 0,$$

where $H_1$ is the 1–dimensional Koszul homology module on a set of generators of $\mathfrak{p}$. Note that $H_1$ is the canonical module $\omega_R$ of $R$. In order to prove the assertion, it suffices to show that $E$ has the $S_2$–property of Serre. Since $H_1$ is Cohen–Macaulay, from the sequence above, we obtain that depth $E \geq \dim R - 1$. Therefore we may assume that $R$ is a local ring of dimension 2 with the maximal ideal $\mathfrak{m}$.

Suppose that $E$ is not reflexive, that is, $E$ is not equal to the bidual $E^{**}$ of $E$. The annihilator of $E^{**}/E$ is an ideal of dimension zero so we may apply to $E^{**}$ the theory of $\mathfrak{m}$–full modules, in particular $\nu(E^{**}) \leq \nu(E) = g+1$. There are two cases to consider. If $\nu(E^{**}) = g$, $E^{**}$ is a free $R$–module. If $e_1, \ldots, e_g$ is one of its bases, then at least one of the $e_i$ does not belong to $E$–say, $e_1 \notin E$. This means that $E \subset \mathfrak{m}e_1 \oplus R^{g-1}$, a module with $(g-1)+\nu(\mathfrak{m})$ generators. But this exceeds $g+1$ since $\nu(\mathfrak{m}) \geq 3$ as $R$ is not a regular local ring. The other possibility is $\nu(E^{**}) = g+1$. It would give rise to a presentation

$$0 \to K \longrightarrow R^{g+1} \longrightarrow E^{**} \to 0.$$

From the rules of computation of divisors, we have that

$$K^{-1} \simeq \det(E^{**}) \simeq \det(E) \simeq (H_1)^{-1},$$

and so $K \simeq H_1$. But all modules in the last exact sequence have depth 2 and $H_1$ is the canonical module. By the standard rules the sequence would split and $E^{**}$ would be free after all, which is not possible by the first case. $\square$

To show the usefulness of the notion of divisors, let us derive two quick applications to integral closure at the end of this seciton. We first extend slightly a result of [3] with a proof based on [4].

**Proposition 4.6** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $E$ a submodule of $R^r$ such that $\mathfrak{m}E :_{R^r} x = E$ for some $x \in \mathfrak{m}$. Let $\ell = \lambda((E :_{R^r} \mathfrak{m})/E)$ and $E :_{R^r} \mathfrak{m} = (y_1, \ldots, y_\ell) + E$ where $y_i \in (E :_{R^r} \mathfrak{m}) \setminus E$ for every $i = 1, \ldots, \ell$. Then*

(i) *$E :_{R^r} \mathfrak{m} = E :_{R^r} x$.*

(ii) *If $\mathfrak{m} \in \mathrm{Ass}\,(R^r/E)$, then $x \notin \mathfrak{m}^2$.*

(iii) *$\{xy_i\}_1^\ell$ is a part of a minimal basis of $E$.*

(iv) *Let $E = (xy_1, \ldots, xy_\ell) + (z_1, \ldots, z_n)$ where $z_i \in E$ and $\ell + n = \nu(E)$. Then*

$$E/xE = \sum_1^\ell R\overline{xy_i} \oplus \sum_1^n R\overline{z_j}$$

*where $^-$ denotes reduction modulo $xE$.*

**Proof.** (i) For $\alpha \in E : x$ and $a \in \mathfrak{m}$, $(a\alpha)x = a(\alpha x) \in \mathfrak{m}E$ so that $a\alpha \in \mathfrak{m}E : x = E$.

(ii) Note that $\mathfrak{m}E :_{R^r} m = E$ since $\mathfrak{m}E :_{R^r} x \supseteq \mathfrak{m}E :_{R^r} m \supseteq E$. Therefore if $x \in \mathfrak{m}^2$, then

$$E = \mathfrak{m}E : x \supseteq \mathfrak{m}E : \mathfrak{m}^2 = (\mathfrak{m}E : \mathfrak{m}) : \mathfrak{m} = E : \mathfrak{m}.$$

Let $\mathfrak{m} = (E :_R \alpha)$ for some $\alpha \in R^r \setminus E$. Then $\alpha \in E :_{R^r} \mathfrak{m} = E$, a contradiction.

(iii) Let $\{a_i\}_1^\ell$ be $R$–elements such that $\sum_1^\ell a_i(xy_i) = x(\sum_1^\ell a_i y_i) \in \mathfrak{m}E$. Then $\sum_1^\ell a_i y_i \in \mathfrak{m}E :_{R^r} x = E$. It follows that $a_i \in E : y_i = \mathfrak{m}$ for each $1 \leq i \leq \ell$.

(iv) Let $\{a_i\}_1^l$ and $\{b_j\}_1^n$ be $R$–elements such that $\sum_1^l a_i(xy_i) + \sum_1^n b_j z_j \in xE$. Since $a_i \in \mathfrak{m}$ for each $i$ and $y_i \in E : \mathfrak{m}$, for each $i$ we have $a_i y_i \in E$ and hence $\sum a_i(xy_i) = x(\sum a_i y_i) \in xE$. It follows that $\sum b_j z_j \in xE$. □

**Corollary 4.7** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $E$ a submodule of $R^r$ such that $\mathfrak{m}E :_{R^r} x = E$ for some $x \in \mathfrak{m}$. Then*

$$E/xE = (E :_{R^r} \mathfrak{m})/E \oplus (E + xR^r)/xR^r = (E :_{R^r} \mathfrak{m})/E \oplus E/x(E :_{R^r} \mathfrak{m}).$$

**Proof.** Consider the following diagram where the rows are exact.

$$\begin{array}{ccccccccc}
0 \longrightarrow & E & \longrightarrow & R^r & \longrightarrow & R^r/E & \longrightarrow & 0 \\
& \cdot x \big\downarrow & & \cdot x \big\downarrow & & \big\downarrow \cdot x & & \\
0 \longrightarrow & E & \longrightarrow & R^r & \longrightarrow & R^r/E & \longrightarrow & 0.
\end{array}$$

By the Snake Lemma, we have the exact sequence

$$0 \to 0 :_E x \to 0 :_{R^r} x \to E :_{R^r} x \to E/xE \to R^r/xR^r \to R^r/(xR^r + E) \to 0.$$

Let $\gamma$ be the map $E :_{R^r} x \to E/xE$ in this exact sequence. Note that $\ker \gamma = \{\alpha \in R^r | \alpha x \in xE\} \supseteq E$. For any $\alpha \in \ker\ \gamma$, $\alpha x = x\beta$ for some $\beta \in E$. Thus, $(\alpha - \beta)x = 0$ so that $\alpha - \beta \in 0 :_{R^r} x \subseteq \mathfrak{m}E : x = E$. Therefore, $\ker \gamma = E$. Since $E :_{R^r} \mathfrak{m} = E :_{R^r} x$, we get an exact sequence

$$0 \to (E : \mathfrak{m})/E \xrightarrow{f} E/xE \to R^r/xR^r \to R^r/(E + xR^r) \to 0$$

If we assume that $E : \mathfrak{m} = (y_1, \ldots y_l) + E$ where $y_i \in E : \mathfrak{m}$, then $f(y \bmod E) = xy \bmod xE$ for each $y \in E : \mathfrak{m}$ so that Image $(f) = \sum_1^l R\overline{xy_i}$, where $^-$ denotes reduction modulo $xE$. By the proposition above,

$$0 \to (E : \mathfrak{m})/E \to E/xE \to E/(x(E : \mathfrak{m})) \to 0$$

is split exact. □

We use these properties of $\mathfrak{m}$–full modules to extend a main result of [4, Theorem (1.1) (1)] to module cases which then describes the completeness of the modules of Kähler differentials in some case(see also [5]).

**Theorem 4.8** *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $E$ a submodule of $R^r$ such that $\mathfrak{m}E :_{R^r} x = E$ for some regular element $x \in \mathfrak{m}$. Suppose that $E$ has finite projective dimension and $\mathfrak{m} \in$ Ass $(R^r/E)$. Then $R$ is a regular local ring.*

**Proof.** It suffices to show that $R/\mathfrak{m}$ has finite projective dimension as an $R$–module. Since $E$ has finite projective dimension, so does $E/xE$. By Corollary 4.7, projective dimension of $(E:\mathfrak{m})/E$ is finite. Since $\mathfrak{m} \in \text{Ass}\,(R^r/E)$, projective dimension of $R/\mathfrak{m}$ is finite. $\square$

**Corollary 4.9** *Let $R = k[x_1, \ldots, x_n]/\mathfrak{p}$ be a normal affine domain over a field $k$ of characteristic zero. If $R$ is a local complete intersection then $\Omega_k(R)$ is integrally closed if and only if $\Omega_k(R)$ is a reflexive module that is free in codimension $\leq 2$.*

**Proof.** The module of differentials has a presentation

$$0 \to \mathfrak{p}/\mathfrak{p}^2 \longrightarrow R^n \longrightarrow \Omega_k(R) \to 0,$$

which in codimension 1 is a exact free complex since $R$ is normal. Since $R$ is a local complete intersection, the conormal module $\mathfrak{p}/\mathfrak{p}^2$ is a projective $R$–module and the sequence gives a projective $R$–resolution of $\Omega_k(R)$. In addition, it yields that $\Omega_k(R)$ is a torsionfree $R$–module. We can then embed $\Omega_k(R)$ into a free $R$–module $R^r$ and the associated prime ideals of the cokernel have codimension two. Let $\mathfrak{q}$ be an associated prime ideal of $R^r/\Omega_k(R)$. According Theorem 4.8, the localization $R_\mathfrak{q}$ is a regular local ring but then $\Omega_k(R)_\mathfrak{q}$ is a free module by the Jacobian criterion. This shows that $\Omega_k(R)$ satisfies the condition $S_2$ of Serre and therefore is reflexive. The converse is clear since reflexive modules over normal domains are always integrally closed. $\square$

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