On the Homology of Two-Dimensional Elimination

Jooyoun Hong

Department de Mathematics, University of California, Riverside
900 Big Springs Drive, Riverside, CA 92521, U.S.A.

Aron Simis [1]

Departamento de Matemática, Universidade Federal de Pernambuco
50740-540 Recife, PE, Brazil

Wolmer V. Vasconcelos [2]

Department of Mathematics, Rutgers University
110 Frelinghuysen Rd, Piscataway, NJ 08854-8019, U.S.A.

Abstract

We study birational maps with empty base locus defined by almost complete intersection ideals. Birationality is shown to be expressed by the equality of two Chern numbers. We provide a relatively effective method of their calculation in terms of certain Hilbert coefficients. In dimension two, after observing that the structure of its irreducible ideals (always complete intersections by a classical theorem of Serre) leads to a natural approach to the calculation of Sylvester determinants, we introduce a computer-assisted method (with a minimal intervention by the computer) which succeeds, in degree ≤ 5, in producing the full sets of equations of the ideals. In the process, it answers affirmatively some questions raised by D. Cox ((9)).

Key words: Almost complete intersection, birational map, elimination, Rees algebra, special fiber, Sylvester determinant

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Email addresses: `jyhong@math.ucr.edu` (Jooyoun Hong), `aron@dmat.ufpe.br` (Aron Simis), `vasconce@math.rutgers.edu` (Wolmer V. Vasconcelos).

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1. Introduction

Let R be a Noetherian ring and $\mathbf{f} = \{f_1, \ldots, f_m\}$ a set of elements of R. Such sets are the ingredients of rational maps between affine and other spaces. At the cost of losing some definition, we choose to examine them in the setting of the ideal I they generate. Specifically, we consider the presentation of the Rees algebra of I

$$0 \to M \longrightarrow S = R[T_1, \ldots, T_m] \xrightarrow{\varphi} R[It] \to 0, \quad T_i \mapsto f_i t.$$

The context of Rees algebra theory allows for the examination of the syzygies of the f_i but also of the relations of all orders, which are carriers of analytic information.

We set $\mathcal{R} = R[It]$ for the Rees algebra of I. The ideal M will be referred to as the *equations* of the f_j, or by abuse of terminology, of the ideal I. If M is generated by forms of degree 1, I is said to be of linear type (this is independent of the set of generators). The Rees algebra $R[It]$ is then the symmetric algebra $\mathcal{S} = \mathrm{Sym}(I)$ of I. Such is the case when the f_i form a regular sequence, M is then generated by the Koszul forms $f_iT_j - f_jT_i$, $i < j$. We will treat mainly *almost complete intersections* in a Cohen-Macaulay ring R, that is, ideals of codimension r generated by $r+1$ elements. Almost exclusively, I will be an ideal of finite co-length in a local ring, or in a ring of polynomials over a field.

Our focus on $\mathcal{R}$ is shaped by the following fact. The class of ideals I to be considered will have the property that both its symmetric algebra $\mathcal{S}$ and the normalization $\mathcal{R}'$ of $\mathcal{R}$ have amenable properties, for instance, one of them (when not both) is Cohen-Macaulay. In such case, the diagram

$$\mathcal{S} \twoheadrightarrow \mathcal{R} \subset \mathcal{R}'$$

gives a convenient dual platform from which to examine $\mathcal{R}$.

There are specific motivations for looking at (and for) these equations. In order to describe our results in some detail, let us indicate their contexts.

(i) Ideals which are almost complete intersections occur in some of the more notable birational maps and in geometric modelling ((3), (4), (5), (6), (7), (8), (9), (10), (17), (18), (21)).

(ii) It is possible interpret questions of birationality of certain maps as an interaction between the Rees algebra of the ideal and its special fiber. The mediation is carried by the first Chern coefficient of the associated graded ring of I. In the case of almost complete intersections the analysis is more tractable, including the construction of suitable algorithms.

(iii) At a recent talk in Luminy ((9)), D. Cox raised several questions about the character of the equations of Rees algebras in polynomial rings in two variables. They are addressed in Section 4 as part of a general program of devising algorithms that produce all the equations of an ideal, or at least some distinguished polynomial (e.g. the 'elimination equation' in it) ((3), (13)).

We now describe our results. Section 2 is an assemblage for the ideals treated here of basics on symmetric and Rees algebras, and on their Cohen-Macaulayness. We also introduce the general notion of a Sylvester form in terms of contents and coefficients in a polynomial ring over a base ring. This is concretely taken up in Section 4 when the base ring is a polynomial ring in 2 variables over a field.

In Section 3 we examine the connection between typical algebraic invariants and the geometric background of rational maps and their images. Here, besides the dimension and the degree of the related algebras, we also consider the Chern number $e_1(I)$ of an ideal. In particular we explain a criterion for a rational map to be birational in terms of an equality of two such Chern numbers, provided the base locus of the map is empty and defined by an almost complete intersection ideal.

In Section 4, we discuss the role of irreducible ideals in producing Sylvester forms. Of a general nature, we describe a method to obtain an irreducible decomposition of ideals of finite co-length. In rings such as $k[s,t]$, due to a theorem of Serre, irreducible ideals are complete intersections, a fact that leads to Sylvester forms of low degree.

Turning to the equations of almost complete intersections, we derive several Sylvester forms over a polynomial ring $R = k[s,t]$, package them into ideals and examine the incident homological properties of these ideals and the associated algebras. It is a computer-assisted approach whose role is to produce a set of syzygies that afford hand computation: the required equations themselves are *not* generated by computation. Concretely, we model a generic class of ideals cases to define 'super-generic' ideals L in rings with several new variables

$$L = (f, g, h_1, \ldots, h_m) \subset A.$$

Using *Macaulay2* ((11)), we obtain the free resolutions of L. In degrees ≤ 5, the resolution has length ≤ 3 (2 when degree $= 4$)

$$0 \to F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \longrightarrow F_0 \longrightarrow L \to 0.$$

It has the property that after specialization the ideals of maximal minors of d_3 and d_2 have codimension 5 and ≥ 4, respectively. Standard arguments of the theory of free resolutions will suffice to show that the specialization of L is a prime ideal.

For ideals in $R = k[s,t]$ generated by forms of degrees ≤ 5, the method succeeds in describing the full set of equations. In higher degree, in cases of special interest, it predicts the precise form of the elimination equation.

For a technical reason–due to the character of irreducible ideals–the method is limited to dimension two. Nevertheless, it is supple enough to apply to non-homogeneous ideals. This may be exploited elsewhere, along with the treatment of ideals with larger numbers of generators in a two-dimensional ring.

2. Preliminaries on symmetric and Rees algebras

We will introduce some basic material of Rees algebras ((2), (12), (22)). Since most of the questions we will consider have a local character, we pick local rings as our setting. Whenever required, the transition to graded rings will be direct.

Throughout we will consider a Noetherian local ring $(R, \mathfrak{m})$ and I an $\mathfrak{m}$-primary ideal (or a graded algebra over a field k, $R = \sum_{n\geq 0} R_n = R_0[R_1]$, $R_0 = k$, and I a homogeneous ideal of finite colength $\lambda(R/I) < \infty$).

We assume that I admits a minimal reduction J generated by $n = \dim R$ elements. This is always possible when k is infinite. The terminology means that for some integer r, $I^{r+1} = JI^r$. This condition in turn means that the inclusion of Rees algebras $R[Jt] \subset R[It]$ is an integral birational extension (birational in the sense that the two algebras have the same total ring of fractions). The smallest such integer, $r_J(I)$, is called the *reduction number* of I relative to J; the infimum of these numbers over all minimal reductions of I is the (absolute) reduction number $r(I)$ of I.

For any ideal, not necessarily $\mathfrak{m}$-primary, the *special fiber* of $R[It]$ – or of I by abuse of terminology – is the algebra $\mathcal{F}(I) = R[It] \otimes_R (R/\mathfrak{m})$. The dimension of $\mathcal{F}(I)$ is called the *analytic spread* of I, and denoted $\ell(I)$. When I is $\mathfrak{m}$-primary, $\ell(I) = \dim R$. A minimal reduction J is generated by $\ell(I)$ elements, and $\mathcal{F}(J)$ is a Noether normalization of $\mathcal{F}(I)$.

Hilbert polynomials

The Hilbert polynomial of I by ($m \gg 0$) is the function ((2)):

$$\lambda(R/I^m) = e_0(I)\binom{m+n-1}{n} - e_1(I)\binom{m+n-2}{n-1} + \text{lower terms.}$$

$e_0(I)$ is the multiplicity of the ideal I. If R is Cohen-Macaulay, $e_0(I) = \lambda(R/J)$, where J is a minimal reduction of I (generated by a regular sequence). For such rings, $e_1(I) \geq 0$.

For instance, if $R = k[x_1, \ldots, x_n]$, $\mathfrak{m} = (x_1, \ldots, x_n)$ and $I = \mathfrak{m}^d$,

$$\begin{aligned}\lambda(R/I^m) = \lambda(R/\mathfrak{m}^{md}) &= \binom{md+n-1}{n} \\ &= d^n\binom{m+n-1}{n} - e_1(I)\binom{m+n-2}{n-1} + \text{lower terms}\end{aligned}$$

where $e_1(I) = \frac{n-1}{2}(d^n - d^{n-1})$.

Both coefficients will be the focus of our interest soon.

Cohen-Macaulay Rees algebras

There is broad array of criteria expressing the Cohen-Macaulayness of Rees algebra (see (1), (14), (19), (23, Chapter 3)). Our needs will be filled by single criterion whose proof is fairly straightforward. We briefly review its related contents.

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension ≥ 1, and let I be an $\mathfrak{m}$-primary ideal with a minimal reduction J. The Rees algebra $R[Jt]$ is Cohen-Macaulay and serves as an anchor to derive many properties of $R[It]$. Here is one that we shall make use of.

Define the *Sally module* $S_J(I)$ of I relative to J to be the cokernel of the natural inclusion of finite $R[Jt]$-modules $I\,R[Jt] \subset I\,R[It]$. Thus,

$$S_J(I) = \sum_{t \geq 2} I^t/IJ^{t-1}.$$

It has a Hilbert function, unlike the algebra $R[It]$, that gives information about the Hilbert function of I (see (22, Chapter 2)). The module on the left, $I \cdot R[Jt]$, is a Cohen-Macaulay $R[Jt]$-module of depth $\dim R + 1$. The Cohen-Macaulayness of $I \cdot R[It]$ is directly related to that of $R[It]$. These considerations lead to the criterion:

Theorem 2.1. If $\dim R \geq 2$ and the reduction number of I is ≤ 1, that is $I^2 = JI$, then $R[It]$ is Cohen-Macaulay. The converse holds if $\dim R = 2$.

Symmetric algebras

Throughout R is a Cohen-Macaulay ring and I is an almost complete intersection. The symmetric algebra $\mathrm{Sym}(I)$ will be denoted by $\mathcal{S}$. Hopefully there will be no confusion between $\mathcal{S}$ and the rings of polynomials $S = R[T_1, \ldots, T_n]$ that we use to give a presentation of either $\mathcal{R}$ or $\mathcal{S}$.

What keeps symmetric algebras of almost complete intersections fairly under control is the following:

Proposition 2.2. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring. If I is an almost complete intersection and depth $R/I \geq \dim R/I - 1$, then $\mathcal{S}$ is Cohen-Macaulay. In particular, if I is $\mathfrak{m}$-primary then $\mathcal{S}$ is Cohen-Macaulay.

Proof. The general assertion follows from (12, Proposition 10.3); see also (16). □

Let R be a Noetherian ring and let I be an R-ideal with a free presentation

$$R^m \xrightarrow{\varphi} R^n \longrightarrow I \to 0.$$

We assume that I has a regular element. If $S = R[T_1, \ldots, T_n]$, the symmetric algebra $\mathcal{S}$ of I is defined by the ideal $M_1 \subset S$ of 1-forms,

$$M_1 = I_1([T_1, \ldots, T_n] \cdot \varphi).$$

The ideal of definition of the Rees algebra $\mathcal{R}$ of I is the ideal $M \subset S$ obtained by elimination

$$M = \bigcup_t (M_1 : x^t) = M_1 : x^\infty,$$

where x is a regular element of I.

Sylvester forms

To get additional elements of M, evading the above calculation, we make use of general Sylvester forms. Recall how these are obtained. Let $\mathbf{f} = \{f_1, \ldots, f_n\}$ be a set of polynomials in $B = R[x_1, \ldots, x_r]$ and let $\mathbf{a} = \{a_1, \ldots, a_n\} \subset R$. If $f_i \in (\mathbf{a})B$ for all i, we can write

$$\mathbf{f} = [f_1 \cdots f_n] = [a_1 \cdots a_n] \cdot A = \mathbf{a} \cdot A,$$

where A is a $n \times n$ matrix with entries in B. By an abuse of terminology, we refer to $\det(A)$ as a *Sylvester form* of $\mathbf{f}$ relative to $\mathbf{a}$, in notation

$$\det(\mathbf{f})_{(\mathbf{a})} = \det(A).$$

It is not difficult to show that $\det(\mathbf{f})_{(\mathbf{a})}$ is well-defined mod $(\mathbf{f})$. The classical Sylvester forms are defined relative to sets of monomials (see (9)). We will make use of them in Section 4. The structure of the matrix A may give rise to finer constructions (lower order Pfaffians, for example) in exceptional cases (see (20)).

In our approach, the f_i are elements of M_1, or were obtained in a previous calculation, and the ideal $(\mathbf{a})$ is derived from the matrix of syzygies φ.

3. Algebraic invariants in rational parametrizations

Let $f_1, \ldots, f_{n+1} \in R = k[x_1, \ldots, x_n]$ be forms of the same degree. They define a rational map

$$\Psi : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^n$$

$$p \to (f_1(p) : f_2(p) : \cdots : f_{n+1}(p)).$$

Rational maps are defined more generally with any number m of forms of the same degree, but in this work we only deal with the case where $m = n + 1$.

There are two basic ingredients to the algebraic side of rational map theory: the ideal theoretic and the algebra aspects, both relevant for the nature of Ψ. First the ideal $I = (f_1, \ldots, f_{n+1}) \subset R$, which in this context is called the *base ideal* of the rational map. Then there is the k-subalgebra $k[f_1, \ldots, f_{n+1}] \subset R$, which is homogeneous, hence a standard k-algebra up to degree renormalization. As such it gives the homogeneous coordinate ring of the (closed) image of Ψ. Finding the irreducible defining equation of the image is known as *elimination* or *implicitization*.

We refer to (21) and (18) (also (20) for an even earlier overview) for the interplay between the ideal and the algebra, as well as its geometric consequences. In particular, the Rees algebra $\mathcal{R} = R[It]$ plays a fundamental role in the theory. A pleasant side of it is that, since I is generated by forms of the same degree, one has $\mathcal{R} \otimes_R k \simeq k[f_1 t, \ldots, f_{n+1} t] \subset \mathcal{R}$, which retro-explains the (closed) image of $\mathbb{P}^{n-1}$ by Ψ as the image of the projection to $\mathbb{P}^n$ of the graph of Ψ. In particular, the fiber cone is reduced and irreducible.

3.1. Elimination degrees and birationality

Although a rational map $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^n$ has a unique set of defining forms $f_1, \ldots, f_{n+1}$ of the same degree and unit gcd, two such maps may look "nearly" the same if they happen to be composite with a birational map of the target $\mathbb{P}^n$ - a so-called Cremona transformation. If this is the case the two maps have the same degree, in particular the final elimination degrees are the same.

However, it may still be the case that the two maps are composite with a rational map of the target which is not birational, so that their degrees as maps do not coincide, yet the degrees of the respective images are the same. In such an event, one would like to pick among all such maps one with smallest possible degree. This leads us to he notion of improper and proper rational parametrizations.

Definition 3.1. Let $\Psi = (f_1 : \cdots : f_{n+1}) : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^n$ be a rational map, where $\gcd(f_1, \ldots, f_{n+1}) = 1$. We will say that Ψ (or the parametrization defined by $f_1, \ldots, f_{n+1}$) is *improper* if there exists a rational map

$$\Psi' = (f'_1 : \cdots : f'_{n+1}) : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^n,$$

with $\gcd(f'_1, \ldots, f'_{n+1}) = 1$, such that:

(1) There is an inclusion of k-algebras $k[f_1, \ldots, f_{n+1}] \subset k[f'_1, \ldots, f'_{n+1}]$;
(2) There is an isomorphism of k-algebras $k[f_1, \ldots, f_{n+1}] \simeq k[f'_1, \ldots, f'_{n+1}]$;
(3) $\deg \Psi' < \deg \Psi$.

We note that if Ψ is improper and Ψ' is as above then the rational map

$$(P_1 : \cdots : P_{n+1}) : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$$

is not birational, where $f_j = P_j(f'_1, \ldots, f'_{n+1})$, for $1 \leq j \leq n+1$. Of course, the transition forms $P_j = P_j(y_1, \ldots, y_{n+1})$ are not uniquely defined.

Example 3.2. The parametrization given by $f_1 = x_1^4, f_2 = x_1^2 x_2^2, f_3 = x_2^4$ is improper since it factors through the parametrization $f'_1 = x_1^2, f'_2 = x_1 x_2, f'_3 = x_2^2$ through either one of the rational maps $(y_1 : y_2 : y_3) \mapsto (y_1^2 : y_2^2 : y_3^2)$ or $(y_1 : y_2 : y_3) \mapsto (y_1^2 : y_1 t_3 : y_3^2)$ neither of which is birational. Moreover, the forms x_1^2, x_1x_2, x_2^2 define a birational map onto its image.

We say that a rational map $\Psi = (f_1 : \cdots : f_{n+1}) : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^n$ is *proper* if it is not improper. The need for considering proper rational maps will become apparent in the context. It is also a basic assumption in elimination theory when one is looking for the elimination degrees (see (9)).

Clearly, if Ψ is birational onto its image then it is proper. The converse does not hold and one seeks for precise conditions under which Ψ is birational onto its image. This is the object of the following parts of this subsection.

When the ideal $I = (f_1, \ldots, f_{n+1})$ has finite co-length – that is, I is $(x_1, \ldots, x_n)$-primary – it is natural to consider another mapping, namely, the corresponding embedding of the Rees algebra $\mathcal{R} = R[It]$ into its integral closure $\widetilde{\mathcal{R}}$. We will explore the attached Hilbert functions into the determinations of various degrees, including the elimination degree of the mapping.

Thus, assume that I has finite co-length. Then we may assume (k is infinite) that $f_1, \ldots, f_n$ is a regular sequence, hence the multiplicity of $J = (f_1, \ldots, f_n)$ is d^n, the same as the multiplicity of $\mathfrak{m}^d$. This implies that J is a minimal reduction of I and of $\mathfrak{m}^d$. We will set up a comparison between $\mathcal{R}$ and $\mathcal{R}' = R[\mathfrak{m}^d]$, where $\mathfrak{m} = (x_1, \ldots, x_n)$, through two relevant exact sequences:

$$0 \to \mathcal{R} \longrightarrow \mathcal{R}' \longrightarrow D \to 0, \tag{1}$$

and its reduction mod $\mathfrak{m}$

$$\overline{\mathcal{R}} \longrightarrow \overline{\mathcal{R}'} \longrightarrow \overline{D} \to 0. \tag{2}$$

$\mathcal{F} = \overline{\mathcal{R}}$ is the *special fiber* of $\mathcal{R}$ (or, of I), and since I is generated by forms of the same degree, one has $\mathcal{F} \simeq k[f_1, \ldots, f_{n+1}]$ as graded k-algebras. By the same token, $\mathcal{F}' = \overline{\mathcal{R}'} \simeq k[\mathfrak{m}^d]$ – the d-th Veronese subring of R. In particular, since $\dim \mathcal{F} = \dim \mathcal{F}'$, the leftmost map in the exact sequence (2) is injective. Also D is annihilated by a power of $\mathfrak{m}$, hence $\dim D = \dim \overline{D}$.

These are the degrees (multiplicities) $\deg(\mathcal{F})$ and $\deg(\mathcal{F}')$ of the special fibers. Since $\mathcal{F}'$ is an integral extension of $\mathcal{F}$, one has

$$\deg(\mathcal{F}') = \deg(\mathcal{F})[\mathcal{F}' : \mathcal{F}], \tag{3}$$

where $[\mathcal{F}':\mathcal{F}]=\dim_K(\mathcal{F}'\otimes_{\mathcal{F}}K)$, where K denotes the fraction field of $\mathcal{F}$ (see, e.g., (21, Proposition 6.1 (b) and Theorem 6.6) for more general formulas). Since $\mathcal{F}'$ is besides integrally closed, the latter is also the field extension degree $[\,k(\mathfrak{m}^d)\,:\,K\,]$. Note that $[\mathcal{F}':\mathcal{F}]=1$ means that the extension $\mathcal{F}\subset\mathcal{F}'$ is birational (equivalently, the rational map Ψ maps $\mathbb{P}^{n-1}$ birationally onto its image). As above, set $L=\mathfrak{m}^d$. We next characterize birationality in terms of both the coefficient e_1 and the dimension of the $\mathcal{R}$-module D.

Proposition 3.3. The following conditions are equivalent:
(i) $[\mathcal{F}':\mathcal{F}]=1$, that is Ψ is birational onto its image;
(ii) $\deg(\mathcal{F})=d^{n-1}$;
(iii) $\dim\overline{D}\leq n-1$;
(iv) $\dim D\leq n-1$
(v) $e_1(L)=e_1(I)$.

Proof. (i) $\Longleftrightarrow$ (ii) This is clear from (3) since $\deg(\mathcal{F}')=d^{n-1}$.

(i) $\Longleftrightarrow$ (iii) Since $\ell(I)=n$ and $\mathcal{F}\subset\mathcal{F}'$ is integral, then $\mathcal{F}\subset\mathcal{F}'$ is a birational extension if and only if its conductor $\mathcal{F}:_{\mathcal{F}}\mathcal{F}'$ is nonzero, equivalently, if and only if $\dim\overline{D}\leq n-1$.

(iv) $\Longleftrightarrow$ (iii) Clearly, $\dim D\leq n$ and in the case of equality its multiplicity is $e_1(L)-e_1(I)>0$. Therefore, the equivalence of the two statements follows suit. ☐

There is some advantage in examining $\overline{D}$ since $\mathcal{F}$ is a hypersurface ring,

$$\mathcal{F}=k[T_1,\ldots,T_{n+1}]/(f)=R[T_1,\ldots,T_{n+1}]/(x_1,\ldots,x_n,f)$$

a complete intersection. Since $\mathcal{F}'$ is also Cohen-Macaulay, with a well-known presentation, it affords an understanding of $\overline{D}$, and sometimes, of D.

3.2. *Calculation of $e_1(I)$ of the base ideal of a rational map*

One objective here is to apply some general formulas for the Chern number $e_1(I)$ of an ideal I to the case of the base ideal of a rational map with source $\mathbb{P}^1=\mathrm{Proj}(k[x_1,x_2])$.

Here is a method put together from scattered facts in the literature of Rees algebras (see (23, Chapter 2)).

Proposition 3.4. Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, let I be an $\mathfrak{m}$-primary ideal with a minimal reduction $J=(a_1,\ldots,a_d)$. Set $R'=R/(a_1,\ldots,a_{d-1})$, $I'=IR'$. Then
(i) $e_0(I)=e_0(I')=\lambda(R/J)$, $e_1(I)=e_1(I')$
(ii) $\mathrm{r}(I')<\deg R'\leq e_0(I)$; in particular, for $n\geq r=\mathrm{r}(I')$, one has $I'^{\,n+1}=a_dI'^{\,n}$
(iii) $\lambda(R'/I'^{\,r+1})=\lambda(R'/I'^{\,r})+\lambda(I'^{\,r}/a_dI'^{\,r})=e_0(I)(r+1)-e_1(I)$
(iv) $e_1(I)=-\lambda(R'/I'^{\,r})+e_0(I)r$

It would be desirable to develop a direct method suitable for the ideal $I=(a,b,c)$ generated by forms of $R=k[s,t]$, of degree n. We may assume that a,b for a regular sequence (i.e. $\gcd(a,b)=1$). We already know that $e_0(I)=n^2$. For regular rings, one knows ((15)) that $e_1(I)\leq\frac{d-1}{2}\,e_0(I)$, $d=\dim R$. Nevertheless the steps above already lead to an efficient calculation for two reasons: the multiplicity $e_0(I)$ is known at the outset and it does not really involve the powers of I. Forms of degree up to 10 are handled well by *Maucalay2* ((11)).

4. Sylvester forms in dimension two

We establish the basic notation to be used throughout. $R = k[s,t]$ is a polynomial ring over the infinite field k, and $I \subset R = k[s,t]$ is a codimension 2 ideal generated by 3 forms of the same degree $n+1$, with free graded resolution

$$0 \longrightarrow R(-n-1-\mu) \oplus R(2(-n-1)+\mu) \xrightarrow{\varphi} R^3(-n-1) \longrightarrow I \longrightarrow 0, \quad \varphi = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix}^t.$$

Then the symmetric algebra of I is $\mathcal{S} \simeq R[T_1,T_2,T_3]/(f,g)$ with

$$f = \alpha_1 T_1 + \beta_1 T_2 + \gamma_1 T_3$$
$$g = \alpha_2 T_1 + \beta_2 T_2 + \gamma_2 T_3.$$

Starting out from these 2 forms, the defining equations of $\mathcal{S}$, following (9), we obtain by elimination higher degrees forms in the defining ideal of $\mathcal{R}(I)$. It will make use of a computer-assisted methodology to show that these algorithmically specified sets generate the ideal of definition M of $\mathcal{R}(I)$ in several cases of interest–in particular answering some questions raised (9). More precisely, the so-called ideal of moving forms M is given when I is generated by forms of degree at most 5. In arbitrary degree, the algorithm will provide the elimination equation in significant cases.

4.1. Basic Sylvester forms in dimension 2

Let $R = k[s,t]$ and let $F, G \in B = R[s,t,T_1,T_2,T_3]$. If $F, G \in (u,v)B$, for some ideal $(u,v) \subset R$, the form derived from

$$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

$$h = ad - bc = \det(F,G)_{(u,v)},$$

will be called a basic Sylvester form.

To explain their naturalness, even for ideals I not necessarily generated by forms, we give an approach to irreducible decomposition of certain ideals.

Theorem 4.1. Let $(R,\mathfrak{m})$ be a Gorenstein local ring and let I be an $\mathfrak{m}$–primary ideal. Let $J \subset I$ be an ideal generated by a system of parameters and let $E = (J : I)/J$ be the canonical module of R/I. If $E = (e_1, \ldots, e_r)$, $e_i \neq 0$, and $I_i = \operatorname{ann}(e_i)$, then I_i is an irreducible ideal and

$$I = \bigcap_{i=1}^{r} I_i.$$

The statement and its proof will apply to ideals of rings of polynomials over a field.

Proof. The module E is the injective envelope of R/I, and therefore it is a faithful R/I–module (see (2, Section 3.2) for relevant notions). For each e_i, Re_1 is a nonzero submodule of E whose socle is contained in the socle of E (which is isomorphic to $R/\mathfrak{m}$) and therefore its annihilator I_i (as an R-ideal) is irreducible. Since the intersection of the I_i is the annihilator of E, the asserted equality follows. □

Corollary 4.2. Let $(R, \mathfrak{m})$ be a regular local ring of dimension two and let I be an $\mathfrak{m}$–primary ideal with a free resolution

$$0 \to R^{n-1} \xrightarrow{\varphi} R^n \longrightarrow I \to 0,$$

$$\varphi = \left[\begin{array}{ccc} & & \\ & \varphi' & \\ & & \\ \hline a_{n-1,1} & \cdots & a_{n-1,n-1} \\ a_{n,1} & \cdots & a_{n,n-1} \end{array}\right],$$

and suppose that the last two maximal minors Δ_{n-1}, Δ_n of φ form a regular sequence. If $e_1, \ldots, e_{n-1}$ are as above, then

$$(\Delta_{n-1}, \Delta_n) : I = I_{n-2}(\xi') = (e_1, \ldots, e_{n-1})$$

and each ideal $(\Delta_{n-1}, \Delta_n) : e_i$ is a complete intersection of codimension 2.

Proof. The assertion that the irreducible I_i is a complete intersection is a result of Serre, valid for all two-dimensional regular rings whose projective modules are free. □

Remark 4.3. In our applications, $I = C(f, g)$, the content ideal of f, g. In some of these cases, $C(f, g) = (s, t)^n$, for some n, an ideal which admits the irreducible decomposition

$$(s, t)^n = \bigcap_{i=1}^{n} (s^i, t^{n+1-i}).$$

One can then process f, g through all the pairs $\{s^i, t^{n-i+1}\}$, and collect the determinants for the next round of elimination. As in the classical Sylvester forms, the inclusion $C(f, g) \subset (s, t)^n$ may be used anyway to start the process, although without the measure of control of degrees afforded by the equality of ideals.

4.2. Cohen-Macaulay algebras

We pointed out in Theorem 2.1 that the basic control of Cohen-Macaulayness of a Rees algebra of an ideal $I \subset k[s, t]$ is that its reduction number be at most 1. We next give a mean of checking this property directly off a free presentation of I.

Theorem 4.4. Let $I \subset R$ be an ideal of codimension 2, minimally generated by 3 forms of the same degree. Let

$$\varphi = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{bmatrix}$$

be the Hilbert-Burch presentation matrix of I. Then $\mathcal{R}$ is Cohen-Macaulay if and only if the equalities of ideals of R hold

$$(\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2) = (u, v),$$

where u, v are forms.

Proof. Consider the presentation

$$0 \to \mathcal{L} \longrightarrow \mathcal{S} = R[T_1, T_2, T_3]/(f, g) \longrightarrow \mathcal{R} \to 0,$$

where f, g are the 1-forms

$$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix} \cdot \varphi.$$

If $\mathcal{R}$ is Cohen-Macaulay, the reduction number of I is 1 by Theorem 2.1, so there must be a nonzero quadratic form h with coefficients in k in the presentation ideal M of $\mathcal{R}$. In addition to h, this ideal contains f, g, hence in order to produce such terms its Hilbert-Burch matrix must be of the form

$$\begin{bmatrix} u & v \\ p_1 & p_2 \\ q_1 & q_2 \end{bmatrix}$$

where u, v are forms of $k[s, t]$, and the other entries are 1-forms of $k[T_1, T_2, T_3]$. Since p_1, p_2 are q_1, q_2 are pairs of linearly independent 1-forms, the assertion about the ideals defined by the columns of φ follow.

4.3. Base ideals generated in degree 4

This is the case treated by D. Cox in his Luminy lecture ((9)). We accordingly change the notation to $R = k[s, t]$, $I = (f_1, f_2, f_3)$, forms of degree 4. The field k is infinite, and we further assume that f_1, f_2 form a regular sequence so that $J = (f_1, f_2)$ is a reduction of I and of $(s, t)^4$. Let

$$0 \to R(-4-\mu) \oplus R(-8+\mu) \xrightarrow{\varphi} R^3(-4) \longrightarrow R \longrightarrow R/I \to 0, \quad \varphi = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{bmatrix} \quad (4)$$

be the Hilbert-Burch presentation of I. We obtain the equations of f_1, f_2, f_3 from this matrix.

Note that μ is the degree of the first column of φ, $4 - \mu$ the other degree. Let us first consider (as in (9)) the case $\mu = 2$.

Balanced case

We shall now give a computer-assisted treatment of the *balanced* case, that is when the resolution (4) of the ideal I has $\mu = 2$ and the content ideal of the syzygies is $(s, t)^2$. Since k is infinite, it is easy to show that there is a change of variables, $T_1, T_2, T_3 \to x, y, z$, so that (s^2, st, t^2) is a syzygy of I. The forms f, g that define the symmetric algebra of I can then be written

$$[f \quad g] = [s^2 \quad st \quad t^2] \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix},$$

where u, v, w are linear forms in x, y, z. Finally, we will assume that the ideal $I_2\left(\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}\right)$ has codimension two. Note that this is a generic condition.

We introduce now the *equations* of I.

- Linear equations f and g:

$$[f \quad g] = [x\ y\ z]\ \varphi = [x\ y\ z] \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{bmatrix}$$

$$= [s^2 \quad st \quad t^2] \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix},$$

where u, v, w are linear forms in x, y, z.

- Biforms h_1 and h_2:

Write Γ_1 and Γ_2 such that

$$[f \quad g] = [x\ y\ z]\ \varphi = [\ s \quad t^2\ ]\ \Gamma_1 = [\ s^2 \quad t\ ]\ \Gamma_2.$$

Then $h_1 = \det \Gamma_1$ and $h_2 = \det \Gamma_2$.

- Implicit equation $F = \det \Theta$, where $[h_1 \quad h_2] = [s \quad t]\ \Theta$.

Using generic entries for φ, in place of the true k-linear forms in old variables x, y, z, we consider the ideal of $k[s, t, x, y, z, u, v, w]$ defined by

$$\begin{aligned} f &= s^2x + sty + t^2z \\ g &= s^2u + stv + t^2w \\ h_1 &= -syu - tzu + sxv + txw \\ h_2 &= -szu - tzv + sxw + tyw \\ F &= -z^2u^2 + yzuv - xzv^2 - y^2uw + 2xzuw + xyvw - x^2w^2 \end{aligned}$$

Proposition 4.5. If $I_2\left(\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}\right)$ specializes to a codimension two ideal of $k[x, y, z]$, then $L = (f, g, h_1, h_2, F) \subset A = R[x, y, z, u, v, w]$ specializes to the defining ideal of $\mathcal{R}$.

Proof. *Macaulay2* ((11)) gives a resolution

$$0 \to A \xrightarrow{d_2} A^5 \longrightarrow A^5 \longrightarrow L \to 0$$

where

$$d_2 = \begin{bmatrix} zv - yw \\ zu - xw \\ -yu + xv \\ -t \\ s \end{bmatrix}.$$

The assumption on $I_2\left(\begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix}\right)$ says that the entries of d_2 generate an ideal of codimension four and thus implies that the specialization LS has projective dimension two and that it is unmixed. Since $LS \not\subset (s,t)S$, there is an element $q \in (s,t)R$ that is regular modulo S/LS. If

$$LS = Q_1 \cap \cdots \cap Q_r$$

is the primary decomposition of LS, the localization LS_q has the corresponding decomposition since q is not contained in any of the $\sqrt{Q_i}$. But now $\mathrm{Sym}_q = \mathcal{R}_q$, so $LS_q = (f,g)_u$, as $I_q = R_q$. □

Non-balanced case

We shall now give a similar computer-assisted treatment of the non-balanced case, that is when the resolution (4) of the ideal I has $\mu = 3$. This implies that the content ideal of the syzygies is (s,t). Let us first indicate how the proposed algorithm would behave.

- Write the forms f, g as

$$f = as + bt$$
$$g = cs + dt,$$

  where

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix} \begin{bmatrix} s^2 \\ st \\ t^2 \end{bmatrix}$$

- The next form is the Jacobian of f, g with respect to (s,t)

$$h_1 = \det(f,g)_{(s,t)} = ad - bc = -bxs^2 - byst - bzt^2 + aus^2 + avst + awt^2.$$

- The next two generators

$$h_2 = \det(f,h_1)_{(s,t)} = b^2xs + b^2yt - abzt - abus - abvt + a^2wt$$

and the elimination equation

$$h_3 = \det(f, h_2)_{(s,t)} = -b^3x + ab^2y - a^2bz + ab^2u - a^2bv + a^3w.$$

Proposition 4.6. $L = (f, g, h_1, h_2, h_3) \subset A = k[s, t, x, y, z, u, v, w]$ specializes to the defining ideal of $\mathcal{R}$.

Proof. *Macaulay2* ((11)) gives the following resolution of L

$$0 \to A^2 \xrightarrow{\varphi} A^6 \xrightarrow{\psi} A^5 \longrightarrow L \to 0,$$

$$\varphi = \begin{bmatrix} s & 0 \\ t & 0 \\ -b & s \\ a & t \\ 0 & -b \\ 0 & a \end{bmatrix},$$

$$\psi = \begin{bmatrix} -b^2x + abu & -b^2y + abz + abv - a^2w & -bsx - bty + asu + atv & -btz + atw & -s^2x - sty - t^2z & -s^2u - stv - t^2w \\ t & -s & 0 & 0 & 0 & 0 \\ a & b & t & -s & 0 & 0 \\ 0 & 0 & a & b & t & -s \\ 0 & 0 & 0 & 0 & a & b \end{bmatrix}$$

The ideal of 2×2 minors of φ has codimension 4, even after we specialize from A to S in the natural manner. Since LS has projective dimension two, it will be unmixed. As $LS \not\subset (s, t)$, there is an element $u \in (s, t)R$ that is regular modulo S/LS. If

$$LS = Q_1 \cap \cdots \cap Q_r$$

is the primary decomposition of LS, the localization LS_u has the corresponding decomposition since u is not contained in any of the $\sqrt{Q_i}$. But now $\mathrm{Sym}_u = \mathcal{R}_u$, so $LS_u = (f, g)_u$, as $I_u = R_u$. $\square$

4.4. Degree 5 *and above*

It may be worthwhile to extend this to arbitrary degree, that is assume that I is defined by 3 forms of degree $n + 1$ (for convenience in the notation to follow). We first consider the case $\mu = 1$. Using the procedure above, we would obtain the sequence of polynomials in $A = R[a, b, x_1, \ldots, x_n, y_1, \ldots, y_n]$

- Write the forms f, g as

$$f = as + bt$$
$$g = cs + dt,$$

where

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix} \begin{bmatrix} s^{n-1} \\ s^{n-2}t \\ \vdots \\ st^{n-2} \\ t^{n-1} \end{bmatrix}$$

- The next form is the Jacobian of f, g with respect to (s, t)

$$h_1 = \det(f, g)_{(s,t)} = ad - bc$$

- Successively we would set

$$h_{i+1} = \det(f, h_i)_{(s,t)}, \quad 1 < n.$$

- The polynomial

$$h_n = \det(f, h_{n-1})_{(s,t)}$$

is the elimination equation.

Proposition 4.7. $L = (f, g, h_1, \dots, h_5) \subset A$ specializes to the defining ideal of $\mathcal{R}$.

In *Macaulay2*, we checked the degrees 5 and 6 cases. In both cases, the ideal L (which has one more generator in degree 6) has a projective resolution of length 2 and the ideal of maximal minors of the last map has codimension four.

Conjecture 4.8. For arbitrary n, $L = (f, g, h_1, \dots, h_n) \subset A$ has projective dimension two and specializes to the defining ideal of $\mathcal{R}$.

In degree 5, the interesting case is when the Hilbert-Burch matrix ϕ has degrees 2 and 3. Let us describe the proposed generators. For simplicity, by a change of coordinates, we assume that the coordinates of the degree 2 column of φ are s^2, st, t^2

$$\begin{aligned} f &= s^2x + sty + t^2z \\ g &= (s^3w_1 + s^2tw_2 + st^2w_3 + t^3w_4)x + (s^3w_5 + s^2tw_6 + st^2w_7 + t^3w_8)y \\ &\quad + (s^3w_9 + s^2tw_{10} + st^2w_{11} + t^3w_{12})z \end{aligned}$$

Let

$$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} x & y & z \\ sA & sB + tC & tD \end{bmatrix} \begin{bmatrix} s^2 \\ st \\ t^2 \end{bmatrix} = \phi \begin{bmatrix} s^2 \\ st \\ t^2 \end{bmatrix}$$

$$= \begin{bmatrix} x & ys + zt \\ sA + tB & stC + t^2D \end{bmatrix} \begin{bmatrix} s^2 \\ t \end{bmatrix} = B_1 \begin{bmatrix} s^2 \\ t \end{bmatrix},$$

$$= \begin{bmatrix} xs + yt & z \\ s^2A + stB & sC + tD \end{bmatrix} \begin{bmatrix} s \\ t^2 \end{bmatrix} = B_2 \begin{bmatrix} s \\ t^2 \end{bmatrix}$$

where A, B, C, D are k-linear forms in x, y, z.

$$\begin{aligned} h_1 &= \det(B_1) \\ &= s^2(-yA) + st(xC - yB - zA) + t^2(xD - zB) \\ &= s^2(-yA) + t(xCs - yBs - zAs + xDt - zBt) \\ &= s(-yAs + xCt - yBt - zAt) + t^2(xD - zB), \end{aligned}$$

$$\begin{aligned} h_2 &= \det(B_2) \\ &= s^2(xC - zA) + st(xD + yC - zB) + t^2(yD) \\ &= s^2(xC - zA) + t(xDs + yCs - zBs + yDt) \\ &= s(xCs - zAs + xDt + yCt - zBt) + t^2(yD). \end{aligned}$$

$$\begin{bmatrix} f \\ h_1 \end{bmatrix} = \begin{bmatrix} x & ys + zt \\ -yA & xCs - yBs - zAs + xDt - zBt \end{bmatrix} \begin{bmatrix} s^2 \\ t \end{bmatrix} = C_1 \begin{bmatrix} s^2 \\ t \end{bmatrix}$$

$$= \begin{bmatrix} xs + yt & z \\ -yAs + xCt - yBt - zAt & xD - zB \end{bmatrix} \begin{bmatrix} s \\ t^2 \end{bmatrix} = C_2 \begin{bmatrix} s \\ t^2 \end{bmatrix}$$

$$\begin{bmatrix} f \\ \\ h_2 \end{bmatrix} = \begin{bmatrix} x & ys + zt \\ \\ xC - zA & xDs + yCs - zBs + yDt \end{bmatrix} \begin{bmatrix} s^2 \\ \\ t \end{bmatrix} = C_3 \begin{bmatrix} s^2 \\ \\ t \end{bmatrix}$$

$$= \begin{bmatrix} xs + yt & z \\ \\ xCs - zAs + xDt + yCt - zBt & yD \end{bmatrix} \begin{bmatrix} s \\ \\ t^2 \end{bmatrix} = C_4 \begin{bmatrix} s \\ \\ t^2 \end{bmatrix}$$

$$c_1 = \det(C_1) = x^2(Cs + Dt) + xy(-Bs) + xz(-As - Bt) + yz(At) + y^2(As)$$

$$c_2 = \det(C_2) = x^2(Ds) + xy(Dt) + xz(-Bs - Ct) + yz(As) + z^2(At)$$

$$c_3 = \det(C_3) = x^2(Ds) + xy(Dt) + xz(-Bs - Ct) + yz(As) + z^2(At)$$

$$c_4 = \det(C_4) = xy(Ds) + xz(-Cs - Dt) + yz(-Ct) + z^2(As + Bt) + y^2(D)$$

$$\begin{bmatrix} f \\ \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} x & y & z \\ \\ -yA & xC - yB - zA & xD - zB \\ \\ xC - zA & xD + yC - zB & yD \end{bmatrix} \begin{bmatrix} s^2 \\ \\ st \\ \\ t^2 \end{bmatrix}$$

Then $F = -x^3D^2 + x^2yCD + xy^2(-BD) + x^2z(2BD - C^2) + xz^2(2AC - B^2) + xyz(BC - 3AD) + y^2z(-AC) + yz^2(AB) + y^3(AD) + z^3(-A^2)$, an equation of degree 5. In particular, the parametrization is birational.

Proposition 4.9. $L = (f,\, g,\, h_1,\, h_2,\, c_1,\, c_2,\, c_4,\, F)$ specializes to the defining ideal of $\mathcal{R}$.

Proof. Using *Macaulay2*, the ideal L has a resolution:

$$0 \longrightarrow S^1 \xrightarrow{d_3} S^6 \xrightarrow{d_2} S^{12} \xrightarrow{d_1} S^8 \longrightarrow L \longrightarrow 0.$$

$$d_3 = [-z \;\; y \;\; x \;\; -t \;\; s \;\; 0]^t$$

$$d_2 = \begin{bmatrix} y & z & 0 & 0 & 0 & 0 \\ x & 0 & z & 0 & 0 & 0 \\ -v & 0 & 0 & z & 0 & x^2w_4 - xzw_7 + xyw_8 + xzw_{12} \\ u & 0 & 0 & 0 & z & -xzw_3 + xyw_4 + z^2w_6 - yzw_7 + y^2w_8 - xzw_8 - z^2w_{11} + yzw_{12} \\ 0 & x & -y & 0 & 0 & 0 \\ 0 & -v & 0 & -y & 0 & xzw_1 - x^2w_3 + yzw_5 + z^2w_9 - xzw_{11} \\ 0 & u & 0 & 0 & -y & xzw_2 - x^2w_4 + z^2w_{10} - xzw_{12} \\ 0 & 0 & u & 0 & -x & xzw_1 + yzw_5 - xzw_6 + x^2w_8 + z^2w_9 \\ 0 & 0 & 0 & u & v & 0 \\ 0 & 0 & v & x & 0 & -xyw_1 + x^2w_2 - y^2w_5 + xyw_6 - x^2w_7 - yzw_9 + xzw_{10} \\ 0 & 0 & 0 & 0 & 0 & -t \\ 0 & 0 & 0 & 0 & 0 & s \end{bmatrix}$$

The ideals of maximal minors give codim $I_1(d_3) = 5$ and codim $I_5(d_2) = 4$ after specialization. As we have been arguing, this suffices to show that the specialization is a prime ideal of codimension two. □

Elimination forms in higher degree

In degrees greater than 5, the methods above are not very suitable. However, in several cases they are still supple enough to produce the elimination equation. We have already seen this when one of the syzygies is of degree 1. Let us describe two other cases.

- Degree $n = 2p$, f and g both of degree p. We use the decomposition

$$(s,t)^p = \bigcap_{i=1}^{p} (s^i, t^{p+1-i}).$$

For each $1 \leq i \leq p$, let

$$h_i = \det(f,g)_{(s^i,t^{p+1-i})}.$$

These are quadratic polynomials with coefficients in $(s,t)^{p-1}$. We set

$$[h1, \cdots, h_p] = [s^{p-1}, \cdots, t^{p-1}] \cdot A,$$

where A is a $p \times p$ matrix whose entries are 2-forms in $k[x,y,z]$. The Sylvester form of degree n, $F = \det(A)$, is the required elimination equation.

- Degree $n = 2p+1$, f of degree p. We use the decomposition

$$(s,t)^p = \bigcap_{i=1}^{p} (s^i, t^{p+1-i}).$$

For each $1 \leq i \leq p$, let

$$h_i = \det(f,g)_{(s^i,t^{p+1-i})}.$$

These are quadratic polynomials with coefficients in $(s,t)^p$. We set

$$[f, h1, \cdots, h_p] = [s^p, \cdots, t^p] \cdot B,$$

where A is a $(p+1)\times(p+1)$ matrix with one column whose entries are linear forms and the remaining columns with entries 2-forms in $k[x,y,z]$. The Sylvester form $F=\det(B)$ is the required elimination equation.

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