TANGENT ALGEBRAS

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ABSTRACT. One studies the Zariski tangent cone $T_X \xrightarrow{\pi} X$ to an affine variety X and the closure $\overline{T}_X$ of $\pi^{-1}(\mathrm{Reg}(X))$ in T_X. One focuses on the comparison between T_X and $\overline{T}_X$, giving sufficient conditions on X in order that $T_X = \overline{T}_X$. One considers, in particular, the question of when this equality takes place in the presence of the reducedness of the Zariski tangent cone. Another problem considered here is to understand the impact of the Cohen–Macaulayness or normality of $\overline{T}_X$ on the local structure of X.

1. INTRODUCTION

Let A be an affine algebra over a field k, and let $\Omega_{A/k}$ be the A-module of Kähler k-differentials. Classically, the properties of this module are closely related to the local singularities of A, embodying in particular the well-known Jacobian criterion for the smoothness of A. Its sheaf version for an algebraic variety is fundamental in intersection theory and its cohomology is a major vehicle for the study of the global geometry of the variety.

In this paper we focus on two basic algebras associated to $\Omega_{A/k}$, where A is a reduced ring essentially of finite type over a perfect field k: the Rees algebra $\mathbb{R}_{A/k}$ of $\Omega_{A/k}$ (defined in Section 2) and its close predecessor $\mathbb{S}_{A/k}$, the symmetric algebra of $\Omega_{A/k}$. If A is regular these algebras coincide, but otherwise they may be quite apart and their respective properties have different impact on the nature of A. In [22] the ring $\mathbb{S}_{A/k}$ was called the *Zariski tangent algebra* of A because the closed fibers of the map $\mathrm{Spec}(\mathbb{S}_{A/k}) \longrightarrow \mathrm{Spec}(A)$ are the Zariski tangent spaces to closed points of $\mathrm{Spec}(A)$, when A is an affine algebra over an algebraically closed field. Alternatively, $\mathrm{Spec}(\mathbb{S}_{A/k})$ is the first jet scheme of $\mathrm{Spec}(A)$. As to $\mathbb{R}_{A/k}$, it plays the role of the coordinate ring of a correspondence in biprojective space (see [19]). A variation on $\mathbb{R}_{A/k}$ is the Rees algebra of the top wedge product of $\Omega_{A/k}$ which, as was argued in [21], gives a full-fledged algebraic version of Zak's inequality for the dimension of the Gauss image of a projectively embedded variety.

Yet another version of such "tangent" algebras is the *tangent star algebra* $\mathbb{T}_{A/k}$, defined as the associated graded ring of the kernel $\mathbb{D}_{A/k}$ of the multiplication map $A \otimes_k A \longrightarrow A$ – also known as the *diagonal ideal*. Looking at the relation of this algebra to $\mathbb{S}_{A/k}$ was the core of [22]. Under mild conditions, it can be shown that $\mathbb{T}_{A/k}$ modulo its A-torsion is isomorphic to $\mathbb{R}_{A/k}$ (see [19, 4.6]). Therefore a certain knowledge about the structure of $\mathbb{R}_{A/k}$ might be borrowed from results in

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[22]. However, since this would not make up for the main bulk in the present work, we have decided to study $\mathbb{R}_{A/k}$ *ab initio*, in its own right.

There is a natural surjection

$$\mathbb{S}_{A/k} \twoheadrightarrow \mathbb{R}_{A/k}.$$

A great deal of the present work has to do with this map, whose kernel measures the failure of describing $\mathbb{R}_{A/k}$ solely in terms of linear equations, or to use a recent terminology for modules, the failure of $\Omega_{A/k}$ being of *linear type*. We also study how the singularities of A are reflected in the normality, Cohen–Macaulayness and Gorensteiness of $\mathbb{R}_{A/k}$.

Let us now describe the main results in some detail.

The paper is divided into three sections. In the first section we focus on the Zariski tangent algebra $\mathbb{S}_{A/k}$. Recall that $\Omega_{A/k} \simeq \mathbb{D}_{A/k}/\mathbb{D}_{A/k}^2$. Now, quite generally, let S be a Noetherian ring and let $\mathbf{D} \subset S$ be an ideal such that the symmetric algebra $\mathcal{S}_A(\mathbf{D}/\mathbf{D}^2)$ is torsionfree over $A = S/\mathbf{D}$. If one assumes, moreover, that $\mathbf{D}$ is generically a complete intersection and A is reduced then $\mathcal{S}_A(\mathbf{D}/\mathbf{D}^2)$ is reduced. The converse to this statement, namely, that $\mathcal{S}_A(\mathbf{D}/\mathbf{D}^2)$ is torsionfree if it is reduced, is known to be false in general. The corresponding question for the associated graded ring $\mathrm{gr}_{\mathbf{D}}(S)$ was treated in [14, 1.10] and shown to be affirmative provided S is a quasi-unmixed local ring and A has finite projective dimension over S. Although the diagonal ideal $\mathbb{D}_{A/k}$ does not have finite projective dimension for singular A, our goal in the first part of the section is to consider this converse in the framework of $\mathbb{S}_{A/k} = \mathcal{S}_A(\mathbb{D}_{A/k}/\mathbb{D}_{A/k}^2)$, trading off the homological restriction on A for a condition on its defining equations (Theorem 2.1 and Corollary 2.2).

In the second part of this section we deal with the same question in the case of an analytically irreducible curve singularity over an algebraically closed field. Here, without any assumptions on the defining equations, we prove that such curves are non-singular provided $\mathbb{S}_{A/k}$ is reduced (Theorems 2.6 and 2.8). This can be regarded as an analogue of Berger's conjecture (see [4]) in which the reducedness of $\mathbb{S}_{A/k}$ replaces the torsionfreeness of $\Omega_{A/k}$.

In the last part of the section we study the relationship between the normality of $\mathbb{S}_{A/k}$ and its "reflexivity", defined in terms of the double duals of the symmetric powers of $\Omega_{A/k}$. Quite generally, the latter is known to imply the former, at least when A is normal. We will prove that the converse holds under suitable conditions on the equations of order 2 defining A (Proposition 2.9 and Theorem 2.10).

The second section is mainly devoted to the behavior of the Rees algebra $\mathbb{R}_{A/k}$. Our first fundamental result deals with the case where A is locally everywhere a complete intersection. Notice that, in this case, the module of differentials $\Omega_{A/k}$ has projective dimension one. We first recall in parallel the case of the symmetric algebra. Namely, if A is a local Cohen–Macaulay ring and E is a finite A-module of projective dimension one, then E satisfies condition (F_0) if and only if the symmetric algebra $\mathcal{S}_A(E)$ is a complete intersection over A, and E satisfies condition (F_1) if and only if $\mathcal{S}_A(E)$ is A-torsionfree ([1, Proposition 4], [13, 1.1], [24, 3.4]) – see Section 2 for an explanation of the condition (F_t). As the torsionfreeness of $\mathcal{S}_A(E)$ gives a natural isomorphism $\mathcal{S}_A(E) \simeq \mathcal{R}_A(E)$, it follows that condition (F_1) implies the Rees algebra $\mathcal{R}_A(E)$ to be Cohen–Macaulay. However, the converse does not hold in general, even if one assumes the preliminary condition (F_0). Nevertheless we are able to prove this converse in the case where E is the module of differentials of

a complete intersection over a field of characteristic zero (Theorem 3.1). Our result shows that Cohen–Macaulayness is a rather restrictive property for $\mathbb{R}_{A/k}$. It implies for instance that if $X \subset \mathbb{P}_k^{2d+1}$ is a d-dimensional non-degenerate smooth complete intersection over a field k of characteristic zero with homogeneous coordinate ring A, then $\mathbb{R}_{A/k}$ is never a Cohen–Macaulay ring. It would be interesting to have a geometric explanation for this phenomenon.

The second kind of structural results concerns the normality of $\mathbb{R}_{A/k}$, in which the condition (F_2) will play a predominant role. Throughout this part A is assumed to be a normal domain. To enlarge the picture we consider yet another algebra associated to a finite A-module E, namely, $\mathcal{B}_A(E) := \bigoplus_{i \geq 0} (E^i)^{**}$, where E^i denotes the ith graded component of $\mathcal{R}_A(E)$ and $-^*$ stands for $\mathrm{Hom}(-, A)$. This algebra could be dubbed the *reflexive closure* of the Rees algebra $\mathcal{R}_A(E)$ – note it is a Krull domain with the same divisor class group as A, but may fail to be Noetherian (see, e.g., [17, Theorem 2]). This algebra has been studied earlier in [11] (see also [28, Chapter 7]). First is a basic criterion for the equality $\mathcal{R}_A(E) = \mathcal{B}_A(E)$ assuming that $\mathcal{R}_A(E)$ is normal or just satisfies Serre's condition (S_2) (Proposition 3.4). The criterion is given in terms of a new condition that controls the growth of the local analytic spreads of E. We then move on to an encore of modules of projective dimension one over normal Cohen–Macaulay domains. For such a module E it is known that the condition (F_2) holds if and only if $\mathcal{S}_A(E) = \mathcal{B}_A(E)$ (see [1, Proposition 4]). In particular, if E satisfies (F_2) then $\mathcal{R}_A(E)$ is normal. Surprisingly, the converse holds as well provided the non-free locus of E be contained in the singular locus of A (Theorem 3.7). An application to $\Omega_{A/k}$ when A is a normal complete intersection now ensues (Corollary 3.8). In fact it turns out that the inequalities $\mathrm{edim}\, A_{\mathfrak{p}} \leq 2 \dim A_{\mathfrak{p}} - 2$ locally on the singular locus of A become equivalent to the powers of $\Omega_{A/k}$ being integrally closed in the range $1 \leq i \leq \mathrm{ecodim}\, A$; here $\mathrm{ecodim}\, A := \mathrm{edim}\, A - \dim A$.

In the last section we consider special situations. A substantial part thereof deals with extending the previous results to cases other than complete intersections. A reasonable bulk of results is obtained if A is of low dimension or low embedding codimension or is sufficiently structured. Thus we find satisfactory answers in the codimension 2 perfect and codimension 3 Gorenstein cases (Proposition 4.1, Theorem 4.8 and Proposition 4.2). Naturally this part requires several developments and the results become somewhat too technical to be described in this introduction. We refer the reader to the appropriate places in the last part of this work.

The section ends with some consequences of the theory so far regarding its relation to Calabi–Yau varieties. We argue that if A is the homogeneous coordinate ring of a Calabi–Yau variety then often $\mathbb{S}_{A/k}$ or $\mathbb{R}_{A/k}$ are the homogeneous coordinate rings of "Calabi–Yau like" varieties.

2. The symmetric algebra of the module of differentials

We need some preliminaries on symmetric and Rees algebras of modules. We will draw upon [23] for terminology and basic notions about Rees algebras of modules. Let E be a finite module over a Noetherian ring A and assume that E is generically free (i.e., free locally at every associated prime of A). In this setting the *Rees algebra* $\mathcal{R}_A(E)$ of E is defined as the symmetric algebra of E modulo A-torsion, $\mathcal{R}_A(E) := \mathcal{S}_A(E)/\tau_A(\mathcal{S}_A(E))$. As is known, this definition retrieves the usual notion of the Rees algebra of an ideal containing a regular element. If in addition

E is torsionfree, then E can be embedded into a finite free module F and the Rees algebra of E can be identified with the image of $\mathcal{S}_A(E)$ in the polynomial ring $\mathcal{S}_A(F)$. In case $\mathcal{S}_A(E) = \mathcal{R}_A(E)$ we say that E is a module of *linear type*.

We recall the following additional notions. Let E be a finite module over a Noetherian ring A. Suppose that E has a rank r (i.e., E is free of rank r locally at every associated prime of A). Given an integer $t \geq 0$, we say that E satisfies condition (F_t) if $\mu(E_{\mathfrak{p}}) \leq \dim A_{\mathfrak{p}} + r - t$ for every $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $E_{\mathfrak{p}}$ is not free; here $\mu(-)$ denotes the minimal number of generators. In terms of Fitting ideals this condition means that $\mathrm{height}\,\mathrm{Fitt}_i(E) \geq i - r + t + 1$ for $i \geq r$. The properties (F_t) play a role in the study of $\mathcal{S}_A(E)$. Thus, e.g., the Cohen–Macaulayness of this algebra implies condition (F_0) for E, and if E is of linear type then (F_1) holds.

By the same token, one can introduce yet another condition based on the *analytic spread* $\ell(E)$ of a finite module E over a local ring $(A, \mathfrak{m})$, defined to be the Krull dimension of the residue algebra $\mathcal{R}_A(E)/\mathfrak{m}\mathcal{R}_A(E)$ (see [23]). We will say that a finite module E of rank r over a Noetherian ring A satisfies condition (L_t) if $\ell(E_{\mathfrak{p}}) \leq \dim A_{\mathfrak{p}} + r - t$ for every $\mathfrak{p} \in \mathrm{Spec}(A)$ with $\dim A_{\mathfrak{p}} \geq t$. Roughly, this condition plays a similar role for the Rees algebra as (F_t) plays for the symmetric algebra. Condition (F_t) implies (L_t), and the converse holds if E is free locally in codimension $\leq t - 1$ and is of linear type. We will only use these conditions in the range $0 \leq t \leq 2$.

Our main interest throughout is the module of Kähler differentials $\Omega_{A/k}$. Let A be a reduced algebra essentially of finite type over a perfect field k – hence $\Omega_{A/k}$ is generically free. We denote by $\mathbb{S}_{A/k} := \mathcal{S}_A(\Omega_{A/k})$ and by $\mathbb{R}_{A/k} := \mathcal{R}_A(\Omega_{A/k})$ the symmetric algebra and the Rees algebra of $\Omega_{A/k}$, respectively. If in addition $(A, \mathfrak{m})$ is local and equidimensional then $\Omega_{A/k}$ has a rank, given by

$$(2.1) \qquad \mathrm{rank}\,\Omega_{A/k} = \dim A + \mathrm{trdeg}_k(A/\mathfrak{m}).$$

In this case the above Fitting conditions can be expressed in terms of local embedding dimensions. Namely, $\Omega_{A/k}$ satisfies (F_t) if and only if $\mathrm{edim}\,A_{\mathfrak{p}} \leq 2\dim A_{\mathfrak{p}} - t$ for every non-regular prime $\mathfrak{p} \in \mathrm{Spec}(A)$ (see [22, the proof of 2.3]).

In the following part we put an emphasis on the structure of the symmetric algebra $\mathbb{S}_{A/k}$ as an ancestor of the corresponding Rees algebra $\mathbb{R}_{A/k}$, which will be the topic of the next section.

2.1. **Reducedness: relation to torsionfreeness.** The main result of this part has a curious geometric consequence. Let A be a local domain with an isolated singularity that is essentially of finite type over a perfect field. If the Zariski tangent algebra of A is reduced, but not a domain, then the defining ideal of A has to contain "many" quadrics. In the language of jet schemes, the same conclusion holds if the first jet scheme of $\mathrm{Spec}(A)$ is reduced, but not irreducible.

Theorem 2.1. *Let A be a reduced ring essentially of finite type over a perfect field k. Assume that for every non-minimal $\mathfrak{p} \in \mathrm{Spec}(A)$, $A_{\mathfrak{p}} \simeq R/I$ where $(R, \mathfrak{n})$ is a regular local ring essentially of finite type over k and $I \subset R$ is an ideal satisfying*

$$\mu(I + \mathfrak{n}^3/\mathfrak{n}^3) \leq \dim R - 1.$$

Then $\mathbb{S}_{A/k}$ is reduced (if and) only if it is A-torsionfree (i.e., $\Omega_{A/k}$ is of linear type).

Proof. The "if" statement was explained in the Introduction.

To show the converse, let $\mathcal{T} \subset \mathbb{S}_{A/k}$ denote the A-torsion submodule of $\mathbb{S}_{A/k}$ and suppose that $\mathcal{T} \neq 0$. Then there exists an associated, hence minimal, prime of $\mathbb{S}_{A/k}$ contracting to a non-minimal prime of A. We may further take such a non-minimal prime to be minimal among all non-minimal primes of A that are contracted from some minimal prime of $\mathbb{S}_{A/k}$. By localizing at this non-minimal prime, we do not change either the hypotheses or the conclusion of the statement, so we can reduce the argument to the situation where $(A, \mathfrak{m}, K)$ is local and $\mathfrak{m}$ is the contraction of a minimal prime of $\mathbb{S}_{A/k}$. Moreover, every minimal prime of $\mathbb{S}_{A/k}$ not containing $\mathfrak{m}$ must contract to a minimal prime of A, hence contains the torsion $\mathcal{T}$. From this follows, since $\mathbb{S}_{A/k}$ is reduced, the crucial relation

$$(2.2) \qquad\qquad \mathcal{T} \cap \mathfrak{m}\mathbb{S}_{A/k} = 0 \,.$$

Now $\mathbb{S}_{A/k}/\mathfrak{m}\mathbb{S}_{A/k}$ is a standard graded polynomial ring over K in $n := \mu(\Omega_{A/k})$ variables. The equality (2.2) implies that $\mathcal{T}$ is mapped isomorphically onto its image in this polynomial ring. Let $h(t) := \dim_K [\mathcal{T}]_t$ and $r := \min\{t \geq 0 \,|\, [\mathcal{T}]_t \neq 0\}$. It follows that $h(t)$ is at least the number of monomials of degree $t - r$, in other words,

$$h(t) \geq \binom{t - r + n - 1}{n - 1} \,.$$

Now consider a presentation $A \simeq R/I$ as given by assumption. Further write $m := \mu(I + \mathfrak{n}^3/\mathfrak{n}^3)$ and $(A', \mathfrak{m}') := (A/\mathfrak{m}^2, \mathfrak{m}/\mathfrak{m}^2)$. The usual A-free presentation of $\Omega_{A/k}$ by means of the transpose of a Jacobian matrix induces a presentation

$$A'^{\,m} \longrightarrow A'^{\,n} \longrightarrow A' \otimes_A \Omega_{A/k} \to 0 \,,$$

yielding, for every $t \geq 1$, an exact sequence

$$A'^{\,m} \otimes_{A'} \mathcal{S}_{t-1}(A'^{\,n}) \longrightarrow \mathcal{S}_t(A'^{\,n}) \longrightarrow A' \otimes_A \mathcal{S}_t(\Omega_{A/k}) \to 0 \,.$$

The equality (2.2) also says that, for every $t \geq 0$, the graded component $[\mathcal{T}]_t$ of $\mathcal{T}$ is a K-vector space of dimension $h(t)$ and a direct summand of $\mathcal{S}_t(\Omega_{A/k})$ as an A-module. Hence $A' \otimes_A \mathcal{S}_t(\Omega_{A/k})$ too admits $K^{\oplus h(t)}$ as a direct summand. Therefore $\mathfrak{m}'^{\,\oplus h(t)}$ is a direct summand of the image of $A'^{\,m} \otimes_{A'} \mathcal{S}_{t-1}(A'^{\,n})$ in $\mathcal{S}_t(A'^{\,n})$, which implies that

$$m \binom{t - 1 + n - 1}{n - 1} \geq \mu(\mathfrak{m}')h(t) \geq \mu(\mathfrak{m}') \binom{t - r + n - 1}{n - 1}$$

for every $t \geq 1$. Observing that $\dim R = \mu(\mathfrak{m}) = \mu(\mathfrak{m}')$, this inequality contradicts the assumption $m \leq \dim R - 1$. $\qquad\square$

Corollary 2.2. *Let* $(R, \mathfrak{n})$ *be a regular local ring essentially of finite type over a perfect field* k, *and let* $I \subset R$ *be an ideal such that* $A = R/I$ *is reduced. Assume that one of the following conditions holds*:
(i) $\mu(I_\mathfrak{p}) \leq \dim R_\mathfrak{p} - 1$ *for every non-minimal* $\mathfrak{p} \in V(I)$;
(ii) $I \subset \mathfrak{n}^3$ *and* A *is an isolated singularity.*
   Then $\mathbb{S}_{A/k}$ *is reduced* (*if and*) *only if it is* A-*torsionfree.*

Remark 2.3. The full force of Theorem 2.1 takes form in the case of a homogeneous ideal $I \subset k[\mathbf{X}] = k[X_1, \ldots, X_n]$, with k a perfect field, such that $A = k[\mathbf{X}]/I$ is reduced and regular on the punctured spectrum. If $\mathbb{S}_{A/k}$ is reduced, but *not* torsionfree, then $\mathrm{Proj}\,(k[\mathbf{X}]/I) \subset \mathbb{P}_k^{n-1}$ lies on the intersection of n independent quadrics because in this situation $\dim_k [I]_2 \geq n$ by Theorem 2.1. There is a converse

when $I \subset (\mathbf{X})^2$ is equidimensional and $\dim_k[I]_2 \leq 2\,\mathrm{height}\,I$, namely, if $\mathbb{S}_{A/k}$ is torsionfree then $n \geq 2\,\mathrm{height}\,I + 1$ (cf. the beginning of this section), hence $\mathrm{Proj}\,(k[\mathbf{X}]/I) \subset \mathbb{P}_k^{n-1}$ cannot lie on the intersection of n independent quadrics.

The next two examples illustrate the above results. The first shows that Theorem 2.1 is sharp, i.e., that the assumption on the numbers of generators cannot be relaxed. For a geometric insight it may be convenient to recall the interpretation, given in [19, 5.17], of the tangential variety to $\mathrm{Proj}\,(A) \subset \mathbb{P}_k^{n-1}$ as $\mathrm{Proj}\,(k \otimes_A \mathbb{R}_{A/k})$.

Example 2.4. Consider the homogeneous coordinate ring A of the Veronese surface in $\mathbb{P}_k^5$. Thus $A = k[\mathbf{X}]/I_2(\mathbf{X})$ where $\mathbf{X}$ is a symmetric 3×3 matrix of indeterminates over k (assumed to be of characteristic $\neq 2$). Here A is a Cohen–Macaulay domain with an isolated singularity, $I_2(\mathbf{X})$ is generated by quadrics, and $\mu(I_2(\mathbf{X})) = 6 = \dim k[\mathbf{X}]$. It can be seen with the aid of *Macaulay* that $\mathbb{S}_{A/k}$ is reduced. However, it is not A-torsionfree because $\Omega_{A/k}$ fails to have the expected number of generators locally at the irrelevant ideal A_+ (see the beginning of this section) – alternatively, using the interpretation mentioned above, one could argue that the defining equation of the tangential variety (which coincides with the secant variety, hence is defined by the determinant of $\mathbf{X}$), when read in the presentation variables of $\mathbb{S}_{A/k}$, is a nonzero torsion element. Actually, as in the reduction step of the proof of Theorem 2.1, $\mathbb{S}_{A/k}$ has exactly two minimal primes, $\mathcal{T}$ and $(\mathbf{X})\mathbb{S}_{A/k}$, the first of which defines $\mathbb{R}_{A/k}$.

Another computation with *Macaulay* shows that $\mathbb{S}_{A/k}$ is Cohen–Macaulay, but $\mathcal{S}(\mathbb{D}_{A/k})$ is not. This example incidentally answers a question posed in [10, Remark on p. 142] concerning the existence of a Cohen–Macaulay generic complete intersection ideal $\mathbf{D} \subset S$, with S and $S/\mathbf{D}$ Cohen–Macaulay, for which the symmetric algebra $\mathcal{S}(\mathbf{D}/\mathbf{D}^2)$ is Cohen–Macaulay, but the symmetric algebra $\mathcal{S}(\mathbf{D})$ is not. Mark Johnson obtained examples with the latter behavior even in a polynomial ring ([15, Example 1]).

Example 2.5. Let $A = k[\mathbf{X}]/I_2(\mathbf{X})$ where $(\mathbf{X})$ is the r-catalecticant matrix

$$\begin{pmatrix} X_1 & X_2 & X_3 & X_4 \\ X_{r+1} & X_{r+2} & X_{r+3} & X_{r+4} \end{pmatrix}$$

with $1 \leq r \leq 4$. Note that for $r = 4$ we obtain the 2×4 generic matrix and for $r = 1$, the usual 2×4 generic Hankel matrix. Also A is a Cohen–Macaulay (domain) in all cases, being a specialization of the generic situation. Clearly A is an isolated singularity. For values $1 \leq r \leq 2$ the Zariski tangent algebra $\mathbb{S}_{A/k}$ is neither Cohen–Macaulay nor torsionfree. Geometrically, we see the reason for $\mathbb{S}_{A/k} \neq \mathbb{R}_{A/k}$ since in this range the tangential variety $\mathrm{Proj}\,(k \otimes_A \mathbb{R}_{A/k}) \subset \mathbb{P}_k^{r+3}$ to $\mathrm{Proj}(k[\mathbf{X}]/I_2(\mathbf{X})) \subset \mathbb{P}_k^{r+3}$ is a proper subvariety of the ambient space. For values $3 \leq r \leq 4$ the Zariski tangent algebra $\mathbb{S}_{A/k}$ is Cohen–Macaulay (computer calculation), hence torsionfree because $r + 4 \geq 7 = 2\,\mathrm{height}\,I_2(\mathbf{X}) + 1$ (see [24, 3.3]).

2.2. **Reducedness: an analogue of Berger's conjecture.** When $\dim A = 1$ one can make Theorem 2.1 more precise. Here it is natural to work in the more general setting of complete k-algebras. Instead of the module of Kähler differentials we use the universally finite module of differentials, defined in an analogous way as $\mathbb{D}/\mathbb{D}^2$ where $\mathbb{D}$ now is the kernel of the multiplication map $A\widehat{\otimes}_k A \longrightarrow A$. Our result is reminiscent of Berger's well-known conjecture asserting that the module of

differentials of a reduced curve singularity over a perfect field cannot be torsionfree (see [4]). It is clear at least that the entire symmetric algebra $\mathbb{S}_{A/k}$ cannot be A-torsionfree, as otherwise the module of differentials would satisfy condition (F_1), which translates into the inequality $\operatorname{edim} A \leq 2 \dim A - 1 = 1$ (see the discussion at the beginning of Section 2).

Theorem 2.6. *Let k be an algebraically closed field and let $A = k[[x_1, \ldots, x_n]]$ be a one–dimensional domain. Then $\mathbb{S}_{A/k}$ is reduced (if and) only if A is regular.*

Proof. Assume that $\mathbb{S}_{A/k}$ is reduced. To argue by way of contradiction assume that $n = \operatorname{edim} A \geq 2$. Now, the integral closure of A is the power series ring $k[[t]]$ in one variable. Letting $v(-)$ denote the t-adic valuation, we may arrange so that

$$v(x_1) < v(x_2) < v(x_3) \leq \cdots \leq v(x_n)$$

and $v(x_2)/v(x_1)$ is not an integer (the condition on $v(x_i)$, for $i \geq 3$, being vacuous if $n = 2$). Write $A = k[[X_1, \ldots, X_n]]/(f_1, \ldots, f_m)$ with $f_i \in (X_1, \ldots, X_n)^2$, $T_j = dX_j$ and $t_j = dx_j$, where d denotes the universally finite derivation. Notice that

$$\mathbb{S}_{A/k} \simeq A[t_1, \ldots, t_n] = A[T_1, \ldots, T_n]/(\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} T_j \mid 1 \leq i \leq m) \, .$$

Here $\partial f_i/\partial x_j$ denotes the image of $\partial f_i/\partial X_j$ in A. Clearly these elements are contained in the maximal ideal $\mathfrak{m}$ of A.

As before, let $\mathcal{T}$ denote the A-torsion of $\mathbb{S}_{A/k}$. By the reducedness of $\mathbb{S}_{A/k}$, one has $\mathcal{T} \cap \mathfrak{m}\mathbb{S}_{A/k} = 0$ as in (2.2) – note that the localization reduction step there is needless in the present context. In particular $\operatorname{Supp}(\mathfrak{m}\mathbb{S}_{A/k}) \cap V(\mathfrak{m}\mathbb{S}_{A/k}) \subset V(\mathcal{T} + \mathfrak{m}\mathbb{S}_{A/k})$ as subsets of $\operatorname{Spec}(\mathbb{S}_{A/k})$. On the other hand,

$$\dim \mathbb{S}_{A/k}/(\mathcal{T} + \mathfrak{m}\mathbb{S}_{A/k}) = \dim \mathbb{R}_{A/k}/\mathfrak{m}\mathbb{R}_{A/k} = 1 \, .$$

Since $(\mathfrak{m}\mathbb{S}_{A/k}, t_3, \ldots, t_n)$ is a two-dimensional prime ideal of $\mathbb{S}_{A/k}$, it follows that $(\mathfrak{m}\mathbb{S}_{A/k}, t_3, \ldots, t_n) \notin \operatorname{Supp}(\mathfrak{m}\mathbb{S}_{A/k})$.

Thus $(\mathfrak{m}\,\mathbb{S}_{A/k})_{(\mathfrak{m}\,\mathbb{S}_{A/k}, t_3, \ldots, t_n)} = 0$. From the presentation

$$\mathbb{S}_{A/k}/(t_3, \ldots, t_n) = A[T_1, T_2]/\mathcal{J}$$

with

$$\mathcal{J} = \left(\frac{\partial f_i}{\partial x_1} T_1 + \frac{\partial f_i}{\partial x_2} T_2 \mid 1 \leq i \leq m \right),$$

we see that

$$\mathfrak{m}A[T_1, T_2]_{\mathfrak{m}A[T_1, T_2]} = \mathcal{J}A[T_1, T_2]_{\mathfrak{m}A[T_1, T_2]} \, .$$

In particular, there exists a polynomial $g(T_1, T_2) \in A[T_1, T_2] \setminus \mathfrak{m}A[T_1, T_2]$ with

$$g(T_1, T_2) \cdot \mathfrak{m} \subset \left(\frac{\partial f_i}{\partial x_1} T_1 + \frac{\partial f_i}{\partial x_2} T_2 \mid 1 \leq i \leq m \right).$$

Comparing coefficients, we conclude that

$$\mathfrak{m} = \left(\frac{\partial f_i}{\partial x_1}, \frac{\partial f_i}{\partial x_2} \mid 1 \leq i \leq m \right).$$

Since $v(x_1)$ is the smallest positive element in the value semigroup of A, there exists an i such that $v(x_1) = v(\partial f_i/\partial x_1)$, or $v(x_1) = v(\partial f_i/\partial x_2)$. Write $f := f_i$. Assuming the first, we write $f = aX_1^2 + g$, with $a \in k$ and $g \in (X_1^3) + (X_1, \ldots, X_n)(X_2, \ldots, X_n)$. Obviously $v(g(x_1, \ldots, x_n)) > 2v(x_1)$, hence $v(\partial g/\partial x_1) >$

$v(x_1)$. Therefore $v(\partial f/\partial x_1) = v(x_1)$ implies $a \neq 0$, while $f(x_1, \ldots, x_n) = 0$ implies $a = 0$. This is a contradiction.

Next assume $v(x_1) = v(\partial f/\partial x_2)$ and write

$$f = \sum_{i=2}^{\infty} a_i X_1{}^i + b X_1 X_2 + h,$$

where $a_i \in k$, $b \in k$ and $h \in (X_1^2, X_2) X_2 + (X_1, \ldots, X_n)(X_3, \ldots, X_n)$. Here $v(h(x_1, \ldots, x_n)) > v(x_1) + v(x_2)$, hence $v(\partial h/\partial x_2) > v(x_1)$. Therefore $v(\partial f/\partial x_2) = v(x_1)$ implies $b \neq 0$. But then $f(x_1, \ldots, x_n) = 0$ implies

$$v(x_1) + v(x_2) = v(b x_1 x_2) = v(\sum_{i=2}^{\infty} a_i x_1^i) = l v(x_1)$$

for some integer l. But this contradicts our assumption that $v(x_2)/v(x_1)$ is not an integer. □

Remark 2.7. The above result means that the first jet scheme of an integral algebroid curve singularity over an algebraically closed field cannot be reduced.

The proof of Theorem 2.6 gives the following stronger result:

Theorem 2.8. *Let k be an algebraically closed field, let $A = k[[x_1, \ldots, x_n]]$ be a one-dimensional ring and assume that $\mathbb{S}_{A/k}$ is reduced. Then $A/\mathfrak{p}$ is regular for every minimal prime ideal $\mathfrak{p}$ of A. If in addition $\mathrm{char}(k) = 0$ and A is quasi-homogeneous (i.e., the completion of a positively graded k-algebra), then A is regular.*

Proof. We use the notation of the previous proof. Of course A is reduced. Suppose there exists a minimal prime $\mathfrak{p}$ of A such that $A/\mathfrak{p}$ is not regular. Then choose a minimal generating set $x_1, \ldots, x_n$ of $\mathfrak{m}/\mathfrak{p}$ such that

$$v(x_1 + \mathfrak{p}) < v(x_2 + \mathfrak{p}) < v(x_3 + \mathfrak{p}) \leq \cdots \leq v(x_n + \mathfrak{p}) \leq \infty$$

and the ratio of the first two values is not an integer (here v is the t–adic valuation on the integral closure $k[[t]]$ of $A/\mathfrak{p}$). The rest of the proof proceeds as before, replacing $\mathbb{S}_{A/k}/(t_3, \ldots, t_n)$ by $\mathbb{S}_{A/k}/(\mathfrak{p}, t_3, \ldots, t_n)$.

Now assume that $\mathrm{char}(k) = 0$ and A is quasi-homogeneous. Let τ denote the kernel of the Euler map $\Omega_{A/k} \twoheadrightarrow \mathfrak{m}$. Since τ is the torsion of $\Omega_{A/k}$, by the reducedness of $\mathbb{S}_{A/k}$ we have $\tau \bigcap \mathfrak{m}\Omega_{A/k} = 0$ as in (2.2), hence τ is a direct summand of $\Omega_{A/k}$. It follows that $\tau = 0$ since $\mu(\Omega_{A/k}) = \mathrm{edim}\, R = \mu(\mathfrak{m})$. But then A is regular because in the quasi-homogeneous case the Berger conjecture is true (see [18, 4.4]). □

2.3. **Normality versus reflexivity.** Let A be a normal domain essentially of finite type over a perfect field k. An interesting algebra related to $\mathbb{S}_{A/k}$ is the "reflexive closure" $\mathbb{B}_{A/k} := \mathcal{B}_A(\Omega_{A/k}) = \bigoplus_{i \geq 0} (\Omega_{A/k}^i)^{**}$ of $\mathbb{R}_{A/k} = \bigoplus_{i \geq 0} \Omega_{A/k}^i$, where $-^*$ stands for A-duals. There is a natural A-algebra inclusion $\mathbb{R}_{A/k} \subset \mathbb{B}_{A/k}$ (see Subsection 3.2, where we study cases when this inclusion is an equality). Now, if $\mathbb{S}_{A/k}$ is normal then it is torsionfree, so $\mathbb{S}_{A/k} = \mathbb{R}_{A/k} \subset \mathbb{B}_{A/k}$. In this part we give sufficient conditions in order that, in the presence of the normality of $\mathbb{S}_{A/k}$, the inclusion $\mathbb{S}_{A/k} \subset \mathbb{B}_{A/k}$ be an equality. The conditions are stated, once again,

in terms of the number of "quadrics" contained in I, when I is the defining ideal of A in a regular local ring $(R, \mathfrak{n})$ essentially of finite type over k.

First we prove a more general result.

Proposition 2.9. *Let $(R, \mathfrak{n})$ be a regular local ring essentially of finite type over a perfect field k and let $I \subset \mathfrak{n}^2$ be an ideal such that $A = R/I$ is reduced. Write g for the height of I and $\mathfrak{m}$ for the maximal ideal of A.*

   (a) *If $\mathbb{S}_{A/k}$ is equidimensional and $(\mathbb{S}_{A/k})_{\mathfrak{m}\mathbb{S}_{A/k}}$ is regular, then $\mu(I + \mathfrak{n}^3/\mathfrak{n}^3) \geq 2g$.*

   (b) *Assume that $\mathrm{char}(k) = 0$ and R is the localization of the polynomial ring $k[X_1, \ldots, X_n]$ at its homogeneous maximal ideal. Let I_2 denote the ideal of R generated by 2-forms in $X_1, \ldots, X_n$ such that $I + \mathfrak{n}^3 = I_2 + \mathfrak{n}^3$. If $\mathbb{S}_{A/k}$ is equidimensional and $(\mathbb{S}_{A/k})_{\mathfrak{m}\mathbb{S}_{A/k}}$ is regular, then $\ell(I_2) = 2g$.*

Proof. Knowingly, $\mathbb{S}_{R/k}$ is a polynomial ring over R. In a more precise way, let $d : R \longrightarrow \Omega_{R/k}$ stand for the universal derivation. Then $\mathbb{S}_{R/k} \simeq R[\mathbf{T}]$, where in part (a), $\mathbf{T} = T_1, \ldots, T_n$ may be chosen to be the differentials (i.e., the images under d) of a separating transcendence basis of R over k and, in part (b), as the differentials $dX_1, \ldots, dX_n$. Set $K := R/\mathfrak{n}$, so that the residue field of $R(\mathbf{T}) = R[\mathbf{T}]_{\mathfrak{n}R[\mathbf{T}]}$ is $K(\mathbf{T})$. Write $r := \mathrm{trdeg}_k(A/\mathfrak{m}) = \mathrm{trdeg}_k K$. There is a presentation

$$(\mathbb{S}_{A/k})_{\mathfrak{m}\mathbb{S}_{A/k}} \simeq R(\mathbf{T})/\mathcal{J}.$$

Let $\mathfrak{p}$ be a minimal prime of A with $\dim A/\mathfrak{p} = \dim A$. Since $\mathbb{S}_{A/k}$ is equidimensional, one has $\dim \mathbb{S}_{A/k} = \dim A/\mathfrak{p} + \mathrm{rank}\, \Omega_{A_\mathfrak{p}/k}$ (see [25, the proof of 1.1.1]). Therefore, by (2.1) one gets

$$\dim \mathbb{S}_{A/k} = \dim A/\mathfrak{p} + \mathrm{trdeg}_k(A/\mathfrak{p}) = 2\dim A/\mathfrak{p} + r = 2(n - r - g) + r = 2n - 2g - r.$$

Hence, again by the equidimensionality of $\mathbb{S}_{A/k}$,

$$\begin{aligned} \mathrm{height}\, \mathcal{J} &= \dim R(\mathbf{T}) - \mathrm{height}\, \mathfrak{m}\mathbb{S}_{A/k} = n - r - (\dim \mathbb{S}_{A/k} - \dim \mathbb{S}_{A/k}/\mathfrak{m}\mathbb{S}_{A/k}) \\ &= n - r - (\dim \mathbb{S}_{A/k} - n) = n - r - ((2n - 2g - r) - n) = 2g. \end{aligned}$$

Thus, since $(\mathbb{S}_{A/k})_{\mathfrak{m}\mathbb{S}_{A/k}}$ is regular, we have

$$\dim_{K(\mathbf{T})}(\mathcal{J} + \mathcal{N}^2/\mathcal{N}^2) = 2g,$$

where $\mathcal{N} = \mathfrak{n}R(\mathbf{T})$.

Now let $f_1, \ldots, f_m$ in R be so chosen that they generate I modulo $\mathfrak{n}^3$ for part (a), and that they be homogeneous quadrics in $k[X_1, \ldots, X_n]$ generating the ideal I_2 for part (b). In either case, one has

$$\mathcal{J} + \mathcal{N}^2/\mathcal{N}^2 = (df_i \,|\, 1 \leq i \leq m) + \mathcal{N}^2/\mathcal{N}^2.$$

It follows that $m \geq 2g$, thus proving (a).

To prove (b), let $f := f(\mathbf{X})$ denote any of the chosen 2-forms $f_1, \ldots, f_m$ in $k[\mathbf{X}] = k[X_1, \ldots, X_n]$. Since the Hessian matrix φ of f is symmetric, one has

$$\mathbf{X} \cdot \varphi \cdot \mathbf{T}^t = \mathbf{T} \cdot \varphi \cdot \mathbf{X}^t.$$

As the Hessians of $f(\mathbf{X})$ and of $f(\mathbf{T})$ coincide this shows that

$$\sum_{j=1}^n \frac{\partial f(\mathbf{X})}{\partial X_j} T_j = \sum_{j=1}^n \frac{\partial f(\mathbf{T})}{\partial T_j} X_j,$$

hence

$$df = \sum_{j=1}^{n} \frac{\partial f(\mathbf{T})}{\partial T_j} X_j\,.$$

We deduce that

$$
\begin{aligned}
\dim_{k(\mathbf{T})}((df_i \mid 1 \le i \le m) + \mathcal{N}^2/\mathcal{N}^2) &= \operatorname{rank}\left(\frac{\partial f_i(\mathbf{T})}{\partial T_j}\right)_{i,j} = \operatorname{rank}\left(\frac{\partial f_i(\mathbf{X})}{\partial X_j}\right)_{i,j} \\
&= n - \operatorname{rank}\Omega_{R/k[f_1,\ldots,f_m]} \\
&= n - \operatorname{trdeg}_{k[f_1,\ldots,f_m]} R \\
&= \operatorname{trdeg}_k k[f_1,\ldots,f_m] = \dim k[f_1,\ldots,f_m]
\end{aligned}
$$

(see also [20, 1.1]). On the other hand, since $f_1,\ldots,f_m$ are forms of the same degree generating I_2, we have $k[f_1,\ldots,f_m] \simeq k\otimes_R \mathcal{R}(I_2)$, hence $\dim k[f_1,\ldots,f_m] = \ell(I_2)$. Summing up, we have proved that

$$2g = \dim_{k(\mathbf{T})}(\mathcal{J} + \mathcal{N}^2/\mathcal{N}^2) = \ell(I_2).$$

$\square$

Theorem 2.10. *Let A be a normal domain essentially of finite type over a perfect field k. Assume that one of the following conditions holds:*

(i) *For every non-regular prime $\mathfrak{p} \in \operatorname{Spec}(A)$, $A_\mathfrak{p} \simeq R/I$ where $(R,\mathfrak{n})$ is a regular local ring essentially of finite type over k and $I \subset \mathfrak{n}^2$ is an ideal satisfying $\mu(I + \mathfrak{n}^3/\mathfrak{n}^3) \le 2\operatorname{height} I - 1$;*

(ii) *(char$(k) = 0$) A is an isolated singularity and has a presentation $A \simeq R/I$ with $(R,\mathfrak{n})$ the localization of $k[X_1,\ldots,X_n]$ at $(X_1,\ldots,X_n)$, such that $\ell(I_2) \ne 2\operatorname{height} I$; here I_2 denotes the ideal of R generated by 2-forms in $X_1,\ldots,X_n$ such that $I + \mathfrak{n}^3 = I_2 + \mathfrak{n}^3$.*

Then $\mathbb{S}_{A/k}$ is normal (if and) only if $\mathbb{S}_{A/k} = \mathbb{B}_{A/k}$.

Proof. Assume that $\mathbb{S}_{A/k}$ is normal. In this case $\mathbb{S}_{A/k}$ is a domain, since $\operatorname{Spec}(\mathbb{S}_{A/k})$ is connected. We prove that the natural inclusion $\mathbb{S}_{A/k} \subset \mathbb{B}_{A/k}$ is an equality. It suffices to show this after localizing at the height one primes of $\mathbb{S}_{A/k}$. Thus, let $\mathfrak{q} \in \operatorname{Spec}(\mathbb{S}_{A/k})$ be a prime of height one. We are done once we prove that $A_{\mathfrak{q}\cap A}$ is regular. Now, replacing A by $A_{\mathfrak{q}\cap A}$, we may assume that $(A,\mathfrak{m})$ is local and $\mathfrak{m} \subset \mathfrak{q}$. By assumption, $(\mathbb{S}_{A/k})_\mathfrak{q}$ is regular, hence so is its further localization $(\mathbb{S}_{A/k})_{\mathfrak{m}\mathbb{S}_{A/k}}$. According to Proposition 2.9 this contradicts our assumptions, unless A is regular. $\square$

Corollary 2.11. *Let $(R,\mathfrak{n})$ be a regular local ring essentially of finite type over a perfect field k and let $I \subset \mathfrak{n}^2$ be an ideal such that $A = R/I$ is an equidimensional isolated singularity. If $\mathbb{S}_{A/k}$ is reduced and $(\mathbb{S}_{A/k})_{\mathfrak{m}\mathbb{S}_{A/k}}$ is regular then*

$$\mu(I + \mathfrak{n}^3/\mathfrak{n}^3) \ge \min\{\dim R, 2\operatorname{height} I\}.$$

Proof. If $\mu(I + \mathfrak{n}^3/\mathfrak{n}^3) \le \dim R - 1$ then $\mathbb{S}_{A/k}$ is torsionfree by Theorem 2.1. Since A is reduced and equidimensional it follows that $\mathbb{S}_{A/k}$ is equidimensional as well (see (2.1) and [25, the proof of 1.1.1]). Therefore $\mu(I + \mathfrak{n}^3/\mathfrak{n}^3) \ge 2\operatorname{height} I$ by Proposition 2.9(a). $\square$

3. The Rees algebra of the module of differentials

3.1. **Cohen–Macaulayness of** $\mathbb{R}_{A/k}$. Let A be a local Cohen–Macaulay ring and let E be a finite A-module of projective dimension one. If E satisfies condition (F_1) then the symmetric algebra $\mathcal{S}_A(E)$ is a Cohen–Macaulay torsionfree A-algebra (see [1, Propositions 3 and 4], [13, 1.1], [24, 3.4]) – in particular, the Rees algebra $\mathcal{R}_A(E)$ is Cohen–Macaulay. The question as to whether, conversely, $\mathcal{R}_A(E)$ being Cohen–Macaulay implies (F_1) has a negative answer in general even if E satisfies (F_0) (see [23, 4.7]). The theorem below will show that this converse holds when E is the module of differentials of a complete intersection over a field of characteristic zero. This shows that Cohen–Macaulayness is a rather restrictive property for $\mathbb{R}_{A/k}$.

Theorem 3.1. *Let k be a field of characteristic zero and let A be a k-algebra essentially of finite type which is locally a complete intersection. Assume the following conditions*:

(i) $\operatorname{edim} A_{\mathfrak{p}} \leq 2 \dim A_{\mathfrak{p}}$ *for every prime* $\mathfrak{p} \in \operatorname{Spec}(A)$;
(ii) $\mathbb{R}_{A/k}$ *is Cohen–Macaulay.*

Then $\operatorname{edim} A_{\mathfrak{p}} \leq 2 \dim A_{\mathfrak{p}} - 1$ *for every non-minimal prime* $\mathfrak{p} \in \operatorname{Spec}(A)$.

Proof. Arguing by way of contradiction, let $\mathfrak{p} \in \operatorname{Spec}(A)$ be minimal such that $\operatorname{edim} A_{\mathfrak{p}} \geq 2 \dim A_{\mathfrak{p}} \geq 2$. By localizing at this prime, we may assume that $(A, \mathfrak{m})$ is local and, by (i), that $\operatorname{edim} A = 2 \dim A$.

Next we reduce to the case where the residue field $K := A/\mathfrak{m}$ is algebraic over k. To do this, let r be the transcendence degree of K over k and suppose $r \geq 1$. Write $A = k[x_1, \ldots, x_m]_{\mathfrak{p}}$ and pick r general k-linear combinations $y_1, \ldots, y_r$ of $x_1, \ldots, x_m$ such that, in particular, their residues yield a transcendence basis of K over k. Notice that $L := k(y_1, \ldots, y_r)$ is a subfield of A. Furthermore, $\Omega_{A/L} = \Omega_{A/k}/(A dy_1 + \cdots + A dy_r)$ with $d : A \longrightarrow \Omega_{A/k}$ denoting the universal derivation. On the other hand, $\operatorname{height}(\Omega_{A/k}\mathbb{R}_{A/k}) = \operatorname{rank} \Omega_{A/k}$ by [23, 2.2] and $\operatorname{rank} \Omega_{A/k} > r$ according to (2.1). Hence the general choice of $y_1, \ldots, y_r$ and the Cohen–Macaulayness of $\mathbb{R}_{A/k}$ yield an isomorphism

$$\mathcal{R}(\Omega_{A/L}) \simeq \mathcal{R}(\Omega_{A/k})/(dy_1, \ldots, dy_r),$$

which shows in particular that this Rees algebra is again Cohen–Macaulay (see [6, 2.2(f)]). Thus, replacing k by L, we may assume henceforth that K is algebraic over k.

Set $d := \dim A$ (a number, not the universal derivation!) and induct on d. If $d = 1$, the Cohen–Macaulayness of $\mathbb{R}_{A/k}$ implies that $\Omega_{A/k}$ modulo torsion is free (see [23, 4.3]), which would force A to be regular (see [16, Theorem]), thus contradicting the equality $\operatorname{edim} A = 2$.

Hence we may suppose that $d \geq 2$. Write $A = R/I$, where $(R, \mathfrak{n})$ is a regular local ring essentially of finite type over k and $I \subset \mathfrak{n}^2$ is an ideal. Write $n := \dim R = \operatorname{edim} A = 2d$ and let $\delta : R \longrightarrow \Omega_{R/k}$ be the universal derivation. Since R is smooth over k and the extension $k \subset K$ is separable algebraic, there exist n elements $X_1, \ldots, X_n$ in $\mathfrak{n}$ so that $\delta X_1, \ldots, \delta X_n$ form a basis of $\Omega_{R/k}$. For an element f in R we can then define the "partial derivatives" $\frac{\partial f}{\partial X_i}$ via the equation $\delta f = \sum \frac{\partial f}{\partial X_i} dX_i$, and we write $\frac{\partial f}{\partial x_i}$ for their images in A. By assumption, I is

generated by an R-regular sequence $f_1, \ldots, f_d$. We consider the n by d matrices

$$\Theta = \left(\frac{\partial f_j}{\partial X_i} \right) \quad \text{and} \quad \theta = \left(\frac{\partial f_j}{\partial x_i} \right).$$

Now θ presents the A-module $\Omega_{A/k}$, which has projective dimension one and satisfies condition (F_1) locally in codimension $d-1 \geq 1$ by the inductive hypothesis. The analytic spread of this module is at most $\dim A + \operatorname{rank} \Omega_{A/k} - 1$ (see [23, 2.3]) and the latter number is $2d - 1 = n - 1$ (see (2.1)). Since $\mathbb{R}_{A/k}$ is Cohen–Macaulay, we may then apply [23, 2.3, 4.7(a) and 4.8] with $s = n - 1$ to conclude that after a k-linear change of basis,

$$I_1(\theta) = \left(\frac{\partial f_1}{\partial x_n}, \ldots, \frac{\partial f_d}{\partial x_n} \right).$$

Thus

$$I_1(\Theta) \subset \left(\frac{\partial f_1}{\partial X_n}, \ldots, \frac{\partial f_d}{\partial X_n}, I \right).$$

Set $J := I_1(\Theta)$ and $J_0 := (\frac{\partial f_1}{\partial X_n}, \ldots, \frac{\partial f_d}{\partial X_n})$, and notice that $J + I = J_0 + I$ by the above. Furthermore height $J_0 \leq d$.

We show that

$$I \subset \overline{\mathfrak{n}J},$$

where $\bar{}$ denotes integral closure. Thus, let f be an element of $\mathfrak{n}$. We argue that

$$f \in \overline{\mathfrak{n} \left(\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n} \right)}.$$

This is well-known in slightly different settings (see, e.g., [26, 7.1.5]), but for lack of the particular reference needed here we include a proof. By the valuative criterion for integral dependence it suffices to show that $f \in \mathfrak{n}(\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n})V$ for every local extension $R \subset V$ with V a complete discrete valuation ring (see, e.g., [26, 6.8.4]). Since the extension $k \subset K$ is separable, there exists a coefficient field K of V that contains k. Clearly $V \simeq K[[t]]$. Write $D : K[[t]] \longrightarrow K[[t]] \, Dt \simeq K[[t]]$ for the universally finite K-derivation of $K[[t]]$, which maps a power series $h(t)$ to $h'(t) \, Dt$. As $R \subset K[[t]]$ and $k \subset K$, one can restrict D to R and this restriction factors though δ. Thus $Df = \sum \frac{\partial f}{\partial X_i} DX_i$ or, equivalently, $f' = \sum \frac{\partial f}{\partial X_i} X_i'$. Since the characteristic of K is zero, one has $hV = th'V$ for every power series $h \in V$ that in not a unit in V. Recall that f and $X_1, \ldots, X_n$ are in $\mathfrak{n}$, hence in the maximal ideal of V. Thus, multiplying both sides of the above equation for f' by t, one sees that indeed $f \in \sum \frac{\partial f}{\partial X_i} X_i V \subset \mathfrak{n}(\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n})V$.

Having shown the inclusion $I \subset \overline{\mathfrak{n}J}$, we conclude that

$$J + I = J_0 + I \subset J_0 + \overline{\mathfrak{n}J} \subset \overline{J_0 + \mathfrak{n}(J + I)}.$$

The valuative criterion for integral dependence now yields $J + I \subset \overline{J_0}$. Therefore height$(J + I) = $ height $J_0 \leq d$. Hence $\dim R/(J + I) \geq 2d - d = d \geq 2$. But this is impossible because $R/(J + I) \simeq A/I_1(\theta)$ has dimension zero by assumption (i). $\quad \square$

Remark 3.2. The theorem implies the following geometric result: let $X \subset \mathbb{P}_k^{2d+1}$ denote a d-dimensional smooth complete intersection over a field k of characteristic zero, with homogeneous coordinate ring A. If $\mathbb{R}_{A/k}$ is Cohen–Macaulay then X is degenerate.

Question 3.3. It seems reasonable to ask whether Theorem 3.1 holds without assumption (i).

3.2. **Normality of** $\mathbb{R}_{A/k}$**.** Let A be a normal domain and let E be a finite A-module of rank r. We write E^i for the ith graded component $\mathcal{R}_A(E)_i$ of $\mathcal{R}_A(E)$ and call it the ith *power* of E. Notice that $E^i = \mathcal{S}_i(E)/\tau_A(\mathcal{S}_i(E))$. Since E has rank r, E^1 can be embedded into $F := A^r$ and any such embedding induces an embedding of $\mathcal{R}_A(E)$ into a polynomial ring $\mathcal{S}_A(F) = A[\mathbf{t}] = A[t_1, \ldots, t_r]$ as a graded A-subalgebra. Let $\overline{\mathcal{R}_A(E)}$ denote the integral closure of $\mathcal{R}_A(E)$ in $A[\mathbf{t}]$, which is again a graded A-subalgebra. Since A is normal, this integral closure coincides with the integral closure of $\mathcal{R}_A(E)$ in its field of fractions, and in particular does not depend on the chosen embedding $E^1 \subset F$. We denote by $\overline{E^i} \subset A[\mathbf{t}]_i$ the ith graded component of $\overline{\mathcal{R}_A(E)}$ and call it the *ith normalized power* of E. We say that E is *integrally closed* if $E^1 = \overline{E^1}$ and that E is *normal* if $E^i = \overline{E^i}$ for every $i \geq 1$, or equivalently, if $\mathcal{R}_A(E)$ is normal. One can see that there are inclusions

$$(3.1) \qquad E^i \subset \overline{E^i} \subset (E^i)^{**} \subset \mathcal{S}_i(A^r)\,.$$

Recall that $\mathcal{B}_A(E) := \bigoplus_{i \geq 0}(E^i)^{**}$; the property (L_2) used in the next proposition has been defined in the preliminaries of Section 2.

Proposition 3.4. *Let A be a universally catenary normal domain and let E be a finite A-module. The following conditions are equivalent:*

(1) *$\mathcal{R}_A(E)$ is normal and E satisfies condition (L_2);*
(2) *$\mathcal{R}_A(E)$ satisfies condition (S_2) of Serre and E satisfies condition (L_2);*
(3) *$\mathcal{R}_A(E) = \mathcal{B}_A(E)$.*

Proof. $(1) \Rightarrow (2)$: This is obvious.

$(2) \Rightarrow (3)$: It suffices to show that all powers of E satisfy Serre's condition (S_2). For that let $\mathfrak{p} \subset A$ be any prime ideal of height at least two. We may assume that A is local with maximal ideal $\mathfrak{m} = \mathfrak{p}$. Then height $\mathfrak{m}\mathcal{R}_A(E) = \dim \mathcal{R}_A(E) - \ell(E) = \dim A + \operatorname{rank} E - \ell(E) \geq 2$, the first equality by the catenarian assumption, the second one by a standard formula (see [23, 2.2]), and the last inequality by (L_2). Since $\mathcal{R}_A(E)$ satisfies condition (S_2), the ideal $\mathfrak{m}$ then contains an $\mathcal{R}_A(E)$-regular sequence of two elements. This is also a regular sequence on each power E^i, hence depth $E^i \geq 2$ for every $i \geq 1$.

$(3) \Rightarrow (1)$: By (3.1), we have equality of powers and normalized powers throughout. This means that $\mathcal{R}_A(E)$ is normal.

Next let $\mathfrak{p} \subset A$ be a prime ideal of height at least two. We may assume that A is local with maximal ideal $\mathfrak{m} = \mathfrak{p}$. To prove condition (L_2) it suffices to show that height $\mathfrak{m}\mathcal{R}_A(E) \geq 2$. We will actually show that $\mathfrak{m}\mathcal{R}_A(E)$ has grade at least two.

To see this, let $a \in \mathfrak{m} \setminus \{0\}$. We may assume that E is torsionfree. Since A is normal, E is free locally in codimension one. Hence each of the $A/(a)$-modules E^i/aE^i has a rank, and this rank is the same as the rank of $(E/aE)^i$. Furthermore, the A-torsion of $\mathcal{S}_A(E)$ vanishes locally at the associated primes in A of $A/(a)$, therefore it maps to the $A/(a)$-torsion of $\mathcal{S}_{A/(a)}(E/aE)$. It follows that there is an induced surjective map $\mathcal{R}_A(E) \twoheadrightarrow \mathcal{R}_{A/(a)}(E/aE)$, hence a surjective map $\mathcal{R}_A(E)/a\mathcal{R}_A(E) \twoheadrightarrow \mathcal{R}_{A/(a)}(E/aE)$ (see [19, 4.5]). Now, for rank reasons the kernel of this map is an $A/(a)$-torsion module. On the other hand, since a is regular on every power E^i and these are assumed to be reflexive, E^i/aE^i is torsionfree

for every $i \geq 1$. It follows that the map $\mathcal{R}_A(E)/a\mathcal{R}_A(E) \twoheadrightarrow \mathcal{R}_{A/(a)}(E/aE)$ is an isomorphism.

Since A is normal of dimension at least two, the grade of $\mathfrak{m}$ is at least two. Therefore the grade of $\mathfrak{m}/(a)$ in $A/(a)$ is at least one, thus the grade of its extension to $\mathcal{R}_{A/(a)}(E/aE)$ is at least one. By the above isomorphism we then get that the ideal $\mathfrak{m}\mathcal{R}_A(E)/a\mathcal{R}_A(E)$ has grade at least one, hence $\mathfrak{m}\mathcal{R}_A(E)$ has grade at least two, as required. $\qquad\qquad\square$

In the special case of projective dimension one, we can improve the above result. For this, we will rely on a basic property of integrally closed modules. Let E be an integrally closed submodule of a free module A^r over a normal domain A. The associated primes of A^r/E are particularly restricted when E is a module of finite projective dimension. This phenomenon was described in the case of ideals by Burch ([5, Corollary 3 on p. 947]). More recently, using a different approach, the question was taken up by Goto and Hayasaka ([7, 1.1 and 2.3]) in the case of ideals and by Hong ([12, 4.8]) in the case of modules. In the latter approaches the condition known as $\mathfrak{m}$-fullness was used. Let $(A, \mathfrak{m})$ be a Noetherian local ring with infinite residue field and let $E \subset A^r$ be a submodule of a free module. One says that E is $\mathfrak{m}$-*full* if there exists an element $x \in \mathfrak{m}$ such that

$$\mathfrak{m}E :_{A^r} x = E\,.$$

The valuative criterion for integral dependence shows that an integrally closed module $E \subset A^r$ is $\mathfrak{m}$-full provided the residue field of the normal local ring A is infinite.

We will make use of the following two results which are restated here for the reader's convenience. Most properties of the notion of $\mathfrak{m}$-fullness arise from the first of these results (see [29, 8.101] for details), including the second one.

Proposition 3.5. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring and* E *a submodule of* A^r *such that* $\mathfrak{m}E :_{A^r} x = E$ *for some* $x \in \mathfrak{m}$. *Then*

$$E/xE \simeq (E :_{A^r} \mathfrak{m})/E \oplus (E + xA^r)/xA^r \simeq (E :_{A^r} \mathfrak{m})/E \oplus E/x(E :_{A^r} \mathfrak{m})\,.$$

Theorem 3.6. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring and* E *a submodule of* A^r *such that* $\mathfrak{m}E :_{A^r} x = E$ *for some regular element* $x \in \mathfrak{m}$. *Suppose that* $\mathrm{projdim}(E) < \infty$ *and* $\mathfrak{m} \in \mathrm{Ass}(A^r/E)$. *Then* A *is regular.*

In general, the normality of the Rees algebra $\mathcal{R}_A(E)$ of a finite module E over a normal domain A fails to imply that E satisfies (L_2), or even (F_2). This may be the case even if E has projective dimension one – for an easy example take E to be the homogeneous maximal ideal of a polynomial ring in two variables over a field. Surprisingly, the implication does hold for modules whose non-free locus is contained in the singular locus of the ring, as is stated in the next theorem.

Theorem 3.7. *Let* A *be a normal Cohen–Macaulay domain and let* E *be a finite* A-*module of rank* r *such that*:
(i) E *has a projective resolution* $0 \to P_1 \longrightarrow P_0 \longrightarrow E \to 0$;
(ii) $E_{\mathfrak{p}}$ *is* $A_{\mathfrak{p}}$-*free for every prime* $\mathfrak{p} \in \mathrm{Spec}(A)$ *such that* $A_{\mathfrak{p}}$ *is regular.*
   The following are equivalent:
   (1) $\mathcal{R}_A(E)$ *is normal*;
   (2) $\overline{E^i} = E^i$ *in the range* $1 \leq i \leq \mathrm{rank}\, P_0 - r$;

(3) E satisfies condition (F_2);
(4) $\mathcal{R}_A(E) = \mathcal{B}_A(E)$.

Proof. (1) $\Rightarrow$ (2): This is obvious.

(2) $\Rightarrow$ (3): Let $\mathfrak{p} \in \mathrm{Spec}(A)$ be a prime minimal with the property that (F_2) fails. Localizing at $\mathfrak{p}$, we may assume that A is a local ring of dimension $d \geq 2$, $\mu(E) \geq d + r - 1$, and E satisfies (F_2) locally on the punctured spectrum of A. Consider a minimal free resolution

$$0 \to A^s \longrightarrow A^n \longrightarrow E \to 0.$$

By assumption $n \geq d + r - 1$, hence $s \geq d - 1$. Since (F_0) holds on the punctured spectrum, it follows that the Weyman complex is a minimal free resolution of the symmetric power $\mathcal{S}_{d-1}(E)$ (see [31, Theorem 1(b)]). This complex has length $d-1$ because $s \geq d - 1$. Moreover E satisfies (F_1) on the punctured spectrum, therefore $\mathcal{S}_{d-1}(E)$ is torsionfree. Thus $E^{d-1} \simeq \mathcal{S}_{d-1}(E)$ has projective dimension $d - 1$. In particular, $\mathfrak{m} \in \mathrm{Ass}\,(\mathcal{S}_{d-1}(A^r)/E^{d-1})$. Since $d - 1 \leq s \leq \mathrm{rank}\, P_0 - r$, we can apply the main assumption to conclude that E^{d-1} is integrally closed. Therefore A is regular by Theorem 3.6. This is a contradiction vis-à-vis (ii) since E is not free.

(3) $\Rightarrow$ (4): This is well-known for modules of projective dimension one over Cohen–Macaulay rings (see, e.g., [1, Proposition 4]).

(4) $\Rightarrow$ (1): This follows from Proposition 3.4. $\qquad\square$

Theorem 3.7 applies naturally to the module $E = \Omega_{A/k}$ via the intervention of the Jacobian criterion, which says that the non-free locus of the module of differentials coincides with the singular locus of a reduced ring A essentially of finite type over a perfect field k. If in addition A is locally a complete intersection, then $E = \Omega_{A/k}$ also has projective dimension ≤ 1.

Corollary 3.8. *Let R be a regular domain essentially of finite type over a perfect field, let I be an R-ideal of height g that is locally a complete intersection, and assume that $A = R/I$ is a normal domain. The following are equivalent*:

(1) $\mathbb{R}_{A/k}$ *is normal*;
(2) $\overline{\Omega^i_{A/k}} = \Omega^i_{A/k}$ *in the range* $1 \leq i \leq g$;
(3) $\mathrm{edim}\, A_{\mathfrak{p}} \leq 2\dim A_{\mathfrak{p}} - 2$ *for every non-regular prime* $\mathfrak{p} \in \mathrm{Spec}(A)$;
(4) $\mathbb{R}_{A/k} = \mathbb{B}_{A/k}$.

Proof. Note that condition (3) translates into $\Omega_{A/k}$ satisfying (F_2) according to the preliminaries of Section 2. The rest follows immediately from Theorem 3.7. $\qquad\square$

The next result treats the case of rings that are not necessarily complete intersections. Here we show, in particular, that if $\mathbb{S}_{A/k}$ is normal then the failure of the equality $\mathbb{S}_{A/k} = \mathbb{B}_{A/k}$ forces the existence of "quadrics" in the defining ideal of A, in a way reminiscent of Proposition 2.9.

Proposition 3.9. *Let R be a regular ring essentially of finite type over a perfect field k and let I be an ideal such that $A = R/I$ is a normal domain. If, for some $i \geq 1$, $\mathcal{S}_i(\Omega_{A/k}) \simeq \Omega^i_{A/k}$ and $\Omega^i_{A/k}$ is integrally closed, then $I \not\subset \mathfrak{p}^{(3)}$ for every $\mathfrak{p} \in \mathrm{Ass}_R((\Omega^i_{A/k})^{**}/\Omega^i_{A/k})$.*

Proof. We replace R by $R_{\mathfrak{p}}$ to assume that $(R, \mathfrak{n})$ and $(A, \mathfrak{m})$ are local. We may also suppose that the residue field is infinite. Notice that $\mathfrak{m} \in \mathrm{Ass}_A((\Omega^i_{A/k})^{**}/\Omega^i_{A/k})$, in

particular $\dim A \geq 2$. Since $\mathcal{S}_i(\Omega_{A/k}) \simeq \Omega^i_{A/k}$, the presentation

$$I/I^2 \longrightarrow A^n = A \otimes_R \Omega_{R/k} \longrightarrow \Omega_{A/k} \to 0$$

induces an exact sequence

(3.2) $$I/I^2 \otimes_A S_{i-1}(A^n) \longrightarrow S_i(A^n) \longrightarrow \Omega^i_{A/k} \to 0 \,.$$

Embed $\Omega^i_{A/k} \subset (\Omega^i_{A/k})^{**} \subset A^t$. Since $\Omega^i_{A/k}$ is assumed to be integrally closed, it is $\mathfrak{m}$-full. Thus there is an element $x \in \mathfrak{m}$ satisfying the hypothesis of Proposition 3.5 with $E = \Omega^i_{A/k}$. Clearly $\mathfrak{m} \in \mathrm{Ass}_A(A^t/\Omega^i_{A/k})$, hence $\Omega^i_{A/k} :_{A^t} \mathfrak{m} \neq \Omega^i_{A/k}$. Therefore the proposition implies that $\Omega^i_{A/k}/x\,\Omega^i_{A/k}$ has $A/\mathfrak{m}$ as a direct summand over $A/(x)$. On the other hand, tensoring (3.2) with $A/(x)$, we see that the syzygies of $\Omega^i_{A/k}/x\,\Omega^i_{A/k}$ have coefficients in the $A/(x)$-ideal generated by the entries of the Jacobian matrix of the generators of I. As $\mathfrak{m}/(x) \neq 0$, we cannot have $I \subset \mathfrak{n}^3$. $\square$

4. Structure results in low codimension

4.1. **Normality revisited.** We now focus on the behavior of $\mathbb{R}_{A/k}$ in low dimension or low embedding codimension. A highlight is the closer relationship between the normality of $\mathbb{R}_{A/k}$ and the local structure of A than is typical of more general Rees algebras.

When A is a Noetherian local ring, we write $\mathrm{ecodim}\, A := \mathrm{edim}\, A - \dim A$.

Let A be a local Cohen–Macaulay ring essentially of finite type over a perfect field k and assume $\mathrm{ecodim}\, A \leq 2$. Write $A \simeq R/I$, where R is a regular local ring essentially of finite type over k and I an ideal of height ≤ 2. In this case the first Koszul homology module $H_1(I)$ on a generating set of I is Cohen–Macaulay (see [2, 2.1(a)]). Thus, if I is generically a complete intersection, then the natural complex

(4.1) $$0 \to H_1(I) \longrightarrow A^m \longrightarrow I/I^2 \to 0$$

is exact. One always has a complex of graded modules over the polynomial ring $B := \mathcal{S}_A(\Omega_{R/k} \otimes_R A) \simeq \mathcal{S}_A(A^n)\,,$

(4.2) $$0 \to \mathrm{Tor}_2^R(A, A) \otimes_A B(-2) \longrightarrow \mathrm{Tor}_1^R(A, A) \otimes_A B(-1) \longrightarrow B \longrightarrow \mathbb{S}_{A/k} \to 0\,,$$

which is nothing but the $\mathcal{M}$-complex of the diagonal ideal $\mathbb{D}_{A/k} \subset A \otimes_k A$ (see [22, Section 2.2] for details). If $\mathrm{height}\, I \leq 1$ then the module $\mathrm{Tor}_2^R(A, A)$ vanishes, whereas for $\mathrm{height}\, I = 2$ it is isomorphic to $(\omega_A)^*$, the A-dual of the canonical module of A. The complex (4.2) can be used to derive properties of $\mathbb{S}_{A/k}$ and of $\mathbb{R}_{A/k}$:

Proposition 4.1. *Let A be a reduced local Cohen–Macaulay ring essentially of finite type over a perfect field k. Assume that $\mathrm{ecodim}\, A \leq 2$ and that $\mathrm{edim}\, A_{\mathfrak{p}} \leq 2 \dim A_{\mathfrak{p}}$ for every prime $\mathfrak{p} \in \mathrm{Spec}(A)$.*

(a) *The sequence (4.2) is exact and $\mathbb{S}_{A/k}$ is Cohen–Macaulay;*
(b) *$\Omega_{A/k}$ is of linear type if and only if $\mathrm{edim}\, A_{\mathfrak{p}} \leq 2 \dim A_{\mathfrak{p}} - 1$ for every non-minimal prime $\mathfrak{p} \in \mathrm{Spec}(A)$;*
(c) *In case A is normal, $\Omega_{A/k}$ is of linear type and normal if and only if $\mathbb{S}_{A/k} = \mathbb{B}_{A/k}$ if and only if $\mathrm{edim}\, A_{\mathfrak{p}} \leq 2 \dim A_{\mathfrak{p}} - 2$ for every non-regular prime $\mathfrak{p} \in \mathrm{Spec}(A)$.*

Proof. Write $A = R/I$ as above, with height $I = 2$.

(a): Notice that

$$(\omega_A)^* \simeq \operatorname{Hom}_A(\omega_A \otimes_A \omega_A, \omega_A) \simeq \operatorname{Hom}_A(S_2(\omega_A), \omega_A).$$

By [30, 1.3], $S_2(\omega_A)$ is a maximal Cohen–Macaulay A-module, hence $\operatorname{Tor}_2^R(A, A)$ is zero or maximal Cohen–Macaulay. Moreover, (4.1) gives depth $I/I^2 \geq \dim A - 1$. Since these depth conditions also hold locally at every prime and since $\Omega_{A/k}$ satisfies condition (F_0), the Acyclicity Lemma implies that (4.2) is exact. It also follows that depth $\mathbb{S}_{A/k} \geq \dim A + n - 2$. On the other hand, $\dim \mathbb{S}_{A/k} = \dim A + n - 2$ again because $\Omega_{A/k}$ satisfies (F_0) (see [24, 2.2]). Therefore $\mathbb{S}_{A/k}$ is Cohen–Macaulay.

(b): From part (a) we have that $\mathbb{S}_{A/k}$ is unmixed. In this case, the module $\Omega_{A/k}$ is of linear type if and only if it satisfies (F_1) (see [24, 3.3 and the first remark on p. 346]). Alternatively, one could use the exact sequence (4.2).

(c): Assume that $\Omega_{A/k}$ is of linear type and normal. We show the equality $\mathbb{S}_{A/k} = \mathbb{B}_{A/k}$. For any non-regular prime $\mathfrak{p} \in \operatorname{Spec}(A)$ write $A_{\mathfrak{p}} \simeq R/I$, where $(R, \mathfrak{n})$ is a regular local ring essentially of finite type over k and $I \subset \mathfrak{n}^2$ is an ideal of height $g \leq 2$. By Theorem 2.10, it suffices to show that $\mu(I + \mathfrak{n}^3/\mathfrak{n}^3) \leq 2g - 1$. This is clear if $g = 1$ or if $g = 2$ and $\mu(I) \leq 3$. Thus we may assume that $g = 2$ and $\mu(I) \geq 4$. But in this situation the Hilbert–Burch theorem gives $I \subset \mathfrak{n}^3$.

If $\mathbb{S}_{A/k} = \mathbb{B}_{A/k}$ then $\Omega_{A/k}$ is of linear type and, in particular, free locally in codimension one. Hence the conditions (F_2) and (L_2) are equivalent. Thus Proposition 3.4 shows that $\Omega_{A/k}$ satisfies (F_2).

Finally, if $\Omega_{A/k}$ has the (F_2) property then it is of linear type by part (b). Moreover part (a) implies that $\mathbb{S}_{A/k}$ is Cohen–Macaulay. Thus $\mathbb{S}_{A/k} = \mathbb{R}_{A/k}$ satisfies condition (2) of Proposition 3.4, hence the proposition implies that $\Omega_{A/k}$ is normal. $\qquad\square$

We now treat the case where A is a local Gorenstein algebra of embedding codimension 3. Write $A \simeq R/I$, where R is a regular local ring essentially of finite type over a perfect field k and I is an ideal of height 3. By [8, 2.8], I/I^2 is a maximal Cohen–Macaulay A-module and therefore $(I/I^2)^{**} \simeq I/I^2$. Thus, starting from a presentation of $\Omega_{A/k}$, we obtain the $\mathcal{Z}$-complex

$$(4.3) \qquad 0 \to (\wedge^3 I/I^2)^{**} \otimes_A B(-3) \longrightarrow (\wedge^2 I/I^2)^{**} \otimes_A B(-2) \longrightarrow (I/I^2) \otimes_A B(-1)$$
$$\longrightarrow B \longrightarrow \mathbb{S}_{A/k} \to 0,$$

where B is the polynomial ring $\mathcal{S}_A(\Omega_{R/k} \otimes_R A)$.

Proposition 4.2. *Let A be a reduced local Gorenstein ring essentially of finite type over a perfect field k. Assume that $\operatorname{ecodim} A \leq 3$ and that $\operatorname{edim} A_{\mathfrak{p}} \leq 2 \dim A_{\mathfrak{p}}$ for every prime $\mathfrak{p} \in \operatorname{Spec}(A)$.*

(a) *The sequence (4.3) is exact and $\mathbb{S}_{A/k}$ is Gorenstein;*

(b) *$\Omega_{A/k}$ is of linear type if and only if $\operatorname{edim} A_{\mathfrak{p}} \leq 2 \dim A_{\mathfrak{p}} - 1$ for every non-minimal prime $\mathfrak{p} \in \operatorname{Spec}(A)$;*

(c) *In case A is normal, $\Omega_{A/k}$ is of linear type and normal if and only if $\mathbb{S}_{A/k} = \mathbb{B}_{A/k}$ if and only if $\operatorname{edim} A_{\mathfrak{p}} \leq 2 \dim A_{\mathfrak{p}} - 2$ for every non-regular prime $\mathfrak{p} \in \operatorname{Spec}(A)$.*

Proof. Since $\operatorname{edim} A_{\mathfrak{p}} \leq 2 \dim A_{\mathfrak{p}}$ for every prime $\mathfrak{p} \in \operatorname{Spec}(A)$ and A is Gorenstein, it follows that A is a complete intersection locally in codimension two. We note

that $(\wedge^3 I/I^2)^{**} \simeq \omega_A \simeq A$ and that the pairing

$$\wedge^2 I/I^2 \times I/I^2 \longrightarrow \wedge^3 I/I^2 \longrightarrow A$$

identifies $(\wedge^2 I/I^2)^{**}$ with $(I/I^2)^*$. As I/I^2 is a maximal Cohen–Macaulay A-module we conclude that the three left most modules in (4.3) are maximal Cohen–Macaulay B-modules. Therefore (4.3) is exact and, since it is a self-dual complex of B-modules, $\mathbb{S}_{A/k}$ is Gorenstein. Now the argument proceeds as in the proof of Proposition 4.1. To show (c) one replaces the Hilbert–Burch theorem used there by the Buchsbaum–Eisenbud structure theorem. □

One ought to observe that parts (a) and (b) of Propositions 4.1 and 4.2 were obtained in [22, 3.4 and 3.2] by passing through the tangent star algebra $\mathbb{T}_{A/k}$. Since the present approach is self-contained and perhaps more direct we have chosen to keep it for completeness.

The previous results motivate the following question:

Question 4.3. Let A be a normal Cohen–Macaulay domain essentially of finite type over a perfect field k. If $\Omega_{A/k}$ is of linear type and normal does it follow that $\operatorname{edim} A_{\mathfrak{p}} \leq 2 \dim A_{\mathfrak{p}} - 2$ for every non-regular prime $\mathfrak{p} \in \operatorname{Spec}(A)$ (or, equivalently, is $\mathbb{S}_{A/k} = \mathbb{B}_{A/k}$)?

Our aim for most of the remainder of the section is a refined version of Proposition 4.1(c) that does not require the (F_0) assumption on the module of differentials – see Theorem 4.8 below. For this we need several auxiliary results that may be of independent interest. Recall that if A is a Noetherian ring and $U \subset E$ are finite modules having a rank then U is said to be a *reduction* of E if the induced inclusion $\mathcal{R}(U) \subset \mathcal{R}(E)$ is an integral ring extension (see, e.g., [23, p. 613]).

Proposition 4.4. *Let A be a Noetherian ring and let $\varphi : A^{g+1} \longrightarrow A^g$ be a homomorphism such that $\operatorname{im}(\varphi)$ has rank g and $\operatorname{Ext}^1_A(\operatorname{im}(\varphi), A) = 0$. Then $I_g(\varphi) \simeq (\ker(\varphi))^*$. In particular, if A satisfies (S_2) it follows that $I_g(\varphi)$ is either the unit ideal or an unmixed ideal of height one. If in addition A is Gorenstein locally in codimension one, then $\operatorname{im}(\varphi)$ is a reduction of $(\operatorname{im}(\varphi))^{**}$.*

Proof. There is an exact sequence

$$A^n \xrightarrow{\psi} A^{g+1} \xrightarrow{\varphi} A^g,$$

so that the entries of the first column of ψ are the signed maximal minors of φ, hence generate $I_g(\varphi)$, an ideal of positive grade. Set $K := \ker(\varphi)$ and $L := \operatorname{im}(\varphi)$. From the assumption we obtain a short exact sequence

$$0 \to L^* \longrightarrow A^{g+1*} \longrightarrow K^* \to 0,$$

which shows that $K^* \simeq \operatorname{im}(\psi^*)$. Next we project A^{n*} onto the free module generated by the first basis element, thus getting a commutative diagram

$$
\begin{array}{ccc}
A^{g+1*} & \xrightarrow{\psi^*} & A^{n*} \\
\parallel & & \downarrow \pi \\
A^{g+1*} & \xrightarrow{\pi\psi^*} & A .
\end{array}
$$

Notice that $\operatorname{im}(\pi\psi^*) = I_g(\varphi)$. The projection π induces a surjection $\operatorname{im}(\psi^*) \twoheadrightarrow \operatorname{im}(\pi\psi^*) = I_g(\varphi)$, which is necessarily an isomorphism since the two modules are torsionfree of rank one.

Moreover, if A satisfies (S_2) then $I_g(\varphi)$ satisfies (S_2), hence it is the unit ideal or an unmixed ideal of height one. The last assertion is a property of the integral closure of modules with divisorial determinantal ideal (see [12, the proof of 2.5]).   □

Corollary 4.5. *Let R be a regular local ring and let $A = R/I$ be an almost complete intersection which is a complete intersection locally in codimension one and satisfies (S_2). If A is non-obstructed (i.e., $T^2(A/R, A) = \operatorname{Ext}_A^1(I/I^2, A) = 0$), then I/I^2 is a reduction of $(I/I^2)^{**}$.*

Proof. Write $g := \operatorname{height} I$. First notice that A is equidimensional because it is catenary and satisfies (S_2). Therefore I/I^2 has rank g as an A-module. Since I is generated by $g + 1$ elements, the first Koszul homology $H_1(I)$ of these elements is the canonical module of A, hence satisfies (S_2). Therefore the sequence

$$0 \to H_1(I) \longrightarrow A^m \longrightarrow I/I^2 \to 0$$

is exact and I/I^2 is a torsionfree A-module. Embedding I/I^2 into A^g and applying Proposition 4.4 with $\operatorname{im}(\varphi) = I/I^2$, we deduce the result.   □

Next is a curious result in embedding codimension 2 that, like Theorem 2.6, can be regarded as an analogue of Berger's conjecture.

As in previous instances, $\tau_A(E)$ will denote the A-torsion of an A-module E.

Let $A = R/I$ be a normal Cohen–Macaulay ring, where R is a regular local ring essentially of finite type over a perfect field k and height $I \leq 2$. The exact sequence (4.1) and the Cohen–Macaulayness of $H_1(I)$ imply that I/I^2 is torsionfree. Thus the natural complex

$$(4.4) \qquad 0 \to I/I^2 \longrightarrow \Omega_{R/k} \otimes_R A \simeq A^n \longrightarrow \Omega_{A/k} \to 0$$

is exact as well.

Theorem 4.6. *Let $(A, \mathfrak{m})$ be a normal local Cohen–Macaulay ring essentially of finite type over a perfect field k. Assume the following conditions:*

(i) $\operatorname{ecodim} A = 2$;
(ii) *A is an almost complete intersection locally in codimension 2.*

If $\Omega_{A/k}/\tau_A(\Omega_{A/k})$ is a Cohen–Macaulay module then A is regular.

Proof. Set $\Omega := \Omega_{A/k}$ and $\omega := \omega_A$. Write $A \simeq R/I$, where R is a regular local ring essentially of finite type over k and I a perfect ideal of height 2. It suffices to prove that I is a complete intersection, for in this case (4.4) shows that Ω is torsionfree of projective dimension one – thus $\Omega = \Omega/\tau_A(\Omega)$ is a maximal Cohen–Macaulay module of finite projective dimension, hence necessarily free.

We first consider the case $\dim A = 2$. Supposing I is not a complete intersection we have that this ideal is minimally generated by 3 elements. Consider a minimal presentation

$$(4.5) \qquad 0 \to R^2 \xrightarrow{\ \varphi\ } R^3 \longrightarrow I \to 0\,.$$

From the exact sequences (4.1) and (4.4) we derive an exact sequence

$$(4.6) \qquad 0 \to \omega \simeq \operatorname{H}_1(I) \longrightarrow A^3 \xrightarrow{\ \Theta\ } A^n \longrightarrow \Omega \to 0\,,$$

with Θ the transposed Jacobian matrix of the 3 generators of I. Moreover, since $\Omega/\tau_A(\Omega)$ is Cohen–Macaulay and $\tau_A(\Omega)$ has grade at least 2, one concludes that

$\operatorname{Ext}_A^1(\Omega, \omega) = 0$. Thus, applying $-^\vee = \operatorname{Hom}_A(-, \omega)$ to (4.6), one can see that there is an exact sequence

$$\omega^n \xrightarrow{\Theta^\vee} \omega^3 \longrightarrow \omega^\vee \simeq A \longrightarrow A/I_1(\varphi)A \to 0\,.$$

Hence we obtain the exact sequence

(4.7) $$\omega^n \xrightarrow{\Theta^\vee} \omega^3 \longrightarrow I_1(\varphi)/I \to 0\,.$$

By (4.5) and the Hilbert–Burch theorem, one has $I \subset I_1(\varphi)^2$. In particular, $I_1(\Theta) \subset I_1(\varphi)$. Thus tensoring (4.7) with $R/I_1(\varphi)$ gives

$$(\omega/I_1(\varphi)\,\omega)^{\oplus 3} \simeq (I_1(\varphi)/I) \otimes_R R/I_1(\varphi) \simeq I_1(\varphi)/I_1(\varphi)^2\,.$$

On the other hand, dualizing (4.5) into R and tensoring with $R/I_1(\varphi)$ yields

$$\omega/I_1(\varphi)\,\omega \simeq (R/I_1(\varphi))^{\oplus 2}\,.$$

It follows that

$$I_1(\varphi)/I_1(\varphi)^2 \simeq (\omega/I_1(\varphi)\,\omega)^{\oplus 3} \simeq (R/I_1(\varphi))^{\oplus 6}\,.$$

Since R is regular, [27, 1.1] implies that $I_1(\varphi)$ is generated by a regular sequence of length 6. But this is impossible in R, a ring of dimension four.

We now consider the case of arbitrary dimension. By the above, A is regular locally in codimension 2. Therefore (4.6) implies that Ω is torsionfree, hence Cohen–Macaulay by assumption. But then also I/I^2 is Cohen–Macaulay by (4.4), which forces I to be a complete intersection (see [8, 2.4]). $\qquad\square$

The next proposition leads into our main result about the normality of the module of differentials in the case of embedding codimension two.

Proposition 4.7. *Let $(A, \mathfrak{m})$ be a normal local ring essentially of finite type over a perfect field k. Assume the following conditions:*

(i) *$\dim A = \operatorname{ecodim} A = 2\,$;*
(ii) *A is either a complete intersection or else an almost complete intersection defined by an ideal of order ≥ 3.*

Then $\Omega_{A/k}$ is not integrally closed.

Proof. Write $A \simeq R/I$, where $(R, \mathfrak{n})$ is a regular local ring essentially of finite type over k and I a perfect ideal of height 2. We may assume that R has infinite residue field. Set $\Omega := \Omega_{A/k}$ and $\Omega^1 := \Omega/\tau_A(\Omega)\,$.

There is an exact sequence

(4.8) $$0 \to (I/I^2)^{**} \longrightarrow A^n \longrightarrow \Omega^1 \to 0\,.$$

By Theorem 4.6, Ω^1 is not Cohen–Macaulay. Thus $\mathfrak{m} \in \operatorname{Ass}_A(A^2/\Omega^1)$ for any embedding $\Omega^1 \subset A^2$. Suppose that Ω is integrally closed, which means that Ω^1 is integrally closed. In particular, Ω^1 is $\mathfrak{m}$-full. If A is a complete intersection, then Ω^1 has finite projective dimension by (4.8), which leads to a contradiction according to Theorem 3.6. Thus we may assume that A is an almost complete intersection and that $I \subset \mathfrak{n}^3$. By Proposition 3.5, there exists an $x \in \mathfrak{m}$ such that $A/\mathfrak{m}$ is a direct summand of $\Omega^1/x\Omega^1$. Write $(A', \mathfrak{m}')$ for the local ring $A/(x)$ and let $\{e_1, \ldots, e_n\}$ be the canonical basis of A'^n. Tensoring (4.8) with A' we obtain a presentation

(4.9) $$0 \to M \longrightarrow A'^n \longrightarrow \Omega^1/x\Omega^1 \to 0\,,$$

where we may assume that $M = \mathfrak{m}' e_1 \oplus N$ with $N \subset \oplus_{i=2}^n A' e_i$. Hence the image of M under the projection $A'^n \twoheadrightarrow A' e_1 \simeq A'$ is $\mathfrak{m}'$.

On the other hand, as A is a Cohen–Macaulay ring of embedding codimension 2 it is non-obstructed (see [9, 3.2(a)]). Therefore by Corollary 4.5, I/I^2 is a reduction of $(I/I^2)^{**}$. Since $I \subset \mathfrak{n}^3$ it follows that $I/I^2 \subset \mathfrak{m}^2 A^n$. Hence, projecting onto $A' e_1 \simeq A'$, we see that $(\mathfrak{m}')^2$ is a reduction of $\mathfrak{m}'$. This is impossible because $\dim A' > 0$. $\qquad\square$

Theorem 4.8. *Let $(A, \mathfrak{m})$ be a normal local Cohen–Macaulay ring essentially of finite type over a perfect field k. Assume the following conditions:*

   (i) ecodim $A = 2$;

   (ii) *Locally in codimension 2, the ring A is either a complete intersection or else an almost complete intersection defined by an ideal of order ≥ 3.*

Then the following conditions are equivalent:

   (1) $\Omega_{A/k}$ *is normal;*

   (2) edim $A_{\mathfrak{p}} \leq 2 \dim A_{\mathfrak{p}} - 2$ *for every non-regular prime $\mathfrak{p} \in \mathrm{Spec}(A)$.*

Moreover, if the equivalent conditions (1) or (2) hold then $\Omega_{A/k}$ is of linear type, $\mathbb{R}_{A/k}$ is Cohen–Macaulay, and $\mathbb{R}_{A/k} = \mathbb{B}_{A/k}$.

Proof. $(1) \Rightarrow (2)$: Proposition 4.7 implies that $\Omega_{A/k}$ satisfies (F_1). Hence by Proposition 4.1(b), $\Omega_{A/k}$ is of linear type and then according to Proposition 4.1(c), it satisfies (F_2).

$(2) \Rightarrow (1)$: This follows from Proposition 4.1(c).

The remaining assertions follow from the same proposition. $\qquad\square$

4.2. Relation to Calabi–Yau varieties.

In this last part we explain to what extent the present results relate to Calabi–Yau varieties.

Let $X \subset \mathbb{P}_{\mathbb{C}}^{n-1}$ be an arithmetically normal projective variety and let A stand for its homogeneous coordinate ring. We say that X is of *Calabi–Yau type* if there exists a homogeneous isomorphism $\omega_A \simeq A$. The notion of Calabi–Yau variety would also require that X be smooth and $H^1(X, \mathcal{O}_X) = 0$. If X is of Calabi–Yau type it often turns out that $\mathrm{Proj}(\mathbb{R}_{A/\mathbb{C}}) \subset \mathbb{P}_{\mathbb{C}}^{2n-1}$ has the same property. Here and in what follows we use the natural embeddings $\mathrm{Proj}(\mathbb{R}_{A/\mathbb{C}}) \subset \mathrm{Proj}(\mathbb{S}_{A/\mathbb{C}}) \subset \mathrm{Proj}(\mathbb{S}_{R/\mathbb{C}}) = \mathbb{P}_{\mathbb{C}}^{2n-1}$, where $R = \mathbb{C}[X_1, \ldots, X_n]$ and $\mathbb{S}_{R/\mathbb{C}}$ is a polynomial ring in $2n$ variables over $\mathbb{C}$ that is regarded as standard graded.

We first look into the case of complete intersections.

Proposition 4.9. *Let $X \subset \mathbb{P}_{\mathbb{C}}^{n-1}$ be a non-degenerate smooth projective variety that is a complete intersection of $g \geq 1$ hypersurfaces of degrees $d_1 \geq \cdots \geq d_g$, and let A stand for its homogeneous coordinate ring. Assume that X is of Calabi–Yau type and consider the subschemes $Y := \mathrm{Proj}(\mathbb{R}_{A/\mathbb{C}}) \subset Z := \mathrm{Proj}(\mathbb{S}_{A/\mathbb{C}}) \subset \mathbb{P}_{\mathbb{C}}^{2n-1}$.*

   (a) *The subscheme Z is the complete intersection of $2g$ hypersurfaces of degrees $d_1, \ldots, d_g, d_1, \ldots, d_g$ and $\omega_{\mathbb{S}_{A/\mathbb{C}}} \simeq \mathbb{S}_{A/\mathbb{C}}$ as graded modules;*

   (b) *If $d_1 = 2$ then Z is neither reduced nor irreducible, and Y is not arithmetically Cohen–Macaulay;*

   (c) *If $d_1 \geq 3$ then $Y = Z$ is reduced and irreducible;*

   (d) *The subscheme Y is arithmetically normal if and only if $d_1 \geq 4$ or $d_2 \geq 3$ (formally setting $d_2 = 2$ when $g = 1$). In this case $Y = Z$ is of Calabi–Yau type.*

Proof. Since $\omega_A \simeq A(-n + \sum_{i=1}^g d_i)$ and X is of Calabi–Yau type, we have $n = \sum_{i=1}^g d_i$. Since X is non-degenerate, $d_g \geq 2$. Therefore $n \geq 2g$, hence $\Omega_{A/\mathbb{C}}$ satisfies condition (F_0). Since $\text{projdim}(\Omega_{A/\mathbb{C}}) \leq 1$ this implies that $\mathbb{S}_{A/\mathbb{C}}$ is a complete intersection of dimension $2(n-g)$ (see [1, Proposition 4]). Moreover, $\mathbb{S}_{A/\mathbb{C}}$ is defined by $2g$ forms of degrees $d_1, \ldots, d_g, d_1, \ldots, d_g$ in a standard graded polynomial ring in $2n$ variables over $\mathbb{C}$. Therefore $\omega_{\mathbb{S}_{A/\mathbb{C}}} \simeq \mathbb{S}_{A/\mathbb{C}}(-2n + 2\sum_{i=1}^g d_i) = \mathbb{S}_{A/\mathbb{C}}$ as graded modules. This shows (a).

To prove (b), note that the hypothesis forces the equality $n = 2g$ and thus $\Omega_{A/\mathbb{C}}$ does not satisfy condition (F_1). It follows from [3, the proof of 2.2] that Z is not irreducible, whereas Theorem 2.1 (or Remark 2.3) implies that Z is not reduced either. Furthermore, Y is not arithmetically Cohen–Macaulay according to Theorem 3.1 (or Remark 3.2).

As to (c), the assumption implies that $n \geq 2g+1$, hence $\Omega_{A/\mathbb{C}}$ satisfies condition (F_1). Again since $\text{projdim}(\Omega_{A/\mathbb{C}}) \leq 1$ we have $\mathbb{S}_{A/\mathbb{C}} = \mathbb{R}_{A/\mathbb{C}}$ (see [1, Proposition 4], [13, 1.1], [24, 3.4]). Finally, notice that $d_1 \geq 4$ or $d_2 \geq 3$ if and only if $n \geq 2g+2$. Thus the equivalence in (d) follows from Corollary 3.8. Parts (a) and (c) then imply the rest. $\square$

Next is the non-complete intersection case.

Proposition 4.10. *Let $X \subset \mathbb{P}_{\mathbb{C}}^{n-1}$ be a smooth arithmetically Cohen–Macaulay projective variety that is not a complete intersection, and let A stand for its homogeneous coordinate ring. Assume that $\dim X \geq 2$, $\text{ecodim } X \leq 3$, and X is of Calabi–Yau type. Consider the subschemes $Y := \text{Proj}(\mathbb{R}_{A/\mathbb{C}}) \subset Z := \text{Proj}(\mathbb{S}_{A/\mathbb{C}}) \subset \mathbb{P}_{\mathbb{C}}^{2n-1}$.*

(a) *The subscheme Z is arithmetically Gorenstein and $\omega_{\mathbb{S}_{A/\mathbb{C}}} \simeq \mathbb{S}_{A/\mathbb{C}}$ as graded modules;*
(b) *If $\dim X = 2$ then Z is neither reduced nor irreducible;*
(c) *If $\dim X \geq 3$ then $Y = Z$ is reduced and irreducible;*
(d) *The subscheme Z is arithmetically normal if and only if $\dim X \geq 4$, in which case $Y = Z$ is of Calabi–Yau type.*

Proof. Notice that $A \simeq R/I$, where $R = \mathbb{C}[X_1, \ldots, X_n]$ and I is a homogeneous Gorenstein ideal of height at most 3, hence of height 3 because X is not a complete intersection; for the same reason, $I \subset (X_1, \ldots, X_n)^2$. Since X is smooth and $n = \text{height } I + \dim A \leq 2 \dim A$, the module $\Omega_{A/\mathbb{C}}$ satisfies (F_0). Proposition 4.2(a) then shows that $\mathbb{S}_{A/\mathbb{C}}$ is Gorenstein and the complex (4.3) is exact. In fact, as a complex of graded modules over the standard graded $\mathbb{C}$-algebra $B := A[T_1, \ldots, T_n]$ this exact sequence now reads

$$0 \to (\wedge^3 I/I^2)^{**} \otimes_A B \longrightarrow (\wedge^2 I/I^2)^{**} \otimes_A B \longrightarrow (I/I^2) \otimes_A B \longrightarrow B \longrightarrow \mathbb{S}_{A/\mathbb{C}} \to 0,$$

where $-^* = \text{Hom}_A(-, A) = \text{Hom}_A(-, \omega_A)$. One has $(\wedge^3 I/I^2)^{**} \simeq \text{Tor}_3^R(A, A) \simeq A(-n)$. Now the arguments in the proof of Proposition 4.2(a) show that $\omega_{\mathbb{S}_{A/\mathbb{C}}} \simeq \mathbb{S}_{A/\mathbb{C}}$ as graded modules, proving part (a).

As to (b), if $\dim X = 2$ then $n = \text{height } I + \dim A = 6 = 2 \dim A$. In particular, $\Omega_{A/\mathbb{C}}$ does not satisfy (F_1). Thus Z is not irreducible according to [3, the proof of 2.2]. On the other hand, by considering the Hilbert function of A modulo a linear system of parameters one sees that $\dim_{\mathbb{C}}[I]_2 \leq 5 = n - 1$. Now an application of Theorem 2.1 (or Remark 2.3) yields that Z is not reduced either, proving (b).

Finally, part (c) follows from Proposition 4.2(b) and part (d) from Proposition 4.2(c). $\square$

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