Normalization of Modules

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Abstract

We introduce techniques to derive estimates for the degrees of the generators of the integral closure of several classes of Rees algebras of modules, and to bound the length of normalization processes. In the case of regular base rings, the bounds are expressed in terms of Buchsbaum–Rim multiplicities and a module version of Briançon–Skoda numbers.

Normalization of modules is the study of the integral closure of the Rees algebra of a module. The questions that arise are natural extensions of those that occur in the normalization of ordinary blowups and also of the so–called multi Rees algebras of ideals. In this paper, we focus on numerical aspects of the process of normalization of a module. Let R be a Noetherian ring and E a finitely generated torsionfree R–module having a rank. To enable the extension, we consider two technical devices to attach an ideal I of a ring S to the R–module E so that the comparison can be made between the Rees algebra of E and the Rees algebra of I. One of the ideals that can be used is the ideal generated by the module in the polynomial ring which contains the Rees algebra of the module. The other is known as the generic Bourbaki ideal of E. More details on these ideals will be given later.

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Let us recall the notion of the Rees algebra of a module. Let R be a Noetherian ring, let E be a finitely generated torsionfree R–module having a rank, and choose an embedding $\varphi : E \hookrightarrow R^e$. The *Rees algebra* $\mathcal{R}(E)$ of E is the R–subalgebra of the polynomial ring $R[t_1, \ldots, t_e]$ generated by all linear forms $a_1t_1 + \cdots + a_et_e$, where $(a_1, \ldots, a_e)$ is the image of an element of E in R^e under the embedding φ. The Rees algebra $\mathcal{R}(E)$ is a standard graded algebra whose n–th component is denoted by E^n. It is independent of the embedding φ since E is torsionfree and has a rank. There is a more general notion of the Rees algebra of a module ([5]), but it is in agreement with the one above for the class of modules we consider. In general, there is a surjection from the symmetric algebra $\mathrm{Sym}(E)$ of E onto the Rees algebra of E and the module E is said to be of *linear type* if this surjection is an isomorphism. Let U be a submodule of E. The module E is *integral* over the module U if the Rees algebra of E is integral over the R-subalgebra generated by U. In this case we say that U is a *reduction* of E. The *integral closure* $\overline{E}$ of E (in R^e) is the largest submodule of R^e which is integral over the module E. If E is equal to $\overline{E}$, then E is called *integrally closed* or *complete* in R^e. If the Rees algebra $\mathcal{R}(E)$ of E is integrally closed in $R[t_1, \ldots, t_e]$, then the module E is said to be *normal* in R^e. If R is normal, then these notions do not depend on the embedding φ as the integral closure of $\mathcal{R}(E)$ in $R[t_1, \ldots, t_e]$ is simply the integral closure of $\mathcal{R}(E)$ in this case. Furthermore the integral closure $\overline{\mathcal{R}(E)}$ of $\mathcal{R}(E)$ is a positively graded R–algebra whose n–th component is the integral closure $\overline{E^n}$ of the module E^n ([16]).

Our objective is to derive numerical properties of the integral closure of the Rees algebra of a module and to measure the length of the algorithmic procedures used to compute it. This paper is organized as follows. In Section 1, we show how the normalization of a module can be obtained from the normalization of an ideal in another ring. It turns out that the integral closure of the Rees algebra of a module is a direct summand of the integral closure of the Rees algebra of the ideal generated by the module. In Section 2, we deal with Buchsbaum–Rim coefficients of modules having finite colength. In particular, we derive bounds for modified Buchsbaum–Rim coefficients of the normalization of a module. They are used along with the notion of Briançon–Skoda number to study the chains of algebras that occur in the normalization of a module. Several bounds for the length of such chains are obtained and they are similar to those of [14] in the case of ideals. In Section 3, we show how the results of Section 2 can be extended to equimultiple modules. Finally in Section 4, we pay attention to the bounds for the degrees of the generators of a Cohen–Macaulay graded algebra which is integral over the Rees algebra of a module.

1 Turning modules into ideals

In this section we show how the issues of normalization of ideals and modules resemble one another. We begin by providing a source of integrally closed modules.

Proposition 1.1 *Let R be a normal domain and let $\mathcal{I} = \bigoplus_{n\geq 0} \mathcal{I}_n$ be a homogeneous ideal of* *the polynomial ring $S = R[t_1, \ldots, t_e]$. If $\mathcal{I}$ is integrally closed as an S–ideal, then each $\mathcal{I}_n$ is integrally closed as an R–module.*

Proof. Write $S = \bigoplus_{n \geq 0} S_n$. Then $\mathcal{I}_n$ is a submodule of the free R–module S_n and we have

$$\mathcal{R}(\mathcal{I}_n) \subseteq \overline{\mathcal{R}(\mathcal{I}_n)} \subseteq \mathcal{R}(S_n) = \mathrm{Sym}(S_n) \xrightarrow{\Phi} S,$$

where Φ is the natural map induced by the identity on S_n. Let $u \in \overline{\mathcal{I}_n} \subset S_n$. Then there is an equation in $\mathcal{R}(S_n)$ of the form

$$u^m + a_1 u^{m-1} + \cdots + a_m = 0, \quad a_i \in {\mathcal{I}_n}^i.$$

Applying Φ to this equation, it converts into an equation of integrality of $u \in S$ over the ideal $\mathcal{I}$. Since $\mathcal{I} = \overline{\mathcal{I}}$, we obtain $u \in \mathcal{I}_n$. $\square$

In the next proposition we explain how to turn an R–module E into an ideal. By doing so, we relate the normalization of a module E to that of the ideal generated by E.

Proposition 1.2 *Let R be a normal domain and E a finitely generated torsionfree R–module. Let $S = R[t_1, \ldots, t_e]$ be a polynomial ring which contains the Rees algebra $\mathcal{R}(E)$ as a homogeneous R–subalgebra and denote by I the S–ideal generated by $E \subset S$. For any positive integer n,*

$$\overline{E^n} = \left[\, \overline{I^n} \,\right]_n,$$

where $[\ ]_n$ denotes the n–th component.

Proof. It is clear that $E^n = [\, I^n \,]_n \subseteq \left[\, \overline{I^n} \,\right]_n$. Since $\left[\, \overline{I^n} \,\right]_n$ is integrally closed by Proposition 1.1, we have $\overline{E^n} \subseteq \overline{\left[\, \overline{I^n} \,\right]_n} = \left[\, \overline{I^n} \,\right]_n$. Now it suffices to verify the equality of the two integrally closed modules $\overline{E^n}$ and $\left[\, \overline{I^n} \,\right]_n$ at the discrete valuation rings containing R. For V any discrete valuation ring containing R we write $VS = V \otimes_R S = V[t_1, \ldots, t_e]$, and consider the VS–ideal (VE) generated by VE. It suffices to show

$$(VE)^n = \overline{(VE)^n},$$

for then

$$\left[\, \overline{I^n} \,\right]_n \subseteq \left[\, \overline{(VE)^n} \,\right]_n = [\, (VE)^n \,]_n = VE^n,$$

and hence $\left[\, \overline{I^n} \,\right]_n \subseteq \overline{E^n}$.

We now prove that $(VE)^n = \overline{(VE)^n}$ for every V. We may assume that the VS–ideal (VE) is generated by the forms $a_1 t_1, \ldots, a_e t_e$ with $a_i \in V$. We may suppose that a_1 divides all a_i, say $a_i = a_1 b_i$, so that $(VE) = a_1(t_1, b_2 t_2, \ldots, b_e t_e)$. Now (VE) is normal if and only if $(t_1, b_2 t_2, \ldots, b_e t_e)$ is normal. We claim that $(t_1, b_2 t_2, \ldots, b_e t_e)$ is normal if and only if $(b_2 t_2, \ldots, b_e t_e)$ is normal. Then we can drop the indeterminate t_1 and iterate. As for the claim, we show more generally that an R–ideal I is normal if and only if (I, x) is normal in the polynomial ring $R[x]$. This follows from the fact that the extended Rees ring of (I, x) is $R[x]\left[It, xt, t^{-1}\right] = R[It, t^{-1}][xt]$, where t is an indeterminate. $\square$

The next corollary is an immediate consequence of Proposition 1.2.

Corollary 1.3 *Let R be a normal domain and E a finitely generated torsionfree R–module. Let S be a polynomial ring which contains the Rees algebra $\mathcal{R}(E)$ as a homogeneous R–subalgebra, and let I denote the S–ideal generated by $E \subset S$. If I is normal, then E is normal.*

Example 1.4 The converse of Corollary 1.3 is not true in general. Let $R = k[x, y]$ and let E be the submodule of the free module $Re_1 \oplus Re_2$ generated by x^2e_1 and y^2e_2. Then E is a free module and hence it is normal. However the ideal I of $S = R[e_1, e_2]$ is generated by two cubic forms. They are a regular sequence. A presentation of $S[It]$ is $S[T_1, T_2]$ modulo the form $f = y^2e_2T_1 - x^2e_1T_2$. Notice that the ideal $\mathfrak{p} = (x, y)S[It]$ is a prime ideal of height 1, but that $S[It]_{\mathfrak{p}}$ is not a discrete valuation ring.

Let R be a Noetherian ring and E a finitely generated torsionfree R–module having a rank. Let S be a polynomial ring which contains the Rees algebra $\mathcal{R}(E)$ as a homogeneous R–subalgebra, and let I denote the S–ideal generated by E. Then the symmetric algebra of E is a direct summand of the symmetric algebra of I. Hence if I is of linear type, then E is of linear type. But the converse is not true. For example, let $R = k[x, y, z]$ and let E be the image of R–linear map $\varphi = \begin{bmatrix} z^2 & x^2 & y^2 & xy \\ x^2 & y^2 & xz & xz + z^2 \end{bmatrix}$. By [19, 5.6], the module E is of linear type. But the ideal I is not of linear type.

Let $(R, \mathfrak{m})$ be a Noetherian local ring and let E be a finitely generated R–module having a rank e. The *analytic spread* of E is defined to be the dimension of the special fiber ring $\mathcal{R}(E)/\mathfrak{m}\mathcal{R}(E)$ and is denoted by $\ell(E)$. We recall that $\ell(E) \leq d + e - 1$ if $d = \dim R > 0$ ([19, 2.3]). For a reduction U of E, the *reduction number* $\mathrm{r}_U(E)$ of E with respect to U is the least integer $r \geq 0$ such that $E^{r+1} = UE^r$. Let R be a Noetherian local ring with infinite residue field. A reduction is said to be *minimal* if it is minimal with respect to containment. For any minimal reduction U of E, one has $\nu(U) = \ell(E)$ where $\nu(\cdot)$ denotes minimal number of generators. We define the *reduction number* $\mathrm{r}(E)$ of E to be the minimum of $\mathrm{r}_U(E)$, where U ranges over all minimal reductions of E.

Remark 1.5 Let $(R, \mathfrak{m})$ be a normal local ring with infinite residue field and let E be a finitely generated torsionfree R–module. Let $S = R[t_1, \ldots, t_e]$ be a polynomial ring containing the Rees algebra $\mathcal{R}(E)$ as a homogeneous R–subalgebra, and let I denote the S–ideal generated by $E \subset S$. An R–module $U \subseteq E$ is a reduction of E if and only if the S–ideal J generated by U is a reduction of I. Moreover $\ell(E) = \ell(IS_{\mathfrak{n}})$ and $\mathrm{r}(E) = \mathrm{r}(IS_{\mathfrak{n}})$, where $\mathfrak{n}$ is the maximal homogeneous ideal of S.

Proof. Notice that $E^n = UE^{n-1}$ if and only of $I^n = JI^{n-1}$. The ideal $IS_{\mathfrak{n}}$ has a general minimal reduction generated by linear forms in S. Such a reduction is given by $JS_{\mathfrak{n}}$ for some U. Hence $\ell(E) = \ell(IS_{\mathfrak{n}})$ and $\mathrm{r}(E) = \mathrm{r}(IS_{\mathfrak{n}})$. □

Remark 1.6 Let R be an equidimensional universally catenary Noetherian local ring of dimension d and let E be a finitely generated torsionfree R–module having a rank e. Let $S = R[t_1, \dots, t_e]$ be a polynomial ring which contains the Rees algebra $\mathcal{R}(E)$ as a homogeneous R–subalgebra, and let I be the S–ideal generated by $E \subset S$. Suppose that the colength $\lambda(R^e/E)$ is finite. Then

$$\mathrm{ht}(I) = \min\{\ e,\ d + e - \nu\,(R^e/E)\ \}.$$

In particular if $e \leq d$, then $\mathrm{ht}(I) = e$.

Proof. The quotient ring S/I can be identified with the symmetric algebra $\mathrm{Sym}(R^e/E)$. Since $\lambda(R^e/E) < \infty$, using [9, 2.6], we obtain

$$\mathrm{ht}(I) = \dim S - \dim(S/I) = d + e - \max\{\ d,\ \nu(R^e/E)\ \} = \min\{\ e,\ d + e - \nu(R^e/E)\ \}.$$

□

2 Briançon-Skoda number for modules

Our objective in this section is to find bounds for a chain of algebras which occur in the process of normalization of a module. Let us first review the definition of Buchsbaum–Rim polynomials. Let R be a Noetherian local ring of dimension d and let $E \subsetneq R^e$ be a submodule such that the colength $\lambda(R^e/E)$ is finite. Let E^n denote the image of the R–linear map $\mathrm{Sym}_n(E) \longrightarrow S = \bigoplus_{n \geq 0} S_n = \mathrm{Sym}(R^e) = R[t_1, \dots, t_e]$. Buchsbaum and Rim proved that the colength $\lambda(S_n/E^n)$ is a polynomial in n of degree $d + e - 1$ for sufficiently large n ([4, 3.1 and 3.4]). This polynomial is called the *Buchsbaum–Rim polynomial* of E. It is of the form

$$P(n) = \mathrm{br}(E)\binom{n + d + e - 2}{d + e - 1} - \mathrm{br}_1(E)\binom{n + d + e - 3}{d + e - 2} + \text{lower terms}.$$

The positive integer $\mathrm{br}(E)$ is called the *Buchsbaum–Rim multiplicity* of E. For any reduction U of E, it is known that $\mathrm{br}(E) = \mathrm{br}(U)$ ([12, 5.3]).

Let R be an analytically unramified normal local ring. Since the integral closure $\overline{\mathcal{R}(E)}$ is finite over $\mathcal{R}(E)$, it follows that $\lambda(\overline{E^n}/E^n)$ is a polynomial of degree at most $d + e - 2$ for $n \gg 0$. Hence one obtains the following polynomial expression for $n \gg 0$,

$$\lambda(S_n/\overline{E^n}) = \overline{P}(n) = \overline{\mathrm{br}}(E)\binom{n + d + e - 2}{d + e - 1} - \overline{\mathrm{br}}_1(E)\binom{n + d + e - 3}{d + e - 2} + \text{lower terms}.$$

It also follows that $\mathrm{br}(E) = \overline{\mathrm{br}}(E)$. More generally, if $\mathcal{A} = \bigoplus_{n\geq 0} A_n$ is a graded R–subalgebra such that $\mathcal{R}(E) \subset \mathcal{A} \subset \overline{\mathcal{R}(E)}$, then for $n \gg 0$,

$$\lambda(S_n/A_n) = P_{\mathcal{A}}(n) = \mathrm{br}(\mathcal{A})\binom{n+d+e-2}{d+e-1} - \mathrm{br}_1(\mathcal{A})\binom{n+d+e-3}{d+e-2} + \text{lower terms.}$$

Let R be a Noetherian local ring and let $E \neq 0$ be a finitely generated torsionfree R–module. The module E is called an *ideal module* if E^{**} is free, where * denotes dualizing into the ring R. For example, if R is Cohen–Macaulay ring of dimension at least 2, then a submodule $E \subsetneq R^e$ having finite colength is an ideal module ([19, 5.1]). An ideal module E has a rank ([19, 5.1]), say e, and affords a *natural* embedding $E \hookrightarrow E^{**} = R^e$. Composing an epimorphism $R^n \twoheadrightarrow E$ with this embedding, we obtain an R–linear map $\varphi : R^n \to R^e$ with $\mathrm{image}(\varphi) = E$. Notice that the i–th Fitting ideal $\mathrm{Fitt}_i(R^e/E)$ of R^e/E is the R–ideal generated by the $(e-i) \times (e-i)$–minors of φ, and that these ideals only depend on E. Hence we may write $\det_0(E)$ for $\mathrm{Fitt}_0(R^e/E)$. We define the *codimension* of E as the height of $\det_0(E)$, the *deviation* of E as $\mathrm{d}(E) = \nu(E) - e + 1 - \mathrm{grade}(\det_0(E))$, and the *analytic deviation* as $\mathrm{ad}(E) = \ell(E) - e + 1 - \mathrm{ht}(\det_0(E))$. An ideal module E is said to be a *complete intersection* or *equimultiple* if $\mathrm{d}(E) \leq 0$ or $\mathrm{ad}(E) \leq 0$, respectively. At this point we need the notion of $\mathfrak{m}$–full modules. A submodule $E \subset R^e$ over a local ring $(R, \mathfrak{m})$ is called $\mathfrak{m}$*–full* if there is an element $x \in \mathfrak{m}$ such that $\mathfrak{m}E :_{R^e} x = E$. We are now ready to give bounds for the Buchsbaum–Rim multiplicity.

Proposition 2.1 *Let $(R, \mathfrak{m})$ be an analytically unramified Cohen–Macaulay normal local ring of dimension $d \geq 2$ and let $E \subsetneq R^e$ be a submodule with $\lambda(R^e/E) < \infty$. Let s be the integer such that $\mathrm{Fitt}_{e-1}(R^e/E)$ is contained in $\mathfrak{m}^s$ but not in $\mathfrak{m}^{s+1}$. Then*

$$e(\mathfrak{m})s^{d-1} \cdot \binom{d+e-2}{e-1} \leq \mathrm{br}(E) \leq e(\det_0(E)) \cdot \binom{d+e-1}{e-1},$$

where $e(\mathfrak{m})$ and $e(\det_0(E))$ are the Hilbert–Samuel multiplicities of the ideals $\mathfrak{m}$ and $\det_0(E)$ respectively.

Proof. We may assume that the residue field of R is infinite. Let $S = \bigoplus_{n\geq 0} S_n = \mathrm{Sym}(R^e) = R[t_1, \ldots, t_e]$ and let $R^n \xrightarrow{\varphi} R^e \longrightarrow R^e/E \longrightarrow 0$ be a presentation of R^e/E. Since $\mathrm{Fitt}_{e-1}(R^e/E) = I_1(\varphi) \subseteq \mathfrak{m}^s$ and $E = \mathrm{image}(\varphi)$, we have

$$\overline{E^n} \subseteq \overline{\mathfrak{m}^{ns}} S_n \quad \text{for all } n \geq 0.$$

As integrally closed modules are $\mathfrak{m}$–full by [3, 2.6], [3, 2.7] shows that

$$\nu(\overline{\mathfrak{m}^{ns}} S_n) \leq \nu(\overline{E^n}).$$

Now for sufficiently large n, we have a polynomial expression

$$\lambda\left(\overline{\mathfrak{m}^{ns}}/\overline{\mathfrak{m}^{ns+1}}\right) = \overline{e_0}\binom{ns+d-1}{d-1} - \overline{e_1}\binom{ns+d-2}{d-2} + \cdots + (-1)^{d-1}\overline{e_{d-1}} \leq \nu(\overline{\mathfrak{m}^{ns}}).$$

Let U be a minimal reduction of E. Then U is a complete intersection, and using [2, 4.4 and 3.1] we obtain

$$\nu(\overline{\mathfrak{m}^{ns}})\cdot\binom{n+e-1}{e-1} = \nu(\overline{\mathfrak{m}^{ns}}S_n) \leq \nu(\overline{E^n}) = \nu(\overline{U^n}) \leq \mathrm{br}(U)\cdot\binom{n+d+e-3}{d+e-2}+\binom{n+d+e-3}{d+e-3}.$$

It follows that

$$\frac{\overline{e_0}s^{d-1}}{(d-1)!(e-1)!}n^{d+e-2} + \text{lower terms} \leq \frac{\mathrm{br}(U)}{(d+e-2)!}n^{d+e-2} + \text{lower terms}.$$

Since $\mathrm{br}(U) = \mathrm{br}(E)$ and $\overline{e_0} = e(\mathfrak{m})$, we have

$$\mathrm{br}(E) \geq e(\mathfrak{m})s^{d-1}\cdot\binom{d+e-2}{e-1}.$$

The other inequality follows from the fact that $\det_0(E)R^e \subset E$ and hence

$$\lambda(S_n/E^n) \leq \lambda(S_n/\det_0(E)^n S_n) \quad \text{for all } n \geq 0.$$

□

We recall the notion of Briançon–Skoda number of an ideal I of an analytically unramified Noetherian local ring R with infinite residue field. It is the smallest integer $c(I) = s$ such that $\overline{I^{n+s}} \subset J^n$ for all n and for any minimal reduction J of I ([8, 4.13]). In a similar way, we can define the *Briançon–Skoda number* of an R–module E.

Definition 2.2 Let R be an analytically unramified normal local ring with infinite residue field and let E be an ideal module over R. The embedding $E \hookrightarrow E^{**} = R^e$ identifies the Rees algebra $\mathcal{R}(E)$ as a R–subalgebra of $S = \bigoplus_{n\geq 0} S_n = \mathrm{Sym}(R^e) = R[t_1,\ldots,t_e]$. The *Briançon–Skoda number* $c = c(E)$ of E is the smallest integer c such that

$$\overline{E^{n+c}} \subseteq U^n S_c \quad \text{for all } n \geq 0$$

and for any minimal reduction of U of E.

It is clear from the definition that the Briançon–Skoda number of the R–module E is at most the Briançon–Skoda number of the $S_{\mathfrak{n}}$–ideal generated by $E \subset S$, where $\mathfrak{n}$ is the maximal homogeneous ideal of S. However these two numbers are not equal in general as can be seen from Example 1.4. Now we show how the Briançon–Skoda number leads to a bound for $\overline{\mathrm{br}_1}(E)$. For ideals this result was proved in [14, 2.2].

Theorem 2.3 *Let R be an analytically unramified Cohen–Macaulay normal local ring of dimension $d \geq 2$ with infinite residue field and let $E \subsetneq R^e$ be a submodule with $\lambda(R^e/E) < \infty$. If the Briançon–Skoda number of E is c, then*

$$0 \leq \mathrm{br}_1(E) \leq \overline{\mathrm{br}_1}(E) \leq \mathrm{br}(E)\cdot\binom{e+c-1}{e}.$$

Proof. We use the notation of Definition 2.2. Let U be a minimal reduction of E. Then U is a complete intersection and $\mathrm{br}_1(U) = 0$ ([2, 3.4]). Since $\lambda(S_n/\overline{E^n}) \leq \lambda(S_n/E^n) \leq \lambda(S_n/U^n)$ and $\overline{\mathrm{br}}(E) = \mathrm{br}(E) = \mathrm{br}(U)$, we have $0 \leq \mathrm{br}_1(E) \leq \overline{\mathrm{br}}_1(E)$. It remains to prove the last inequality in the theorem.

By the definition of $c = c(E)$ we obtain

$$\lambda(S_{n+c}/U^{n+c}) - \lambda(S_{n+c}/\overline{E^{n+c}}) = \lambda(\overline{E^{n+c}}/U^{n+c}) \leq \lambda(U^n S_c/U^{n+c}). \tag{1}$$

Write $\mathcal{C} = \bigoplus_{n\geq 0} U^n S_c/U^{n+c}$. This is a finitely generated graded module over $\mathcal{R}(U)/\mathfrak{m}^t\mathcal{R}(U)$ for some $t > 0$, where $\mathfrak{m}$ denotes the maximal ideal of R. Hence it has a Hilbert polynomial $P(n)$ of the form

$$P(n) = e_0(\mathcal{C}) \cdot \binom{n+d+e-2}{d+e-2} - e_1(\mathcal{C}) \cdot \binom{n+d+e-3}{d+e-3} + \text{lower terms.}$$

Since the polynomial $P(n)$ has degree at most $d + e - 2$ and $\mathrm{br}_1(U) = 0$, inequality (1) implies that $\overline{\mathrm{br}}_1(E) \leq e_0(\mathcal{C})$.

In order to estimate $e_0(\mathcal{C})$, we consider the filtration

$$U^{n+c} \subset U^{n+c-1}S_1 \subset \cdots \subset U^n S_c.$$

We define the $\mathcal{R}(U)$–modules

$$D_i = \bigoplus_{n\geq c-i} U^n S_i/U^{n+1}S_{i-1}, \quad 1 \leq i \leq c,$$

which give the factors of a filtration of $\mathcal{C}$. Any epimorphism $R^{d+e-1} \twoheadrightarrow U$ induces a surjective map

$$\phi : \mathrm{Sym}_n(R^{d+e-1}) \otimes S_{i-1} \otimes (S_1/U) \twoheadrightarrow U^n S_i/U^{n+1}S_{i-1}.$$

Since $\lambda(S_1/U) = \mathrm{br}(U) = \mathrm{br}(E)$ ([4, 4.5], [12, 5.3]), we have

$$\lambda\left(\mathrm{Sym}_n(R^{d+e-1}) \otimes S_{i-1} \otimes (S_1/U)\right) = \binom{n+d+e-2}{d+e-2}\binom{e+i-2}{i-1}\mathrm{br}(E).$$

Therefore we obtain

$$e_0(\mathcal{C}) \leq \mathrm{br}(E)\sum_{i=1}^{c}\binom{e+i-2}{i-1} = \mathrm{br}(E) \cdot \binom{e+c-1}{e},$$

which establishes the last inequality in the theorem. □

An important instance is given by the Briançon–Skoda theorem in [13, Theorem 1]. Lipman and Sathaye showed that if R is a regular local ring, then $\overline{I^{n+\ell-1}} \subset I^n$ for all n, where ℓ is the analytic spread of $I \neq 0$. This shows that the Briançon–Skoda number of $I \neq 0$ is at most $\ell(I) - 1$. Using this theorem, we are able to give a bound for the length n of a chain $\mathcal{R}(E) \subseteq \mathcal{A}_0 \subsetneq \cdots \subsetneq \mathcal{A}_n = \overline{\mathcal{R}(E)}$ of graded algebras satisfying (S_2) between the Rees algebra $\mathcal{R}(E)$ of a module E and its integral closure $\overline{\mathcal{R}(E)}$.

Theorem 2.4 *Let R be a regular local ring of dimension $d \geq 2$ and let $E \subsetneq R^e$ be a submodule with $\lambda(R^e/E) < \infty$. For any distinct graded R–subalgebras $\mathcal{A}$ and $\mathcal{B}$ such that*

$$\mathcal{R}(E) \subseteq \mathcal{A} \subsetneq \mathcal{B} \subseteq \overline{\mathcal{R}(E)},$$

where $\mathcal{A}$ satisfies Serre's condition (S_2)*, we have*

$$0 \leq \mathrm{br}_1(\mathcal{A}) < \mathrm{br}_1(\mathcal{B}) \leq \mathrm{br}(E) \cdot \binom{2e+d-3}{e}.$$

In particular, any chain of graded (S_2) *R–subalgebras containing $\mathcal{R}(E)$ and contained in $\overline{\mathcal{R}(E)}$ has length at most* $\mathrm{br}(E) \cdot \binom{2e+d-3}{e}$.

Proof. We may assume that the residue field of R is infinite. Let $\mathcal{A}$ and $\mathcal{B}$ be any graded R–subalgebras such that

$$\mathcal{R}(E) \subseteq \mathcal{A} \subseteq \mathcal{B} \subseteq \overline{\mathcal{R}(E)},$$

and let $\mathcal{D} = \mathcal{B}/\mathcal{A}$. As in the proof Theorem 2.3, the module $\mathcal{D} = \mathcal{B}/\mathcal{A}$ has dimension at most $d+e-1$ and hence $\mathrm{br}_1(\mathcal{A}) \leq \mathrm{br}_1(\mathcal{B})$. Moreover if $\mathcal{A}$ satisfies (S_2) and $\mathcal{A} \neq \mathcal{B}$, then $\mathcal{D}$ has dimension exactly $d+e-1$ and hence $\mathrm{br}_1(\mathcal{B}) = \mathrm{br}_1(\mathcal{A}) + e_0(\mathcal{D})$ with $e_0(\mathcal{D}) > 0$. In particular $\mathrm{br}_1(\mathcal{A}) < \mathrm{br}_1(\mathcal{B})$. Combining this with Theorem 2.3, we obtain

$$0 \leq \mathrm{br}_1(E) \leq \mathrm{br}_1(\mathcal{A}) < \mathrm{br}_1(\mathcal{B}) \leq \overline{\mathrm{br}}_1(E) \leq \mathrm{br}(E) \cdot \binom{e+c-1}{e},$$

where c is the Briançon–Skoda number of E.

Let $S = R[t_1, \ldots, t_e]$ be a polynomial ring containing the Rees algebra $\mathcal{R}(E)$ as a homogeneous R–subalgebra, $\mathfrak{n}$ the maximal homogeneous ideal of S, and I the S–ideal generated by $E \subset S$. Then by Remark 1.5 and [13, Theorem 1],

$$c = c(E) \leq c(IS_{\mathfrak{n}}) \leq \ell(IS_{\mathfrak{n}}) - 1 = \ell(E) - 1 \leq d+e-2.$$

Hence we obtain

$$\mathrm{br}_1(\mathcal{B}) \leq \mathrm{br}(E) \cdot \binom{2e+d-3}{e}.$$

□

A simple way to find an integral extension of a ring with the property (S_2), which does not necessarily involve canonical modules, arises in the following manner. Let R be a Noetherian domain satisfying (S_2) and let A be a domain containing R that is a finite R–module. Then $B = \mathrm{Hom}_A(\mathrm{Hom}_R(A, R), \mathrm{Hom}_R(A, R))$ is a finite and birational ring extension of A that satisfies (S_2). This observation can be applied to Rees algebras of modules. Let R be an analytically unramified Cohen–Macaulay normal local ring with infinite residue field and let E be an equimultiple module. Let U be a minimal reduction of E. Then $\overline{\mathcal{R}(U)} = \overline{\mathcal{R}(E)}$, the module U is a complete intersection, and the Rees algebra

$T = \mathcal{R}(U)$ is Cohen–Macaulay ([19, 5.6]). Then for any graded R–subalgebra $\mathcal{A}$ with $T \subset \mathcal{A} \subset \overline{T}$,

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_T(\mathcal{A}, T), \mathrm{Hom}_T(\mathcal{A}, T))$$

is a graded R–subalgebra of $\overline{T}$ containing $\mathcal{A}$ that satisfies (S_2).

3 Extended degree for equimultiple modules

In this section we extend the results of the previous section to equimultiple modules. Let R be an analytically unramified Cohen–Macaulay normal local ring, E an equimultiple R–module of rank e and codimension $g \geq 2$, and $\mathrm{Min}(R^e/E) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ the set of all minimal prime ideals in $\mathrm{Supp}(R^e/E)$. Notice that $\dim R_{\mathfrak{p}_i} = g$ for $1 \leq i \leq s$. Let $S = \bigoplus_{n \geq 0} S_n = \mathrm{Sym}(R^e) = R[t_1, \ldots, t_e]$ be a polynomial ring which contains the Rees algebra $\mathcal{R}(E)$ as a homogeneous R–subalgebra. Consider a graded R–subalgebra $\mathcal{A} = \bigoplus_{n \geq 0} A_n$ with $\mathcal{R}(E) \subset \mathcal{A} \subset \overline{\mathcal{R}(E)} \subset S$. We define a function $\deg(S^n/A_n) : \mathbb{N} \to \mathbb{N}$ by

$$\deg(S^n/A_n) = \sum_{i=1}^{s} \lambda\left((S_n/A_n)_{\mathfrak{p}_i}\right) \deg(R/\mathfrak{p}_i),$$

where $\deg(\cdot)$ is the Hilbert–Samuel multiplicity. For $n \gg 0$, the length $\lambda\left((S_n/A_n)_{\mathfrak{p}_i}\right)$ is given by a polynomial $P_i(n)$ of the form

$$P_i(n) = \mathrm{br}(\mathcal{A}_{\mathfrak{p}_i}) \cdot \binom{n+g+e-2}{g+e-1} - \mathrm{br}_1(\mathcal{A}_{\mathfrak{p}_i}) \cdot \binom{n+g+e-3}{g+e-2} + \text{lower terms.}$$

Hence for $n \gg 0$, we have

$$\deg(S^n/A_n) = \sum_{j=0}^{g+e-1} (-1)^j \mathrm{b}_j(\mathcal{A}) \binom{n+g+e-j-2}{g+e-j-1},$$

where

$$\mathrm{b}_0(\mathcal{A}) = \sum_{i=1}^{s} \mathrm{br}(\mathcal{A}_{\mathfrak{p}_i}) \deg(R/\mathfrak{p}_i) \quad \text{and} \quad \mathrm{b}_j(\mathcal{A}) = \sum_{i=1}^{s} \mathrm{br}_j(\mathcal{A}_{\mathfrak{p}_i}) \deg(R/\mathfrak{p}_i) \text{ if } j > 0.$$

We define $\mathrm{b}_j(E) = \mathrm{b}_j(\mathcal{R}(E))$ and $\overline{\mathrm{b}}_j(E) = \mathrm{b}_j(\overline{\mathcal{R}(E)})$.

Proposition 3.1 *Let R be an analytically unramified Cohen–Macaulay normal local ring with infinite residue field and let E be an equimultiple R–module of rank e and codimension $g \geq 2$. Let $\mathrm{Min}(R^e/E) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ and let U be a minimal reduction of E. Then*

(a) $\mathrm{b}_0(E) = \deg(R^e/U)$.

(b) *Let* $\varepsilon = \min_{1\le i\le s}\{\ s_i \mid \mathrm{Fitt}_{e-1}((R^e/E)_{\mathfrak{p}_i})$ is contained in $\mathfrak{p}_i^{s_i} R_{\mathfrak{p}_i}$ but not in $\mathfrak{p}_i^{s_i+1} R_{\mathfrak{p}_i}\}$. *Then*

$$\varepsilon^{g-1}\binom{g+e-2}{e-1}\sum_{i=1}^{s}\deg(R_{\mathfrak{p}_i})\deg(R/\mathfrak{p}_i) \le \mathrm{b}_0(E).$$

(c) *For any graded* R*–subalgebra* $\mathcal{A}$ *with* $\mathcal{R}(E) \subset \mathcal{A} \subset \overline{\mathcal{R}(E)}$, *we have* $\mathrm{b}_0(\mathcal{A}) = \mathrm{b}_0(E)$.

Proof. To prove (a) notice that

$$\mathrm{b}_0(E) = \sum_{i=1}^{s}\mathrm{br}(E_{\mathfrak{p}_i})\deg(R/\mathfrak{p}_i) = \sum_{i=1}^{s}\lambda\left((R^e/U)_{\mathfrak{p}_i}\right)\deg(R/\mathfrak{p}_i) = \deg(R^e/U).$$

Part (b) follows from Proposition 2.1, and (c) is a consequence of the fact that the Buchsbaum–Rim multiplicity does not change when passing to the integral closure of a module ([12, 5.3]). □

Proposition 3.2 *Let* R *be a regular local ring and let* E *be an equimultiple module of rank* e *and codimension* $g \ge 2$. *Then*

$$0 \le \mathrm{b}_1(E) \le \overline{\mathrm{b}}_1(E) \le \mathrm{b}_0(E)\cdot\binom{2e+g-3}{e}.$$

Proof. Let $\mathrm{Min}(R^e/E) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. By Theorems 2.3 and 2.4, we have

$$0 \le \mathrm{br}_1(E_{\mathfrak{p}_i}) \le \overline{\mathrm{br}}_1(E_{\mathfrak{p}_i}) \le \mathrm{br}(E_{\mathfrak{p}_i})\cdot\binom{2e+g-3}{e}.$$

Therefore we obtain

$$\begin{aligned} 0 &\le \mathrm{b}_1(E) = \sum_{i=1}^{s}\mathrm{br}_1(E_{\mathfrak{p}_i})\deg(R/\mathfrak{p}_i) \le \sum_{i=1}^{s}\overline{\mathrm{br}}_1(E_{\mathfrak{p}_i})\deg(R/\mathfrak{p}_i) = \overline{\mathrm{b}}_1(E), \\ \overline{\mathrm{b}}_1(E) &\le \sum_{i=1}^{s}\mathrm{br}(E_{\mathfrak{p}_i})\cdot\binom{2e+g-3}{e}\deg(R/\mathfrak{p}_i) = \mathrm{b}_0(E)\cdot\binom{2e+g-3}{e}. \end{aligned}$$

□

We are now going to use Proposition 3.2 to bound the length of any chain of graded algebras satisfying (S_2) which appear in the process of normalization of an equimultiple module. For ideals this was done in [14, 4.2].

Theorem 3.3 *Let* R *be a regular local ring and let* E *be an equimultiple module of rank* e *and codimension* $g \ge 2$. *For any distinct graded* R*–subalgebras* $\mathcal{A}$ *and* $\mathcal{B}$ *such that*

$$\mathcal{R}(E) \subseteq \mathcal{A} \subsetneq \mathcal{B} \subseteq \overline{\mathcal{R}(E)}$$

and $\mathcal{A}$ satisfies (S_2), *we have*

$$0 \le \mathrm{b}_1(\mathcal{A}) < \mathrm{b}_1(\mathcal{B}) \le \overline{\mathrm{b}}_1(E) \le \mathrm{b}_0(E) \cdot \binom{2e+g-3}{e}.$$

In particular, any chain of graded (S_2) *R–subalgebras containing $\mathcal{R}(E)$ and contained in $\overline{\mathcal{R}(E)}$ has length at most* $\mathrm{b}_0(E) \cdot \binom{2e+g-3}{e}$.

Proof. By Proposition 3.2, it suffices to show that $\mathrm{b}_1(\mathcal{A}) < \mathrm{b}_1(\mathcal{B})$. Consider the exact sequence of $\mathcal{R}(E)$–modules

$$0 \to \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{D} \to 0.$$

Let $\mathrm{Min}(R^e/E) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. Suppose $\mathrm{b}_1(\mathcal{A}) = \mathrm{b}_1(\mathcal{B})$. Then by Theorem 2.4, we have $\mathcal{A}_{\mathfrak{p}_i} = \mathcal{B}_{\mathfrak{p}_i}$ for each i. Hence $L = \mathrm{ann}_R(\mathcal{D})$ has height at least $g+1$. Let $\mathfrak{P}$ be any minimal prime ideal of $L\mathcal{A}$ and $\mathfrak{p} = \mathfrak{P} \cap R$. Notice that $R_{\mathfrak{p}}$ has dimension at least $g+1$. Hence $\dim \mathcal{A}_{\mathfrak{p}}$ is at least $g+1+e$. On the other hand $\dim(\mathcal{A}_{\mathfrak{p}}/L\mathcal{A}_{\mathfrak{p}}) \le \ell(E_{\mathfrak{p}}) = g+e-1$ as E is equimultiple. Therefore the ideal $L\mathcal{A}_{\mathfrak{p}}$ has height at least 2 and hence $L\mathcal{A}$ does. Since $\mathcal{A}$ satisfies (S_2), we have $\mathrm{grade}(L\mathcal{A}) \ge 2$, and therefore $\mathcal{A} = \mathcal{B}$. $\square$

4 Cohen–Macaulay algebras

For any monomial ideal in a polynomial ring in d variables over a field, it is known that the integral closure of the Rees algebra of the ideal is Cohen–Macaulay ([6, Theorem 1]) and that if the first $d-1$ powers of the ideal are integrally closed then it is normal ([17, 3.1]). Similar results are known for an integrally closed module over a 2–dimensional regular local ring ([11, 4.1]). Now we show for a graded R–subalgebra $\mathcal{B}$ of $\overline{\mathcal{R}(E)}$ containing $\mathcal{R}(E)$ that if $\mathcal{B}$ is Cohen–Macaulay then $\mathcal{B}$ is generated by its components of degrees at most $\dim R - 1$.

Theorem 4.1 *Let R be an analytically unramified Cohen–Macaulay normal local ring of dimension $d > 0$ with infinite residue field and let E be a finitely generated torsionfree R–module. Let $\mathcal{B} = \bigoplus_{n \ge 0} B_n$ be a graded R–subalgebra with $\mathcal{R}(E) \subset \mathcal{B} \subset \overline{\mathcal{R}(E)}$. If $\mathcal{B}$ is Cohen–Macaulay, then*

$$B_{n+1} = EB_n \quad \text{for all } n \ge d-1.$$

In particular $\mathcal{B}$ is generated as an $\mathcal{R}(E)$–module by forms of degrees at most $d-1$.

When the module E has a rank 1, which is the case of ideals, the theorem is proved in [15]. That result was based on the characterization of the Cohen–Macaulayness of the Rees algebra of an ideal I in terms of its associated graded ring and its reduction number ([1], [10], [18]). It turns out that Theorem 4.1 is a direct consequence of the case of ideals and the technique of Bourbaki sequences.

Proof. We assume that $\text{rank}(E) = e \geq 2$. Let $a_1, \ldots, a_n$ be a set of R–generators of E and let $\mathbf{Z}$ be a $n \times (e-1)$ matrix of distinct indeterminates over R. Let R' be $R[\mathbf{Z}]_{\mathfrak{m}[\mathbf{Z}]}$. Set

$$E' = R' \otimes_R E, \;\; x_j = \sum_{i=1}^{n} z_{ij} a_i, \;\; F = \sum_{j=1}^{e-1} R' x_j, \;\; \mathcal{B}' = R' \otimes_R \mathcal{B}.$$

Notice that E'/F has rank 1 as an R'–module and that the natural map $\mathcal{R}(E')/F\mathcal{R}(E') \longrightarrow \mathcal{B}'/F\mathcal{B}'$ becomes an isomorphism after tensoring with the quotient field of R. Furthermore as $\text{ht}(E\mathcal{R}(E)) = e$, we have $\text{ht}(E\mathcal{B}) = e$. As $\mathcal{B}$ is a Cohen–Macaulay domain, [7, Theorem and Proposition 3] shows that $\mathcal{B}'/F\mathcal{B}'$ is again a Cohen–Macaulay domain. Thus writing I' for the image of E'/F in $\mathcal{B}'$, we see that I' is an R'–ideal and $\mathcal{R}(I') \subset \mathcal{B}'/F\mathcal{B}'$ is a finite and birational extension. Hence $\mathcal{B}'/F\mathcal{B}'$ is a Cohen–Macaulay ring contained in $\overline{\mathcal{R}(I')}$. Now the assertion follows from the result in the case of ideals ([15]) and an application of Nakayama's Lemma. $\square$

References

[1] I. Aberbach, C. Huneke, and N. Trung, Reduction numbers, Briançon–Skoda theorems and the depth of Rees rings, *Comp. Math.* **97** (1995), 403–434.

[2] J. Brennan, B. Ulrich, and W.V. Vasconcelos, The Buchsbaum–Rim polynomial of a module, *J. Algebra* **241** (2001), 379–392.

[3] J. Brennan and W.V. Vasconcelos, Effective normality criteria for algebras of linear type, *J. Algebra* **273** (2004), 640–656.

[4] D. Buchsbaum and D.S. Rim, A generalized Koszul complex, II. Depth and multiplicity, *Trans. Amer. Math. Soc.* **111** (1964), 197–224.

[5] D. Eisenbud, C. Huneke, and B. Ulrich, What is the Rees algebra of a module?, *Proc. Amer. Math. Soc.* **131** (2003), 701–708.

[6] M. Hochster, Rings of invariants of tori, Cohen–Macaulay rings generated by monomials, and polytopes, *Annals of Math.* **96** (1972), 318–337.

[7] M. Hochster, Properties of Noetherian rings stable under general grade reduction, *Arch. Math.* **24** (1973), 393–396.

[8] C. Huneke, Uniform bounds in Noetherian rings, *Invent. Math.* **107** (1992), 203–223.

[9] C. Huneke and M.E. Rossi, The dimension and components of symmetric algebras, *J. Algebra* **98** (1986), 200–210.

[10] B. Johnston and D. Katz, Castelnuovo regularity and graded rings associated to an ideal, *Proc. Amer. Math. Soc.* **123** (1995), 727–734.

[11] D. Katz and V. Kodiyalam, Symmetric powers of complete modules over a two–dimensional regular local ring, *Trans. Amer. Math. Soc.* **349** (1997), 481–500.

[12] S. Kleiman and A. Thorup, A geometric theory of the Buchsbaum–Rim multiplicity, *J. Algebra* **167** (1994), 168–231.

[13] J. Lipman and A. Sathaye, Jacobian ideals and a theorem of Briançon–Skoda, *Michigan Math. J.* **28** (1981), 97–116.

[14] C. Polini, B. Ulrich, and W.V. Vasconcelos, Normalization of ideals and Briançon–Skoda Numbers, *Math. Res. Lett.* **12** (2005), 827–842.

[15] C. Polini, B. Ulrich, W.V. Vasconcelos, and R. Villarreal, Indices of normalization of ideals, in preparation.

[16] D. Rees, Reductions of modules, *Math. Proc. Camb. Phil. Soc.* **101** (1987), 431–449.

[17] L. Reid, L.G. Roberts, and M.A. Vitulli, Some results on normal homogeneous ideals, *Comm. Alg.* **31** (2003), 4485–4506.

[18] A. Simis, B. Ulrich, and W.V. Vasconcelos, Cohen–Macaulay Rees algebras and degrees of polynomial relations, *Math. Ann.* **301** (1995), 421–444.

[19] A. Simis, B. Ulrich, and W.V. Vasconcelos, Rees algebras of modules, *Proc. London Math. Soc.* **87** (2003), 610–646.