Effective Normality Criteria for Algebras of Linear Type

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December 14, 2004

Dedicated to Jürgen Herzog on the occasion of his sixtieth-second birthday

Abstract

The algebras studied here are subalgebras of rings of polynomials generated by 1–forms (so–called Rees algebras), with coefficients in a Noetherian ring. Given a normal domain R and a torsionfree module E with a free resolution,

$$\cdots \longrightarrow F_2 \stackrel{\psi}{\longrightarrow} F_1 \stackrel{\varphi}{\longrightarrow} F_0 \longrightarrow E \to 0,$$

we study the role of the matrices of syzygies in the normality of the Rees algebra of E. When the Rees algebra $\mathcal{R}(E)$ and the symmetric algebra $S(E)$ coincide, the main results characterize normality in terms of the ideal $I_c(\psi)S(E)$ and of the completeness of the first s symmetric powers of E, where $c = \text{rank } \psi$, and $s = \text{rank } F_0 - \text{rank } E$. It requires that R be a regular domain. Special results, under broader conditions on R, are still more effective.

1 Introduction

Let R be a commutative, Noetherian ring and let A be a (commutative) finitely generated graded R–algebra generated by its component of degree one,

$$A = R + A_1 + A_2 + \cdots = R[A_1].$$

[0]2000 AMS *Mathematics Subject Classification.* Primary 13B22; Secondary 13H05, 13H10.
[0]*Key Words and Phrases.* Algebra of linear type, complete module, normal algebra, Rees algebra, symmetric algebra.
*Partially supported by the NSF.

There is a natural surjection $\tau : S_R(A_1) \to A$, from the symmetric algebra of A_1 onto A, and A is said to be of *linear type* if τ is an isomorphism. Such algebras are naturally defined in terms of generators and relations of its degree 1 component. Setting $E = A_1$, if

$$R^m \xrightarrow{\varphi} R^n \longrightarrow E \to 0$$

is a presentation of E as an R–module,

$$S(E) = S_R(E) \simeq R[T_1, \ldots, T_n]/(f_1, \ldots, f_m),$$

where the f_i's are linear forms in the T_i–variables, obtained from the matrix multiplication $[T_1, \ldots, T_n] \cdot \varphi$. The equalities $S_t(E) \simeq A_t$, $t \geq 1$, impose several known restrictions on the matrix φ (see [17, Chapter 1]). We are going to assume that R is a (normal) domain and the A_t are torsionfree R–modules, when these algebras can also be described as follows. Let K be the field of fractions of R, and let E be a finitely generated torsionfree of rank r, that is $K \otimes_R E \cong K^r$. This provides an embedding $E \hookrightarrow R^r$ which allows us to define the *Rees algebra of* E as the subalgebra $\mathcal{R}(E)$ of $S(R^r)$ generated by the forms in E ($S(\cdot)$ denotes the symmetric algebra functor): $\mathcal{R}(E)$ is a standard graded algebra, whose component of degree n, $\mathcal{R}(E)_n \subset S_n(R^r)$, is a module of n–forms of the ring of polynomials $S(R^r) = R[T_1, \ldots, T_r]$. It is a natural extension of the blowup algebra of an ideal I, $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n$. When we have the isomorphism $S(E) \simeq \mathcal{R}(E)$ (which with the fixed embedding we view simply as an equality), it is standard terminology to refer to E as an R–module of *linear type*. Looking at $\mathcal{R}(E)$ as a symmetric algebra or as a subring of a ring of polynomials provides us with two windows through which the properties of the algebra can be examined.

In this paper we develop effective criteria of normality for Rees algebras, especially of modules of linear type. In particular, given a normal domain R and a torsionfree module E with a free resolution,

$$\cdots \longrightarrow F_2 \xrightarrow{\psi} F_1 \xrightarrow{\varphi} F_0 \longrightarrow E \to 0,$$

we study the role of the matrices of syzygies in the normality of the Rees algebra of E. This goal being, in general, too open ended, we will restrict ourselves to those algebras which are of linear type, that is, $\mathcal{R}(E) = S(E)$.

Let us describe the contents of this paper. In Section 2 we introduce the order determinant of a module E, and discuss some of its properties in relation to the integral closure of E. Since it is needed later, we establish the $\mathfrak{m}$–fullness property of complete modules. (At this point, we warn the reader for using the qualifier *complete* in very distinct concepts: complete modules, complete intersection modules and complete intersection algebras.)

In Section 3, we embark on our analysis of the normality of the following families of algebras:

- Algebras of complete intersection modules

- Complete intersection algebras
- Almost complete intersection algebras
- General algebras of linear type

The first three classes are described by data very amenable to verification of normality. Furthermore, should the algebras turn out to be not normal, the processes involved provide proper integral extensions.

The main results of this paper (Theorems 6.1 and 6.2) characterize normality of $\mathcal{R}(E)$, over a regular domain R, in terms of the ideal $I_c(\psi)S(E)$ and of the completeness of the first s symmetric powers of E, where $c = \text{rank } \psi$, and $s = \text{rank } F_0 - \text{rank } E$. It requires that R be a regular domain. The other results (Theorems 3.1, 4.1, and 4.2), are variations of this result for special classes of modules but under less restrictive hypotheses on the base rings. Moreover, they are conveniently framed for computation. For this class of algebras, certification of normality of this method outperforms the direct approach of Jacobian criteria. In addition, the methods are mostly characteristic free.

The authors are very grateful to the referee for the detailed reading and numerous suggestions that lead to a more transparent manuscript.

2 Complete modules

This section has a preliminary character in which we extend some properties of complete ideals to modules. Most of the facts we are going to review quickly can be traceable to [11]. For basic facts and terminology we shall use [3] and [5]. As for notation: let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely generated R–module; we set $\nu(M)$ for the minimal number of generators of M, $\lambda(M)$ for its length if M has a finite composition series. If M is a torsionfree R–module with Rees algebra $\mathcal{R}(M)$, its analytic spread $\ell(M)$ is the dimension of the special fiber of the Rees algebra of M, $\mathcal{R}(M) \otimes_R (R/\mathfrak{m})$. If M is a module of linear type, $\ell(M) = \dim S_R(M)/\mathfrak{m}S_R(M) = S_{R/\mathfrak{m}}(M/\mathfrak{m}M) = \nu(M)$.

Throughout R will be a Noetherian domain with a field of fractions K. Let E be a submodule of R^r, of rank r, and let $\mathcal{R}(E)$ be its Rees algebra. In analogy with the case of ideals, one has the following key notion.

Definition 2.1 Let $U \subset E$ be a submodule. We say that U is a *reduction* of E or, equivalently, E is *integral* over U, if $\mathcal{R}(E)$ is integral over the R-subalgebra generated by U.

Alternatively, the integrality condition is expressed by the equations $\mathcal{R}(E)_{s+1} = U \cdot \mathcal{R}(E)_s$, $s \gg 0$. The least integer $s \geq 0$ for which this equality holds is called the *reduction number* of E *with respect to* U and denoted by $r_U(E)$. For any reduction U of E the module E/U is torsion, hence U has the same rank as E. This follows from the fact that a module of linear type, such as a free module, admits no proper reductions.

Let E be a submodule of R^r. The *integral closure* of E in R^r is the largest submodule $\overline{E} \subset R^r$ having E as a reduction. E is *integrally closed* or *complete* if $E = \overline{E}$. If $\mathcal{R}(E)$ is the Rees algebra of E, E is said to be *normal* if all components $\mathcal{R}(E)_n$ are complete. Since $\mathcal{R}(E)$ is a standard graded R–algebra, and $\mathcal{R}(E)_1 = E$, without risk of confusion we set $\mathcal{R}(E)_n = E^n$. When E is of linear type, $E^n = S_n(E)$.

We introduce an ideal measuring the embedding $E \hookrightarrow R^r$, and discuss its role.

Definition 2.2 Let E be a finitely generated R–module of rank r. The *order determinant* of the embedding $E \stackrel{\varphi}{\longrightarrow} R^r$ is the ideal I defined by the image of the mapping $\wedge^r \varphi$,

$$\text{image } (\wedge^r \varphi) = I \cdot \wedge^r (R^r).$$

When the embedding is clear we will write $I = \det_0(E)$.

Proposition 2.3 ([11]) *Let $F \subset E$ be torsionfree R–modules of rank r and $E \hookrightarrow R^r$ an embedding. Denote by* $\det_0(F)$ *and* $\det_0(E)$ *the corresponding order determinants. F is a reduction of E if and only if* $\det_0(F)$ *is a reduction of* $\det_0(E)$.

Proof. Let $S = \mathcal{R}(R^r) = R[T_1, \ldots, T_r]$, and $\mathcal{R}(F) \subset \mathcal{R}(E)$ the Rees algebras of F and E. Since the three algebras have the same fields of fractions, they have the same valuation overrings. For a valuation ring $\mathbf{V}$ of S, let $V = \mathbf{V} \cap K$. We note that $V\mathcal{R}(F), V\mathcal{R}(E), VS$ are rings of polynomials over the free modules $VF \subset VE \subset V^r$, and therefore are integrally closed. It follows that F is a reduction of E if and only if $VF = VE$ for each of the valuations such as V.

We check that the order determinants detect this situation. First, note that for any domain T that is an overring of R, one has that $\det_0(E)T = \det_0(TE)$. This follows because TE is the image of $T \otimes_R E$ in $T \otimes_R R^r = T^r$ and exterior powers commute with base change.

Finally we observe that in the embedding of free modules

$$VF \stackrel{\alpha}{\longrightarrow} VE \stackrel{\beta}{\longrightarrow} V^r,$$

the two ideals $V \det_0(E)$ and $V \det_0(F)$ are generated by $\det(\beta)$ and $\det(\beta \cdot \alpha)$, respectively. Thus our conditions are equivalent to $\det(\alpha)$ being a unit of V. $\square$

Making use of the notion of Rees valuations (see a discussion in [6]) we reduce the number of valuation rings in the description of the integral closure of a module.

Corollary 2.4 *Let R be an integral domain and let E be a submodule of rank r of the free module R^r. The integral closure of E in R^r is given by*

$$\overline{E} = R^r \cap \bigcap_V VE,$$

where V runs over the Rees valuations of $\det_0(E)$.

We have the following application to the natural extension of $\mathfrak{m}$–full ideals to modules.

Definition 2.5 Let $(R, \mathfrak{m})$ be a Noetherian local ring and let E be a submodule of R^r. E is an $\mathfrak{m}$*–full module* if there is an element $x \in \mathfrak{m}$ such that $\mathfrak{m}E :_{R^r} x = E$.

Proposition 2.6 *Suppose* $(R, \mathfrak{m})$ *is a Noetherian local domain and* $E \hookrightarrow R^r$ *is a module of rank* r*. If the residue field of* R *is infinite, complete modules are* $\mathfrak{m}$*–full, that is there exists* $x \in \mathfrak{m}$ *such that* $\mathfrak{m}E :_{R^r} x = E$*.*

Proof. Set $L = \det_0(E)$. The proof is similar to the ideal case, according to [6], but takes into account the role of the order determinantal ideal in the description of the integral closure observed above.

Let $(V_1, \mathfrak{p}_1), \ldots, (V_n, \mathfrak{p}_n)$ be the Rees valuations associated to L. Observe that since, for each i, $\mathfrak{m}V_i \neq 0$, we have that $\mathfrak{m}$ contains $L_i = \mathfrak{p}_i\mathfrak{m}V_i \cap R$ properly. Since $R/\mathfrak{m}$ is infinite, we have the comparison of vector spaces

$$\mathfrak{m}/\mathfrak{m}^2 \neq \sum_{i=1}^{n} L_i + \mathfrak{m}^2/\mathfrak{m}^2.$$

Thus we can choose $x \in \mathfrak{m}$ that is not contained in $\bigcup_{i=1}^{n} \mathfrak{p}_i\mathfrak{m}V_i$. This means that for each i, $\mathfrak{m}V_i = xV_i$. We claim this element will do. Let $y \in \mathfrak{m}E :_{R^r} x$. For each V_i we have $yx \in \mathfrak{m}V_iE = xV_iE$. Thus $y \in \bigcap_{i=1}^{n} V_iE \cap R^r$, and therefore $y \in \overline{E} = E$. □

As a consequence one has:

Corollary 2.7 *Let* E *be an* $\mathfrak{m}$*–full submodule of* R^r *of rank* r*. Suppose* G *is a module,* $E \subset G \subset R^r$ *that satisfies* $\lambda(G/E) < \infty$*. Then* $\nu(G) \leq \nu(E)$*.*

Proof. Using the notation above, like in the ideal case, consider the exact sequence

$$0 \to E/\mathfrak{m}E \longrightarrow G/\mathfrak{m}E \xrightarrow{\cdot x} G/\mathfrak{m}E \longrightarrow G/(xG, \mathfrak{m}E) \to 0.$$

We have

$$\nu(E) = \nu(E/\mathfrak{m}E) = \nu(G/(xG + \mathfrak{m}E)) \geq \nu(G),$$

as desired. □

3 Complete intersection modules

Let R be a Cohen–Macaulay ring and let E be a *complete intersection module.* This means that there exists a mapping of rank r

$$\varphi : R^m \longrightarrow R^r,$$

with $E = \text{image } \varphi$, with $m = r + c - 1$, for $c \geq 2$, and the ideal $I = \det_0(E)$ has codimension c.

Our purpose is to describe, entirely in terms of the ideal I, when E is integrally closed. It turns out that more is achieved.

Theorem 3.1 *Let R be a Cohen–Macaulay integrally closed domain and let E be a complete intersection module. The following conditions are equivalent:*

(a) *E is integrally closed.*

(b) *E is normal.*

(c) $\det_0(E)$ *is an integrally closed generic complete intersection.*

If moreover $\det_0(E)$ *is a prime ideal the conditions above hold.*

The proof of the theorem for modules will be inspired by the arguments in [6], with some technicalities stripped away due to the assumption that R is a Cohen–Macaulay domain.

Proof. Let $\mathcal{R} = \mathcal{R}(E)$ be the Rees algebra of the module E. According to [8, Lemma 3.2] or [14, Theorem 5.6], E is a module of linear type and $\mathcal{R}$ is a Cohen–Macaulay ring.

To check the normality of $\mathcal{R}$, we need to understand the height 1 prime ideals P of $\mathcal{R}$. Set $\mathfrak{p} = P \cap R$ and localize at $R_{\mathfrak{p}}$. If $\mathfrak{p} = 0$, $\mathcal{R}(E)_{\mathfrak{p}}$ is a ring of polynomials over the field of fractions of R, and we have nothing to be concerned about. We may assume that $(R, \mathfrak{m})$ is a local ring and that $P = \mathfrak{m}\mathcal{R}(E)$. If $\dim R = 1$, R is a discrete valuation ring and E is a free R–module.

We may thus assume $\dim R \geq 2$. If $E \subset R^r$ contains a free summand of R^r, $E = E_0 \oplus R^s$, $\mathcal{R}(E)$ would be a Rees algebra of a complete intersection module of rank $r - s$, over the ring of polynomials $R[T_1, \ldots, T_s]$. Thus all conditions would be preserved in E_0. We may finally assume that $E \subset \mathfrak{m}R^r$.

We claim that $\dim R = d = c$, that is $\mathfrak{m}$ is a minimal prime of I, in particular the module R^r/E has finite length. This follows simply from the equality

$$\begin{aligned} \text{height } \mathfrak{m}\mathcal{R} &= \dim \mathcal{R} - \ell(E) \\ &= (d + r) - (c + r - 1) = (d - c) + 1, \end{aligned}$$

since E is of linear type and therefore its analytic spread $\ell(E)$ is equal to its minimal number of generators.

Let us assume that condition (a) holds, that is E is integrally closed. We denote by d_0 the embedding dimension of R. Among other things, we must show that $d = d_0$, which means that R is a regular local ring. According to Proposition 2.7, E is $\mathfrak{m}$–full. Since $\lambda(\mathfrak{m}R^r/E) < \infty$, by Corollary 2.7, we must have

$$\nu(E) \geq \nu(\mathfrak{m}R^r).$$

Because the module $\mathfrak{m}R^r$ is minimally generated by rd_0 elements, while E itself is generated by $r + d - 1$ elements, we have

$$r + d - 1 \geq rd_0,$$

which means that

$$(r - 1)(d - 1) \leq r(d - d_0) \leq 0.$$

Since we assumed $d \geq 2$, this means that $d = d_0$ and that $r = 1$. In other words, E is isomorphic to a system of parameters of a regular ring. The assertion thus follows from the ideal case ([6]).

To prove the other implications, we observe that given an embedding $E \hookrightarrow R^r$, to verify that $E = \overline{E}$, it suffices to consider localizations $R_{\mathfrak{p}}$ for the associated primes of R^r/E. Thus to show (c) $\Rightarrow$ (a), or (b), we first recall the equality of associated primes (see [8, Theorem 3.3], [2, Proposition 3.3])

$$\forall n, \quad \text{Ass } (S_n(R^r)/E^n) = \text{Ass } (R/\det_0(E)).$$

Localizing at one of the associated primes of $\det_0(E)$, we may assume that $(R, \mathfrak{m})$ is a local ring and $\det_0(E)$ is a $\mathfrak{m}$–primary parameter ideal. Since we may assume that $\dim R \geq 2$, by the criterion of [6], $\det_0(E) \not\subset \mathfrak{m}^2$. This gives a decomposition $E \simeq R^{r-1} \oplus \det_0(E)$, and therefore $\mathcal{R}(E) \simeq \mathcal{R}(\det_0(E))[T_1, \ldots, T_{r-1}]$, which is a normal ring as $\det_0(E)$ is a normal ideal, again by [6].

Finally, the ideal $\mathfrak{p} = \det_0(E)$ has finite projective dimension (Eagon–Northcott complex) and therefore if $\mathfrak{p}$ is prime, $R_{\mathfrak{p}}$ is a regular local ring and (c) will hold. $\square$

Example 3.1 Let $\mathfrak{p}$ be a perfect prime ideal of codimension 2 generated by the maximal minors of the $n \times (n+1)$ matrix φ. According to the assertion above, the columns of φ generate a normal submodule of R^n. In contrast, $\mathfrak{p}$ is rarely a normal ideal. This also shows that the Rees algebra generated by the column vectors of a generic matrix is always normal.

Complete intersection ideals can be tested for completeness (or even normality) very efficiently:

Theorem 3.2 *Let R be a Cohen–Macaulay integral domain and let I be a height unmixed, generic complete intersection ideal of codimension c. I is integrally closed if and only if the following conditions hold:*

(i) height ann $\wedge^{c+1} \sqrt{I} \geq c+1$;

(ii) height ann $\wedge^2 (\sqrt{I}/I) \geq c+1$.

Proof. The meaning of these numerical conditions is simply the following: (i) For each minimal prime $\mathfrak{p}$ of I, $R_{\mathfrak{p}}$ is a regular local ring. The condition (ii) means that at each such prime $\mathfrak{p}$, $I_{\mathfrak{p}}$ has at least $c-1$ of the elements in a minimal generating set of $\mathfrak{p}R_{\mathfrak{p}}$. Together they imply that the primary components of I are integrally closed according to [6]. $\square$

4 Complete intersection algebras

Let R be a Noetherian ring and A a finitely generated R–algebra, $A = R[T_1, \ldots, T_n]/J$. We will say that A is a complete intersection R–algebra if J is generated by a regular sequence. Let R be an integrally closed Cohen–Macaulay domain and let E be a finitely generated torsionfree R–module. Locally, the condition that the Rees algebra $\mathcal{R}(E)$ be a complete intersection can be characterized by the projective presentation of E: If

$$R^m \xrightarrow{\varphi} R^n \longrightarrow E \to 0, \tag{1}$$

is a minimal presentation, the m 1–forms $(f_1, \ldots, f_m) = [T_1, \ldots, T_n] \cdot \varphi$ are necessarily minimal generators in any (algebra) presentation of $\mathcal{R}(E)$. Since $\dim \mathcal{R}(E) = \dim R + (n - m)$, $\mathcal{R}(E)$ is a complete intersection if and only if the f_j form a regular sequence generating a prime ideal. Furthermore, E is of linear type (see [1]).

Conversely, according to [17, Theorem 3.1.6], for $S(E)$ to be a complete intersection we must have

$$\text{grade } I_t(\varphi) \geq \text{rank}(\varphi) - t + 1 = m - t + 1, \quad 1 \leq t \leq m.$$

Moreover, since $S(E) = \mathcal{R}(E)$ is an integral domain, this requirement will hold modulo any nonzero element of R, so it will be strengthened to

$$\text{grade } I_t(\varphi) \geq \text{rank}(\varphi) - t + 2 = m - t + 2, \quad 1 \leq t \leq m.$$

It can be rephrased in terms of the local number of generators as

$$\nu(E_{\mathfrak{p}}) \leq n - m + \text{height } \mathfrak{p} - 1, \quad \mathfrak{p} \neq 0.$$

For these modules, the graded components of the Koszul complex of the forms of $A = R[T_1, \ldots, T_n]$ (see [1]),

$$[f_1, \ldots, f_m] = [T_1, \ldots, T_n] \cdot \varphi,$$

give R–projective resolutions of the symmetric powers of E:

$$0 \to \wedge^s R^m \to \wedge^{s-1} R^m \otimes R^n \to \cdots \to R^m \otimes S_{s-1}(R^n) \to S_s(R^n) \to S_s(E) \to 0.$$

In particular we have:

Proposition 4.1 *Let R be a Cohen–Macaulay normal domain and let E be a finitely generated torsionfree R–module such that the Rees algebra $\mathcal{R}(E)$ is a complete intersection defined by m equations. The non-normal R–locus of $\mathcal{R}(E)$ has codimension at most $m + 1$.*

Proof. It suffices to consider one observation. For any torsionfree R–module G, contained in a free module F, the embedding

$$\overline{G}/G \hookrightarrow F/G$$

shows that if G has projective dimension $\leq r$, the associated primes of $\overline{G}/G$ have codimension at most $r + 1$. In the case of the symmetric powers $S_s(E)$ of E, the projective dimensions are bounded by m, from the comments above. □

Our next aim is to discuss the normality of the algebra $\mathcal{R}(E)$ versus the completeness of the module E and of a few other symmetric powers.

We will begin our analysis with the special case of a hypersurface. Assume that the module E is torsionfree with one defining relation

$$0 \to R\mathbf{a} \longrightarrow R^n \longrightarrow E \to 0, \quad \mathbf{a} = (a_1, \ldots, a_n).$$

The ideal generated by the a_i has grade at least 2, and

$$S(E) = R[T_1, \ldots, T_n]/(f), \quad f = a_1T_1 + \cdots + a_nT_n.$$

Proposition 4.2 *In the setting above, $S(E)$ is normal if and only if for every prime ideal $\mathfrak{p}$ of codimension two that contains $(a_1, \ldots, a_n)$, the following conditions hold:*

(i) $R_{\mathfrak{p}}$ *is a regular local ring.*

(ii) $(a_1, \ldots, a_n) \not\subset \mathfrak{p}^{(2)}$.

Proof. Since $S(E)$ is Cohen–Macaulay, to test for normality if suffices to check the localizations $S(E)_P$, where P is a prime ideal of $A = R[T_1, \ldots, T_n]$ of height 2 that contains f. Let $\mathfrak{p} = P \cap R$. If height $\mathfrak{p} \leq 1$, $E_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$–module, and $S(E_{\mathfrak{p}})$ is a ring of polynomials over a DVR. Otherwise $P = \mathfrak{p}A$, with height $\mathfrak{p} = 2$.

The listed conditions just express the fact that $(\mathfrak{p}A/(f))_P$ is a cyclic module. □

Corollary 4.3 *Suppose further that $R = k[x_1, \ldots, x_d]$ is a ring of polynomials over a field of characteristic zero. $S(E)$ is normal if and only if*

$$\text{height } (a_1, \ldots, a_n, \frac{\partial a_i}{\partial x_j}, \ i = 1 \ldots n, \ j = 1 \ldots d) \geq 3.$$

While the normality is straightforward, the completeness of E requires a different kind of analysis; see also [9].

Theorem 4.1 *Let R be an integrally closed Cohen–Macaulay domain that is regular in codimension at most two. If E is a module defined by a single relation as above, then E is complete if and only if E is normal.*

Proof. According to the previous discussion, E is not normal precisely when there exists a prime ideal of codimension two $\mathfrak{p} \supset (a_1, \ldots, a_n)$, such that $(\mathfrak{p}A/(f))_{\mathfrak{p}A}$ is not principal. We will argue that in such case there exists an element h in the integral closure of $S(E)$, of degree 1 but not lying in E. In other words, E is not complete.

We replace R by $R_{\mathfrak{p}}$, and denote by x, y a set of generators of its maximal ideal. In the one–dimensional local ring $B = S(E)_{\mathfrak{p}S(E)}$, $P = \mathfrak{p}B$, one have that $\text{Hom}_B(P, P) = P^{-1}$, since P is not principal. It will suffice to find elements of degree 1 in $(\mathfrak{p}S(E))^{-1}$ that do not belong to $E = S_1$.

Let $z \in \mathfrak{p}$, so that x, f is a regular sequence in $\mathfrak{p}A$. Writing

$$\begin{aligned} z &= ax + by \\ f &= cx + dy, \end{aligned}$$

where c and d are elements of E, we obtain

$$(z, f) : \mathfrak{p}A = (z, f, ad - bc).$$

The fraction $h = (ad - bc)z^{-1} \notin E$ and has the desired property. □

When $S(E)$ is defined by more than one hypersurface, the role of the defining matrix is more delicate.

Theorem 4.2 *Let R be an integrally closed Cohen–Macaulay domain and let E be a torsionfree R–module of linear type with a projective resolution*

$$0 \to R^m \xrightarrow{\varphi} R^n \longrightarrow E \to 0.$$

$\mathcal{R}(E)$ is normal if and only if the following conditions hold:

(i) *For every prime ideal $\mathfrak{p} \supset I_t(\varphi)$ of height $m - t + 2$, $R_\mathfrak{p}$ is a regular local ring.*

(ii) *The modules $S_s(E)$ are complete, for $s = 1 \ldots m$.*

Proof. Let us assume that $\mathcal{R}(E)(= S(E))$ is normal, and establish (i). Let $\mathfrak{p}$ be a prime ideal as in (i). Since height $I_{t-1}(\varphi) \geq m - t + 3$, localizing at $\mathfrak{p}$, we obtain a presentation of the module in the form (we set still $R = R_\mathfrak{p}$)

$$0 \to R^{m-t+1} \xrightarrow{\varphi'} R^{n-t+1} \longrightarrow E \to 0,$$

where the entries of φ' lie in $\mathfrak{p}$. Changing notation, this means that we can assume that all entries of φ lie in the maximal ideal $\mathfrak{p}$ Setting $A = R[T_1, \ldots, T_n]$, and $P = \mathfrak{p}A$,

$$S(E)_P = (A/(f_1, \ldots, f_m))_P,$$

and therefore if S_P is a DVR, A_P must be a regular local ring, and therefore R will also be a regular local ring.

For the converse, let P be a prime ideal of the ring of polynomials A, of height $m + 1$, containing the forms f_i's, and set $\mathfrak{p} = P \cap R$. The normality of $S(E)$ means that for all such P, $S(E)_P$ is a DVR. The claim is that failure of this to hold is controlled by either (i) or (ii).

We may localize at $\mathfrak{p}$. Suppose that height $\mathfrak{p} = m + 1$. In this case, there exists $z \in \mathfrak{p}$ so that $z, f_1, \ldots, f_m$ is a regular sequence in $\mathfrak{p}A$. If $x_1, \ldots, x_{m+1}$ is a regular system of parameters of the local ring R, from a representation

$$[z, f_1, \ldots, f_m] = [x_1, \ldots, x_{m+1}] \cdot \psi,$$

as in Theorem 4.1, we obtain the socle equality, according to [10]

$$(z, f_1, \ldots, f_m) : \mathfrak{p}A = (z, f_1, \ldots, f_m, \det \psi).$$

The image u of $z^{-1}\det\psi$, provides us with a nonzero form of degree m, in the field of fractions of $S(E)$. If $S(E)_P$ is not a DVR,

$$\mathrm{Hom}_{S(E)}(PS(E)_P, PS(E)_P) = (PS(E)_P)^{-1},$$

and, given that $u \in (PS(E)_P)^{-1}$, we obtained a fresh element in the integral closure of $S_m(E)$.

On the other hand, if height $\mathfrak{p} \leq m$, we may assume that for some $1 < t \leq m$, $I_t(\varphi) \subset \mathfrak{p}$, but $I_{t-1}(\varphi) \not\subset \mathfrak{p}$. This implies that height $\mathfrak{p} = m - t + 2$. We can localize at $\mathfrak{p}$, and argue as in the previous case. □

5 Almost complete intersection algebras

We shall now consider modules of linear type whose Rees algebras are *almost complete intersections.* Let $(R, \mathfrak{m})$ be a Cohen–Macaulay integral domain and let E be a torsionfree R–module of finite projective dimension. Let

$$R^p \stackrel{\psi}{\longrightarrow} R^m \stackrel{\varphi}{\longrightarrow} R^n \longrightarrow E \to 0$$

be a minimal presentation of E. If E is of linear type and the ideal of relations, $(f_1, \ldots, f_m) = [T_1, \ldots, T_n] \cdot \varphi$, has codimension $m-1$, the linear relations among the f_j has rank 1, and therefore has a single generator being a second syzygy module of finite projective dimension. This means that $p = 1$ and ψ is injective.

For the module E to be of linear type the roles of the determinantal ideals $I_t(\varphi)$ and of $I = I_1(\psi)$ are less well behaved than in the case of complete intersections. Let us give a summary of some of the known results, according to [17, Section 3.4]:

Theorem 5.1 *Let R be a Cohen–Macaulay integral domain and let E be a torsionfree R–module whose second Betti number is* 1 *(in particular of projective dimension* 2*). The following hold:*

(a) *If $S(E) = \mathcal{R}(E)$ then* height $I_1(\psi)$ *is odd.*

(b) *If I is a strongly Cohen–Macaulay ideal of codimension* 3*, satisfying the condition $\mathcal{F}_1$, and E^* is a third syzygy module, then $S(E)$ is a Cohen–Macaulay integral domain.*

This shows the kind of requirement that must be present when one wants to construct integrally closed Rees algebras in this class.

Example 5.1 Let $R = k[x_1, \ldots, x_d]$ be a ring of polynomials. In [19], for each $d \geq 4$, it is described an indecomposable vector bundle on the punctured spectrum of R, of rank $d-2$. Its module E of global sections has a resolution

$$0 \to R \stackrel{\psi}{\longrightarrow} R^d \stackrel{\varphi}{\longrightarrow} R^{2d-3} \longrightarrow E \to 0,$$

with φ having linear forms as entries, and $\psi(1) = [x_1, \ldots, x_d]$. If d is odd, according to [12, Corollary 3.10], E is of linear type and normal.

The analysis of the normality of the two previous classes of Rees algebras was made simpler because they were naturally Cohen–Macaulay. This is not the case any longer with almost complete intersections, requiring that the S_2 condition be imposed in some fashion. We pick one closely related to normality.

Theorem 5.2 *Let R be a regular integral domain and let E be a torsionfree module whose second Betti number is 1. Suppose E is of linear type. If the ideal $I_1(\psi)\mathcal{R}(E)$ is principal at all localizations of $\mathcal{R}(E)$ of depth 1 then $\mathcal{R}(E)$ satisfies the condition S_2 of Serre.*

Proof. The condition on $L = I_1(\psi)\mathcal{R}(E)$ means that for any localization S_P, $S = \mathcal{R}(E)$, with depth $S_P = 1$, the ideal L_P is principal. There are several global ways to recast this, such as $(L \cdot L^{-1})^{-1} = S$.

We may assume that R is a local ring, and that the resolution of E is minimal. Pick in $A = R[T_1, \ldots, T_n]$ a prime ideal P for which depth $S_P = 1$. We must show that $\dim S_P = 1$. As in the other cases, we may assume that $P \cap R$ is the maximal ideal of R.

We derive now a presentation of the ideal $J = J(\varphi) = (f_1, \ldots, f_m)A$, modulo J:

$$0 \to K \to (A/J)^m = S^m \longrightarrow J/J^2 \to 0,$$

and analyze the element of S induced by $v = \psi(1)$. This is a nonzero 'vector' whose entries in S_P generate L_P, which by assumption is a principal ideal. This means that $v = \alpha v_0$, for some nonzero $\alpha \in S_P$, where v_0 is an unimodular element S_P. Since $v \in K$, this means that the image u of v_0 in $(J/J^2)_P$, is a torsion element of the module. Two cases arise. If $u = 0$, $(J/J^2)_P$ is a free S_P since it has also rank $m-1$, and therefore J_P is a complete intersection in the regular local ring A_P, according to [15]. This implies that the Cohen–Macaulay local ring S_P has dimension 1. On the other hand, if $u \neq 0$, u is a torsion element of $(J/J^2)_P$ which is also a minimal generator of the module. We thus have that in the exact sequence

$$0 \to \text{torsion } (J/J^2)_P \to (J/J^2)_P \to C \to 0,$$

C is a torsionfree S_P–module, of rank $m-1$, generated by $m-1$ elements. We thus have that the ideal J_P of the regular local ring A_P has the property

$$J_P/J_P^2 \simeq (A_P/J_P)^{m-1} \oplus (\text{torsion}).$$

According to [15] again, J_P is a complete intersection since it has codimension $m-1$. This shows that S_P must have dimension 1. □

In studying the normality in an algebra $S = S(E)$ (E torsionfree) of linear type, the advantage of S_2 holding is extremely useful. To recall briefly some technical facts from [17, p. 138]. For a prime ideal $\mathfrak{p} \subset R$ there is an associated prime ideal in $S(E)$ defined by

$$T(\mathfrak{p}) = \ker(S_R(E) \longrightarrow S_{R/\mathfrak{p}}(E/\mathfrak{p}E)_{\mathfrak{p}}).$$

Since R is a domain, $T(\mathfrak{p})$ is the contraction $\mathfrak{p}S(E)_{\mathfrak{p}} \cap S(E)$, so height $T(\mathfrak{p})$ = height $\mathfrak{p}S(E)$.

The prime ideals we are interested in are those of height 1. If R is equidimensional, we see that

$$\text{height } T(\mathfrak{p}) = 1 \text{ if and only if } \nu(E_{\mathfrak{p}}) = \text{height } \mathfrak{p} + \text{rank}(E) - 1.$$

Let us quote [17, Proposition 5.6.2]:

Proposition 5.2 *Let R be a universally catenarian Noetherian ring and let E be a finitely generated module such that $S(E)$ is a domain. Then the set*

$$\{T(\mathfrak{p}) \mid \text{height } \mathfrak{p} \geq 2 \textit{ and } \text{height } T(\mathfrak{p}) = 1\}$$

is finite. More precisely, for any presentation $R^m \stackrel{\varphi}{\to} R^n \to E \to 0$, this set is in bijection with

$$\{\mathfrak{p} \subset R \mid E_{\mathfrak{p}} \textit{ not free, } \mathfrak{p} \in \text{Min}(R/I_t(\varphi)) \textit{ and } \text{height } \mathfrak{p} = \text{rank}(\varphi) - t + 2\},$$

where $1 \leq t \leq \text{rank}(\varphi)$.

6 Effective criteria

Having already given effective criteria of normality in the cases of complete intersection modules and modules of projective dimension 1, we now consider more general modules. We have found convenient to break down the proofs into two cases, the special case of modules of second Betti number 1 serving as guide in tracking the more general case.

Remark 6.1 There is also a technical reason for grouping these results. If E is a torsionfree R–module of linear type, a prominent role (checking the condition R_1 of Serre) is played by the prime ideals of $S(E)$ of codimension one, which we denoted $T(\mathfrak{p})$: height $T(\mathfrak{p}) = 1$ if and only if $\dim R_{\mathfrak{p}} + \text{rank } (E) = \nu(E_{\mathfrak{p}}) + 1$, a condition equivalent to (when $E_{\mathfrak{p}}$ has finite projective dimension $\neq 1, 0$) the second Betti number of $E_{\mathfrak{p}}$ is 1.

Theorem 6.1 *Let R be a regular integral domain and let E be a torsionfree module with a free resolution*

$$0 \to R \stackrel{\psi}{\longrightarrow} R^m \stackrel{\varphi}{\longrightarrow} R^n \longrightarrow E \to 0.$$

Suppose E is of linear type. E is normal if and only if the following conditions hold:

(i) *The ideal $I_1(\psi)S(E)$ is principal at all localizations of $S(E)$ of depth* 1.

(ii) *The modules $S_s(E)$ are complete, for $s = 1 \ldots m - 1$.*

Proof. We set $A = R[T_1, \ldots, T_n]$ and $J = J(\varphi) = (f_1, \ldots, f_m) = [T_1, \ldots, T_n] \cdot \varphi$. We only have to show that (i) and (ii) imply that E is normal. In view of Theorem 5.2, it suffices to verify the condition R_1 of Serre.

Let $P \subset A$ be a prime ideal such that its image in $S = A/J$ has height 1. We may assume that $P \cap R$ is the maximal ideal of R and that E has projective dimension 2, that is $I_1(\psi) \subset \mathfrak{m}$ as otherwise we could apply Theorem 4.2.

From Proposition 5.2, and the paragraph preceding it, $\dim R = \text{rank}(\varphi) - t + 2$ on the one hand and $\dim R = n - r + 1$ on the other. Thus, $t = 1$ and $m = \dim R$. Pick $0 \neq a \in \mathfrak{m}$ and consider the ideal $I = (a, f_1, \ldots, f_m) \subset A$. With $P = \mathfrak{m}A$, set $L = I : P$. For a set $x_1, \ldots, x_m$ of minimal generators of $\mathfrak{m}$, we have

$$[a, f_1, \ldots, f_m] = [x_1, \ldots, x_m] \cdot B(\Phi), \tag{2}$$

where $B(\Phi)$ is a $m \times (m+1)$ matrix whose first column has entries in R, and the other columns are linear forms in the T_i's. We denote by L_0 the ideal of A generated by the minors of order m that fix the first column of $B(\Phi)$. These are all forms of degree $m-1$, and $L_0 \subset L$. When we localize at P however, $L_0 A_P = L A_P$, since by condition (i) and the proof of Theorem 5.2, J_P is a complete intersection and the assertion follows from [10]. This means that the image C of $a^{-1}L_0$ in the field of fractions of S is not contained in S and has the property that $C \cdot PS_P \subset S_P$, giving rise to two possible outcomes:

$$C \cdot PS_P = \begin{cases} PS_P \\ S_P \end{cases} \tag{3}$$

In the first case, C would consist of elements in the integral closure of S_P but it not contained in S_P. This cannot occur since by condition (ii) all the symmetric powers of E, up to order $m-1$, are complete. This means that the second possibility occurs, and S_P is a DVR. □

Theorem 6.2 *Let R be a regular integral domain and let E be a torsionfree module of rank r, with a free presentation*

$$R^p \stackrel{\psi}{\longrightarrow} R^m \stackrel{\varphi}{\longrightarrow} R^n \longrightarrow E \to 0.$$

Suppose E is of linear type. $\mathcal{R}(E)$ is normal if and only if the following conditions hold:

(i) *The ideal $I_c(\psi)S(E)$, $c = m + r - n$, is principal at all localizations of $S(E)$ of depth 1.*

(ii) *The modules $S_s(E)$ are complete, for $s = 1 \ldots n - r$.*

Proof. As it was emphasized in Remark 6.1, as far as the condition R_1 is concerned, the Rees algebras we are considering behave as if they are either complete or almost complete intersections. We focus therefore on the condition S_2.

We note that $n - r$ is the height of the defining ideal $J(\varphi)$ of S, while $c = m + r - n$ is the rank of the second syzygy module.

We consider the complex induced by tensoring the tail of the presentation by S,

$$S^p \stackrel{\overline{\psi}}{\longrightarrow} S^m \longrightarrow J/J^2 \to 0.$$

Note that $\overline{\psi}$ has rank c, while J/J^2 has rank $n - r$. This means that the kernel of the natural surjection $C = \text{coker } (\overline{\psi}) \to J/J^2$ is a torsion S–module.

Lemma 6.2 *Let E be a module as above, set $A = R[T_1, \ldots, T_n]$ and $J = (f_1, \ldots, f_m) = [T_1, \ldots, T_n] \cdot \varphi$ the defining ideal of $S(E) = A/J$. Let $P \supset J$ be a prime ideal of A. If $I_c(\overline{\psi})S_P$ is a principal ideal then J_P is a complete intersection ideal.*

Proof. Let C be the module defined above. According to [18, Proposition 2.4.5], since the Fitting ideal $I_c(\overline{\psi})_P$ is principal, the module C_P decomposes as

$$C_P \simeq S_P^{n-r} \oplus (\text{torsion}).$$

This means that we have a surjection

$$S_P^{n-r} \to (J_P/J_P^2)/(\text{torsion})$$

of torsionfree S_P–modules of the same rank. Therefore

$$J_P/J_P^2 \simeq S_P^{n-r} \oplus (\text{torsion}).$$

At this point, since J_P has finite projective dimension, we invoke [15] again to conclude that J_P is a regular sequence. □

The rest of the proof of Theorem 6.2 would proceed as in the proof of Theorem 6.1. Choosing $P = \mathfrak{m}A$ so that S_P has dimension 1, J_P is a complete intersection of codimension $n - r = \dim R - 1 = d - 1$. As in setting up the equation (2), we pick $0 \neq a \in \mathfrak{m}$, pick a minimal set $\{x_1, \ldots, x_d\}$ of generators for $\mathfrak{m}$, and define the matrix $B(\Phi)$

$$[a, f_1, \ldots, f_m] = [x_1, \ldots, x_d] \cdot B(\Phi). \tag{4}$$

Note that when localizing at P the ideals $(a, f_1, \ldots, f_m)_P$ and $(x_1, \ldots, x_d)_P$ are generated by regular sequences and we can use the same argument employed in the proof of Theorem 6.1. □

A significant difference between Theorem 6.1 and Theorem 6.2 lies in the fact that there are natural constructions of algebras arising from modules of second Betti number 1, which is lacking in the general case.

Remark 6.3 It is rather natural to assume regularity in our discussion. It arises, partly from a result in [4] (rediscovered in [7]), that the localizations of the associated primes of complete ideals are regular local rings. Jooyoun Hong has pointed out to us how the proof in [7] extends readily to modules. We are grateful to her for this observation.

Remark 6.4 It may be worthwhile to rephrase this criterion algorithmically: Let R be a regular integral domain and let E be a torsionfree module of rank r, with a free presentation

$$R^p \stackrel{\psi}{\longrightarrow} R^m \stackrel{\varphi}{\longrightarrow} R^n \longrightarrow E \to 0.$$

Set $J = J(\varphi) = [T_1, \ldots, T_n] \cdot \varphi \subset A = R[T_1, \ldots, T_n]$. Suppose E is of linear type, a pretest for which could be formulated as follows:

$$\text{If } 0 \neq z \in I_{n-r}(\varphi) \text{ then } J(\varphi) : z = J(\varphi).$$

Then E is normal if and only if it passes the following tests:

(i) Let $c = m + r - n$, pick $0 \neq b \in I = I_c(\psi)$, set $L = (b, J) : I$. Then

$$\text{grade } (b^{-1}L(J, I)\mathcal{R}(E) \geq 2.$$

This condition is equivalent to saying that S has the S_2 condition of Serre and it is a complete intersection in codimension one. It also sets up, as the proofs of Theorems 6.1 and 6.2 make explicit, procedures providing elements of the integral closure of $S(E)$ should the test fail.

(ii) Pick $0 \neq a \in I_r(\varphi)$. For each integer $1 \leq t \leq r$, let L_t be the component of codimension $r - t + 2$ of the radical of $I_t(\varphi)$. Then

$$\text{height } (a^{-1}((a, J) : L_t)L_t, J) \geq r + 2.$$

If one could be assured of the completeness of the $S_s(E)$, for $s \leq n - r$, by some other means, this cumbersome step, requiring a great deal of work in identifying radical components (fortunately of ideals of R) could be avoided.

This should be contrasted (in characteristic zero) with the Jacobian criterion: If $K = \text{Jac}(\mathcal{R}(E))$, then

$$\text{grade } K\mathcal{R}(E) \geq 2.$$

Its computation could be rather cumbersome. Another point, Theorem 6.2 does not require characteristic zero or that R be a ring of polynomials, just that it be regular up to codimension $r + 1$.

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