# Multiplicities and the Number of Generators of Cohen–Macaulay Ideals

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ABSTRACT. Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d$ and let $I$ be a Cohen–Macaulay ideal of codimension $g$. There are several invariants of $R$ and of $I$ (multiplicities, embedding dimension, index of nilpotency, etc) that may affect the number of generators $\nu(I)$ of $I$. After reviewing results of Sally, Valla and others, we derive estimates for $\nu(I)$ depending on a new mix of some of these invariants.

## 1. Introduction

Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d$ and let $I$ be an ideal of height $g > 0$. If $I$ is a Cohen–Macaulay ideal, there are several approaches that can be used to bound the minimal number of generators of $I$ in terms of the multiplicity data, or some other degree,

$$\nu(I) \leq f(\deg(R), \deg(R/I), e(I), g, d, \mathrm{embdim}(R), \ldots),$$

where $f$ is a linear or quadratic polynomial.

For our purpose here, we set ourselves in the framework of three previous discussions of this issue: [**2**], [**8**], and [**10**]. A different approach to link $\nu(I)$ to mixed multiplicities is undertaken in [**5**]. After reviewing quickly a few of these results, we first introduce some variations on them, and embark on an approach that emphasizes the computation of the lengths of chains

$$I \subset I_1 \subset I_2 \subset \cdots \subset I_n = \mathfrak{m}$$

of irreducible ideals containing $I$, together with an analysis of the number of generators of the subideals in a composition series of $R/I$ containing subchains of irreducible ideals. A more intrusive use of the integral closure $\overline{I}$ of $I$ was sought but it has yet to be realized.

Our two most contrasting results are the following. We state the results for $\mathfrak{m}$–primary ideals, since in the Cohen–Macaulay case the reduction to primary ideals is always possible (unexplained notation defined in the body of the paper).

- Proposition 3.1(b): If $I$ is $\mathfrak{m}$–primary of multiplicity $e(I)$, and $I \subset \mathfrak{m}^2$, then
$$\nu(I) < e(I) \leq d! \cdot \ell(R/\overline{I}) \deg(R).$$
- Theorem 3.7(b): If $\epsilon = \mathrm{embdim}(R) \geq 2$, $I \subset \mathfrak{m}^2$, $\ell(R/I) > 4$ and $t$ is the index of nilpotency of $R/I$, then
$$\nu(I) \leq (\ell(R/I) - t - \frac{\epsilon}{2} + 2)(\epsilon - 1).$$

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The first of these is an assemblage of known parts. The other bound is novel. Together with already established results they suggest that a more comprehensive understanding could be at play. In a short final section we make some observations on the non-primary ideal case.

## 2. Estimating with multiplicities

Let us begin by setting the stage. Throughout $R$ is a local ring of dimension $d$, with infinite residue field and multiplicity $\deg(R)$. $I$ is a Cohen–Macaulay ideal of height $g$ and multiplicity $\deg(R/I)$. Other elements to appear are the embedding dimension $\mathrm{embdim}(R)$ of $R$ and indices of nilpotency.

A very general approach to estimate number of generators with multiplicities uses the method of extended multiplicities (see [**6**]). These are numerical functions (denoted by $\mathrm{Deg}(\cdot)$) on modules that coincide with the classical multiplicity of local rings or graded modules (denoted by $\deg(\cdot)$). A typical expression from [**6**] is:

$$\nu(I) \leq \deg(R) + (g-1)\deg(R/I) + \underbrace{(d-r-1)(\mathrm{Deg}(R/I) - \deg(R/I))}_{\text{non-Cohen-Macaulay correction}}, \tag{1}$$

where $\mathrm{Deg}(\cdot)$ is any extended degree function and $r = \mathrm{depth}\ R/I$.

If $I$ is a Cohen–Macaulay ideal, $\mathrm{Deg}(R/I) = \deg(R/I)$, we get a benchmark for comparison with all other approaches (see also [**10**]):

Proposition 2.1. *If $I$ is a Cohen–Macaulay ideal of codimension $g$,*

$$\nu(I) \leq \deg(R) + (g-1)\deg(R/I). \tag{2}$$

The case $g \leq 1$ is part of a very general result valid for all Cohen–Macaulay modules ([**3**, Corollary 4.7.11]).

The following two results map different aspects of the problem. The first result comes from [**8**]:

Theorem 2.2. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d > 0$.*

(a) *Let $I$ be an $\mathfrak{m}$–primary ideal of nilpotency index $t$. Then $\nu(I) \leq t^{d-1}\deg(R) + d - 1$.*
(b) *Let $I$ be a Cohen–Macaulay ideal of height $g > 0$. Then $\nu(I) \leq \deg(R/I)^{g-1}\deg(R) + g - 1$.*

Exploring the Koszul complex, Valla ([**10**]) derived the following different bounds:

Theorem 2.3. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of multiplicity $e$. Let $I$ be a Cohen–Macaulay ideal of codimension $g$ such that the multiplicity of $R/I$ is $\delta$. Setting $m = \min\{e, \delta\}$, then*

(a) *If $g > 0$, then*

$$\nu(I) \leq e + \frac{\delta(g-1)^2}{g} + \frac{m(g-1)}{g}.$$

(b) *If $g \geq 2$ and $I \subset \mathfrak{m}^2$, then*

$$\nu(I) \leq e + \frac{\delta(g-1)^2}{g} + \frac{\min\{m+g, \delta\}}{g} - \binom{g}{2}.$$

In some cases, instead of $\deg(R/I)$ one has other invariants of the ideal appearing in these estimates. Here is one from [**2**], whose technique will be used again.

Theorem 2.4. *Let $R$ be a Cohen–Macaulay local ring and let $I$ be a Cohen–Macaulay ideal of codimension 2. If $R/I$ has Cohen–Macaulay type $r$, then*

$$\nu(I) \leq (r+1)\deg(R).$$

PROOF. The proof of [**2**] places no restriction on $R$, but here we assume that $R$ contains an appropriate field so that we may complete $R$ and assume the existence of a regular local subring $S \subset R$ over which $R$ is finite. Since $R$ is Cohen–Macaulay, this means that $R$ is a free $S$–module of rank $\deg(R)$. $R/I$ has a minimal free resolution over $S$,

$$0 \to S^p \longrightarrow S^m \longrightarrow S^n \longrightarrow R/I \to 0,$$

in which $n \leq \deg(R)$. The number $m$ is the minimal number of generators of $I$ as an $S$–module. Since $m = n + p$, we need to bound $p$. Applying the functor $\mathrm{Hom}_S(\cdot, S)$, we get that the canonical module $\omega_{R/I} = \mathrm{Ext}^2_S(R/I, S)$ of $R/I$ can be generated minimally by $p$ elements. But $\omega_{R/I}$ is generated by $r$ elements as an $R/I$–module and therefore can be generated by $r \cdot \deg(R)$ over $S$, and we get our estimate. □

The next method has a different character: Let $J_0$ be an ideal generated by a regular sequence of $g-1$ elements, $J_0 \subset I$. Preferably $J_0$ should be part of a minimal reduction of $I$. We have

$$\nu(I) \leq \nu(I/J_0) + \nu(J_0) \leq \deg(R/J_0) + g - 1, \tag{3}$$

since $I/J_0$ is a maximal Cohen–Macaulay module of rank 1 over the Cohen–Macaulay ring $R/J_0$. The issue is to relate $\deg(R/J_0)$ to $\deg(R/I)$.

Let us consider two applications of the method to the case of an ideal $I$ that is equimultiple. The other approach is somewhat not sensitive to this additional information. First, let $R$ be a Gorenstein ring and suppose $J = (J_0, x)$ is a minimal reduction of $I$. Since $x$ is regular modulo $J_0$, we have $\deg(R/J_0) \leq \deg(R/J)$,

$$\nu(I) \leq \deg(R/J) + g - 1.$$

Consider the exact sequence

$$0 \to (J : I)/J \longrightarrow R/J \longrightarrow R/(J : I) \to 0,$$

where $(J : I)/J$ is the canonical module of $R/I$ and therefore the equality $\deg((J : I)/J) = \deg(R/I)$ yields

$$\deg(R/J) = \deg(R/I) + \deg(R/(J : I)).$$

If, for instance, $I$ is equimultiple of reduction number 1, that is $I^2 = JI$, $I \subset J : I$ and therefore $\deg(R/I) \geq \deg(R/(J : I))$, since they are both modules of dimension $d - g$. We thus obtain the estimate

$$\nu(I) \leq 2\deg(R/I) + g - 1$$

which is, usually, for this limited class of ideals, considerably better than the bound (2). We extend this further to arbitrary Cohen–Macaulay rings.

PROPOSITION 2.5. *Let $R$ be a Cohen–Macaulay local ring of type $r$, and let $I$ be a Cohen–Macaulay ideal of height $g \geq 1$. If $I$ is equimultiple and has reduction number $\leq 1$, then*

$$\nu(I) \leq (r+1)\deg(R/I) + g - 1. \tag{4}$$

PROOF. Taking into account the considerations above, it will suffice to show that $\deg(J : I/J) \leq r\deg(R/I)$. We may assume (passing to the completion if required) that $R$ has a canonical module, say $\omega$. The ring $R/J$ has $\overline{\omega} = \omega/J\omega$ for its canonical module. $\overline{\omega}$ is a faithful $R/J$, generated by $r$ elements. In the usual manner, we can build an embedding

$$R/J \hookrightarrow \overline{\omega}^{\oplus r}.$$

Applying the functor $\mathrm{Hom}(R/I, \cdot)$, we obtain an embedding of $J : I/J$ into the direct sum of $r$ copies of $\mathrm{Hom}(R/I, \overline{\omega})$. Since this is the canonical module of $R/I$, it has the same multiplicity as $R/I$. □

## 3. Number of generators of primary ideals

Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring and let $I$ be an $\mathfrak{m}$–primary ideal. We seek bounds between the number of generators of $I$ and other quantities attached to the ideal:

$$\nu(I) \leftrightarrow \left\{ \begin{array}{rcll} d & = & \dim R & \\ \epsilon & = & \nu(\mathfrak{m}) & \\ s & = & \ell(R/I), & \text{the colength of } I \\ e(I) & = & \text{multiplicity of } I & \\ t & = & \mathrm{nil}(R/I), & \text{smallest } n \text{ such that } \mathfrak{m}^n \subset I \end{array} \right.$$

One relation between some of these quantities is put together as follows (see also [**1**]):

Proposition 3.1. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d$ and let $I$ be an $\mathfrak{m}$–primary ideal of multiplicity $e(I)$.*

(a) *Then $\nu(I) \leq e(I) + d - 1$.*
(b) *If $I \subset \mathfrak{m}^2$, then $\nu(I) < e(I) \leq d! \cdot \ell(R/\overline{I}) \deg(R)$.*

Proof. We may assume that $R$ has an infinite residue field. Let $J$ be a minimal reduction of $I$. From the inequalities exhibited by the diagram

$$\begin{tikzcd} & I \arrow[dl, dash, "\nu(I)"'] \arrow[dr, dash, "\ell(I/J)"] & \\ \mathfrak{m}I \arrow[dr, dash] & & J \arrow[dl, dash, "d"] \\ & \mathfrak{m}J & \end{tikzcd}$$

$$\begin{array}{rcl} \nu(I) & = & \ell(I/\mathfrak{m}I) \leq \ell(I/\mathfrak{m}J) \\ & = & \ell(I/J) + \ell(J/\mathfrak{m}J) \\ & = & \ell(R/J) - \ell(R/I) + d, \end{array}$$

the first assertion follows, since $e(I) = \ell(R/J)$ as $R$ is Cohen–Macaulay. For the other bound, just observe that with $I \subset \mathfrak{m}^2$, $\ell(R/I) \geq \ell(R/\mathfrak{m}^2) \geq \epsilon + 1 \geq d + 1$.

The final inequality arises from a well-known theorem of Lech ([**7**]):

Theorem 3.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d$ and let $I$ be an $\mathfrak{m}$–primary ideal. Then*

$$e(I) \leq d! \cdot \ell(R/I) \cdot \deg(R). \tag{5}$$

It gets slightly improved when we replace $I$ by its integral closure $\overline{I}$, since $e(I) = e(\overline{I})$, which we then use in the estimation. $\square$

**Number of generators and the socle.** We will now develop new expressions for bounds on the number of generators of ideals that integrate efficiencies usually associated with Gorenstein ideals.

Proposition 3.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $I$ be an $\mathfrak{m}$–primary ideal.*

(a) *If $I \subset \mathfrak{m}^2$, $z \in (I : \mathfrak{m}) \setminus I$, $I$ is irreducible and $\epsilon \geq 2$, then $\nu(I) = \nu(I, z) - 1$.*
(b) [H. Schoutens] *In general, $\nu(I) \leq \nu(I, z) + \epsilon - 1$.*

A related class of ideals of reduction number one is taken from [**4**]; it will not require that $R$ be Cohen–Macaulay.

We start by quoting [**4**, Lemma 3.5] about the existence of a *dual basis* .

LEMMA 3.4. *Let $(R,\mathfrak{m})$ be a Noetherian local ring with embedding dimension $n$ at least two. Let $I$ be an $\mathfrak{m}$-primary irreducible ideal contained in $\mathfrak{m}^2$. If $\mathfrak{m}=(x_1,\ldots,x_n)$ then there exist $y_1,\ldots,y_n$ such that for all $1\leq i,j\leq n$*

$$x_iy_j \equiv \delta_{ij}\Delta \mod I, \tag{6}$$

*where $\Delta$ is the lift in $R$ of the socle generator of $R/I$ and $\delta_{ij}$ denotes Kronecker's delta.*

PROOF. The first assertion is part of Lemma 3.4: It follows from the equality $\mathfrak{m}(I,z)=\mathfrak{m}I$.

For the other assertion, suppose $I$ is minimally generated by $a_1,\ldots,a_n$; adding $z$ may result in a set (after notation change of the $a_i$) of generators for $(I,z)$ of the form $a_1,\ldots,a_r,z$. This means (after another rewrite) that $a_i=b_iz$, for i$\geq r+1$. As $z$ lies in the socle of $I$, we have $n-r\leq\epsilon$, thus proving the assertion. □

REMARK 3.5. *We are going to use slightly modified versions to take into account the possibility that $I$ is not contained in $\mathfrak{m}^2$. Denote by $\epsilon_0$ the dimension of the vector space $(I+\mathfrak{m}^2)/\mathfrak{m}^2$. (For later usage, we denote it $\epsilon_0(I)$.) If $\epsilon_0>0$, say $\{x_1,\ldots,x_{\epsilon_0}\}\subset I$ is part of a minimal set of generators of $I$, we can mod out these elements, we may still apply Proposition 3.3.a, provided $\epsilon-\epsilon_0\geq 2$ and still get the equality $\nu(I)=\nu(L)-1$. As for Proposition 3.3.b, we get the improved bound $\nu(I)\leq\nu(L)+\epsilon-\epsilon_0-1$, since $\epsilon-\epsilon_0$ is the embedding dimension of the actual ring.*

The second assertion gives a mechanism to bound $\nu(I)$ in terms of $s$ and of $\epsilon$. On the other hand, the first assertion allows for a possible decreased count for the number of generators. This suggests that one should go from $I$ to $\mathfrak{m}$ by steps that meet the largest possible number of irreducible ideals. We look at this issue now.

The question reduces to the Artinian case. Let $I$ be an ideal and consider sequences

$$I\subset I_1\subset I_2\subset\cdots\subset I_n=\mathfrak{m},$$

where each $I_i$ is an irreducible ideal.

PROPOSITION 3.6. *The supremum of the length of these sequences is at most $t$, the nilpotency index of $R/I$. Furthermore, if $A=R[x]_{(\mathfrak{m},x)}$ then for the ideal $IA$ the supremum is exactly $nil(R/I)=nil(A/IA)$.*

PROOF. We first assume that $R$ is a Gorenstein ring and let $L=\mathrm{ann}(I)$. By duality, we get a chain of ideals

$$\mathrm{ann}(I)\supset\mathrm{ann}(I_1)\supset\mathrm{ann}(I_2)\supset\cdots\supset\mathrm{ann}(I_n)=\mathrm{ann}(\mathfrak{m}),$$

where $\mathrm{ann}(I_i)=Rx_i$ is principal for $i=1,\ldots,n$. In particular we obtain that $x_n=rx_1x_2\cdots x_{n-1}$, which shows that $n-1<t$.

Let us show that the lengths of these chains, in the ring $A$, attain the bound $t$. From $\mathfrak{m}^t\subset I$, we have that $L\mathfrak{m}^t=0$. Suppose we had $L\mathfrak{m}^p=0$ for $p<t$. This would imply $\mathfrak{m}^p\subset 0:L=I$, by duality, contrary to the definition of $t$.

If $L$ is principal, $L=(c)$, from $c\mathfrak{m}^{t-1}\neq 0$ we would build a sequence $x_1,\ldots,x_t$ of elements of $\mathfrak{m}$ with $cx_1\cdots x_t\neq 0$, which would give rise to a corresponding sequence of irreducible ideals $I_i=\mathrm{ann}(cx_1\cdots x_i)$, of length t, as desired.

If $I$ is not irreducible, $L$ is not principal, $L = (c_1, \ldots, c_m)$, pass to the ring $A$ and consider the element

$$c = \sum_{j=1} c_i x^i.$$

It clear that $\mathfrak{m}^t c \neq 0$. We can now build a sequence of irreducible ideals containing $I$, of length $t$.

If $R$ is not Gorenstein, let $E$ be the injective envelope of the residue field of $R$ and set $S$ to be the trivial extension of $R$ by $E$,

$$S = R \times E, \quad \text{multiplication defined by } (a,b)(c,d) = (ac, ad + cb).$$

$S$ is a Gorenstein local ring of maximal ideal $M = \mathfrak{m} \times E$. If we put $I' = I \times E$, $\mathrm{nil}(S/I') = \mathrm{nil}(R/I)$ and we can obviously apply the argument above to the pair $(S, I')$ to derive the same assertion for the pair $(R, I)$. □

Let us now derive our first bound for $\nu(I)$. Since none of the measures introduced above are affected by the passage to $A$, we can assume the existence of chains of irreducible ideals containing $I$ with $t$ elements (actually $t+1$ if $t > 1$). We make a few additional observations. The approach we take is to build $\mathfrak{m}$ starting from $I$ by adding successively socle elements along convenient chains–here we will take the chains of irreducible ideals. If anywhere in the process we get an ideal not contained in $\mathfrak{m}^2$, we can reduce the dimension of $R$ and the embedding dimension by 1, at least for as long as $d > 0$. Finally, we remark that if $\ell(R/I)$ is very low, then the number of generators can be easily estimated. For example, if $\ell(R/I) = 3$, then $\ell(\mathfrak{m}/I) = 2$, from which we get $\nu(I) \leq \epsilon + 1$. The next case is not much more complicated, and one gets that if $\ell(R/I) = 4$ then $\nu(I) \leq \epsilon + 3$.

A version of Part (a) below is due to H. Schoutens ([**9**]).

THEOREM 3.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of embedding dimension $\epsilon \geq 2$, and let $I$ be an $\mathfrak{m}$–primary ideal contained in $\mathfrak{m}^2$ of colength $s$.*

(a) *If $\ell(R/I) > 4$ then*

$$\nu(I) \leq (\ell(R/I) - 4)(\epsilon - 1) - \frac{\epsilon(\epsilon - 7)}{2}.$$

(b) *If $t$ is the nilpotency index of $R/I$ then*

$$\begin{aligned} \nu(I) &\leq (\ell(R/I) - t)(\epsilon - 1) - \frac{\epsilon^2 - 5\epsilon + 2}{2} - 1 \\ &\leq (s - t - \frac{\epsilon}{2} + 2)(\epsilon - 1). \end{aligned} \tag{7}$$

**Proof.** (a) We are going to, starting from $I$, successively add socle elements along the chain of irreducible ideals. If at some point, we add to the ideal $J$ a socle element $z \in J : \mathfrak{m}$ so that $L = (J, z)$ is also contained in the chain, we want to use Proposition 3.3 to bound $\nu(J)$ in terms of $\nu(L)$.

The argument here is a simplified version of what is needed in Part (b). We pick a chain of ideals

$$I = L_0 \subset L_1 \subset \cdots \subset L_r \subset L_{r+1} \subset \mathfrak{m},$$

so that $L_{i+1} = (L_i, z_i)$, where $z_i$ is a nonzero socle element for $R/L_i$; furthermore we assume that $\ell(R/L_{r+1}) = 4$.

Using Proposition 3.3.a and the remark that follows it, we have the following relationship between $\nu(L_i)$ and $\nu(L_{i+1})$: If $\epsilon_0 = \epsilon_0(L_i) > 0$, then

$$\nu(L_i) \leq \nu(L_{i+1}) + \epsilon - \epsilon_0 - 1.$$

This means that when we reach $L_r$, that satisfies $\epsilon_0(L_r) \leq 4$ since $\ell(\mathfrak{m}/L_r) = 4$, from the combined count

$$\nu(I) \leq \nu(L_{r+1}) + (\ell(R/4) - 4)(\epsilon - 1),$$

we can further take away

$$1 + 2 + \cdots + (\epsilon - 4) = \binom{\epsilon - 3}{2},$$

without breaking the inequality. Since $\nu(L_{r+1}) \leq \epsilon + 3$, collecting we get the desired estimate.

(b) The assertions are not changed if we reduce $R$ mod $\mathfrak{m}I$, so that we may assume that $R$ is an Artinian local ring. After passage to the extension $A$, we may assume that we have a chain of irreducible ideals

$$I \subset I_1 \subset I_2 \subset \cdots \subset I_t = \mathfrak{m}.$$

To apply the proposition properly, we want to distinguish the cases when $J$ is irreducible into two subcases. An irreducible ideal $J \neq \mathfrak{m}$ is said to be of type 1 if

$$\dim \mathfrak{m}/(J + \mathfrak{m}^2) = 1;$$

otherwise $J$ is said to be of type 2.

We now proceed with the proof. If $J$ is not irreducible, then we have

$$\nu(J) \leq \nu(L) + \epsilon - 1,$$

which is taken into the ledger. If $J$ is irreducible, say $J = I_{t_0}$, and of type 2, then $J$ has a set of minimal generators of the form

$$J = (x_1, \ldots, x_m, y_1, \ldots, y_n),$$

where $x_i \in \mathfrak{m} \setminus \mathfrak{m}^2$, and $y_j \in \mathfrak{m}^2$. Furthermore, because $J$ is of type 2, $R' = R/(x_1, \ldots, x_m)$ has embedding dimension at least two and $JR'$ is an irreducible ideal contained in the square of its maximal ideal. We can again apply Proposition 3.3.b to obtain that $\nu(J) = \nu(L) - 1$.

In case $J$ is of type 1, $R/J$ has embedding dimension 1, and in particular this is a principal ideal ring. Thus $J$ and all ideals containing it are generated by $\epsilon$ elements. Suppose this first occur at the irreducible ideal $I_{t_0}$, $t_0 \leq t$. This means that $I_{t_0}$ and all ideals containing it are generated by $\epsilon$ elements. The number of 'socle' extensions from $I$ until $I_{t_0}$ is given by $\ell(R/I) - \ell(R/I_{t_0})$. This means that we have

$$\nu(I) \leq (\ell(R/I) - \ell(R/I_{t_0}) - t_0 + 1)(\epsilon - 1) + \nu(I_{t_0}) - t_0 - \delta,$$

where $\delta$ represents the discrepancy in using $\epsilon - 1$ instead of $\epsilon - \epsilon_0 - 1$, as done in Part (a). We claim that this contributes at least

$$\delta \geq 1 + 2 + \cdots + (\epsilon - 2) = \frac{(\epsilon - 1)(\epsilon - 2)}{2}.$$

This is the case because whenever we reach an irreducible ideal $J$, the socle element $z$ does not affect $\epsilon_0$, that is $\epsilon_0(J) = \epsilon_0(J, z)$, unless $\epsilon_0(J) = \epsilon - 1$, since socle elements of irreducible ideals in rings of embedding dimension at least two are contained in the square of the maximal ideal.

Finally, considering that $\ell(R/I_{t_0}) = t - t_0 + 1$, we obtain

$$\begin{aligned}
\nu(I) &\leq (\ell(R/I) - t)(\epsilon - 1) + \epsilon - \frac{(\epsilon - 1)(\epsilon - 2)}{2} - t_0 \\
&\leq (\ell(R/I) - t)(\epsilon - 1) - \frac{\epsilon^2 - 5\epsilon + 2}{2} - t_0.
\end{aligned}$$

Since $t_0 \geq 1$, the estimate is established after a straightforward rewrite. $\square$

TABLE 1. Bounds for the number of generators of $\mathfrak{m}$–primary ideals ($s = \ell(R/I)$, $t = \mathrm{nil}(R/I)$, $\epsilon = \mathrm{embdim}(R) \geq 2$).

| | |
|---|---|
| $\deg(R) + (d-1)\ell(R/I)$ | 2.1 |
| $e(I) + d - 1$ | 3.1($a$) |
| $e(I) \leq d! \cdot \ell(R/\overline{I}) \deg(R)$ if $I \subset \mathfrak{m}^2$ | 3.1(b) |
| $(s - t - \frac{\epsilon}{2} + 2)(\epsilon - 1)$ | 3.7(b) |

REMARK 3.8. *This kind of estimation is inherently uneven: If we apply it to the powers $I^q$ of the ideal $I$, for $q \gg 0$ the value $\nu(I^q)$ is given by a polynomial in $q$ of degree $d-1$, while $\ell(R/I^q)$ grows as a polynomial in $q$ of degree $d$. Given that $t(I^q)$ is bounded by a linear polynomial, the expression is asymptotically quite unbalanced.*

## 4. Non–primary Cohen–Macaulay ideals

Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dim $d \geq 1$, and let $I$ be a Cohen–Macaulay ideal of codimension $g < d$. We seek to extend to this class of ideals some of the improvements on the number of generators for $I$ as measured against benchmarks bounds such as (2).

A first step consists in examining the class of equimultiple ideals (we assume that $R$ has infinite residue throughout).

PROPOSITION 4.1. *Let $I$ be an equimultiple Cohen–Macaulay ideal of codimension $\geq 1$, $J$ one of its minimal reductions.*

(a) *If $\sqrt{I}$ is a Cohen–Macaulay ideal*

$$\nu(I) \leq \deg(R/J) + (g-1)\deg(R/\sqrt{I}). \tag{8}$$

(b) *Moreover, if $I \subset (\sqrt{I})^2$*

$$\nu(I) \leq \deg(R/J) - \deg(R/\sqrt{I}). \tag{9}$$

**Proof.** The proof is similar to that of Proposition 3.1. Consider the exact sequences

$$0 \to J/J\sqrt{I} \simeq (R/\sqrt{I})^g \longrightarrow R/J\sqrt{I} \longrightarrow R/J \to 0,$$
$$0 \to I/J\sqrt{I} \longrightarrow R/J\sqrt{I} \longrightarrow R/I \to 0.$$

From the first sequence we get that $R/J\sqrt{I}$ is a Cohen–Macaulay module, which taken into the other sequence shows that $I/J\sqrt{I}$ is also a Cohen–Macaulay module. Thus all the modules in these sequence are Cohen–Macaulay of dimension $d-g$, so that we can compute multiplicities using the rules to compute lengths. From the second sequence we have

$$\deg(I/J\sqrt{I}) = \deg(R/J\sqrt{I}) - \deg(R/I),$$

while from the other sequence,

$$\deg(R/J\sqrt{I}) = g\deg(R/\sqrt{I}) + \deg(R/J).$$

It follows that

$$\deg(I/J\sqrt{I}) = \deg(R/J) + (g-1)\deg(R/\sqrt{I}) - \deg(\sqrt{I}/I).$$

If we drop the last summand and take into account that $\deg(I/J\sqrt{I}) \geq \nu(I/J\sqrt{I})$ (this is a Cohen–Macaulay module), we prove Part (a) since

$$\nu(I/J\sqrt{I}) \geq \nu(I/I\sqrt{I}) = \nu(I).$$

On the other hand, if $I \subset (\sqrt{I})^2$,

$$\deg(\sqrt{I}/I) \geq \deg(\sqrt{I}/(\sqrt{I})^2) \geq g \deg(R/\sqrt{I}),$$

and the estimation above shows that

$$\deg(I/J\sqrt{I}) \leq \deg(R/J) - \deg(R/\sqrt{I}),$$

which proves Part (b). □

Remark 4.2. *The requirement on $\sqrt{I}$ is strict: For example, if $I = (x^2y^2 + z^3w, x^3z + y^2z^2 + xyzw) \subset k[x, y, z, w]$, a computation with Macaulay will show that $\sqrt{I}$ is not Cohen–Macaulay.*

The other approach to the estimation of $\nu(I)$, in the case $g < d$, is that of hyperplane sections: we can pick a regular sequence $\mathbf{z} = z_1, \ldots, z_{d-g}$ that is superficial for both rings, $R/I$ and $R$, and if $J$ is a regular sequence, for $R/J$ as well. This has for effect, $\deg(R/I) = \ell(R/(I, \mathbf{z}))$, and similar formulas for $\deg(R)$ and $\deg(R/J)$ (in case $J$ itself is Cohen–Macaulay, as when $I$ is equimultiple). At the same time, $\nu(I) = \nu(I, \mathbf{z}) - (d - g)$. Note that if $I$ is equimultiple,

$$e(I, \mathbf{z}) = e(J, \mathbf{z}) = \deg(R/J).$$

Let us integrate these observations into Propositions 3.1(a):

Proposition 4.3. *For these ideals,*

$$\nu(I) \leq e(I, \mathbf{z}) + g - 1.$$

*Moreover, if $I$ is equimultiple with a minimal reduction $J$*

$$\nu(I) \leq \deg(R/J) + g - 1.$$

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