Math 552: Abstract Algebra II

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Set 2

Spring 2009

Outline

Composition laws

A composition on a set $\mathbb{X}$ is a function assigning to pairs of elements of $\mathbb{X}$ an element of $\mathbb{X}$,

$$(a, b) \mapsto \mathbf{f}(a, b).$$

That is a function of two variables on $\mathbb{X}$ with values in $\mathbb{X}$. It is nicely represented in a composition table

$\mathbf{f}$	$*$	b	$*$
$*$	$*$	$*$	$*$
a	$*$	$\mathbf{f}(a, b)$	$*$
$*$	$*$	$*$	$*$

We represent it also as

$$\mathbb{X} \times \mathbb{X} \xrightarrow{\mathbf{f}} \mathbb{X}$$

Example: Abelian group

An abelian group is a set $\mathbf{G}$ with a composition law denoted '$+$'

$$\mathbf{G} \times \mathbf{G} \rightarrow \mathbf{G},$$

$$a, b \in \mathbf{G}, \quad a + b \in \mathbf{G}$$

satisfying the axioms

- associative $\forall a, b, c \in \mathbf{G}, \quad (a+b)+c = a+(b+c)$
- commutative $\forall a, b \in \mathbf{G}, \quad a+b = b+a$
- existence of O

$$\exists O \in \mathbf{G} \quad \text{such that } \forall a \quad a + O = a$$

- existence of inverses

$$\forall a \in \mathbf{G} \quad \exists b \in \mathbf{G} \quad \text{such that } a + b = O$$

This element is unique and denoted $-a$.

Rings

A ring R is a set with two composition laws, called 'addition' and 'multiplication', say $+$ and $\times$: $\forall a, b \in R$ have compositions $a + b$ and $a \times b$. (The second composition is also written $a \cdot b$, or simply ab.)

- $(R, +)$ is an abelian group
- $(R, \times)$: multiplication is associative, and distributive over $+$, that is $\forall a, b, c \in R$,

$$(ab)c = a(bc), \quad ab = ba, \quad a(b + c) = ab + ac$$

- existence of identity: $\exists e \in R$ such that

$$\forall a \in R \quad e \times a = a \times e = a$$

- If $ab = ba$ for all $a, b \in R$, the ring is called commutative

There is a unique identity element e, usually we denote it by 1:

$$e = ee' = e'e = e'$$

Rings and Modules

A ring R is a set with two composition laws $+$ and $\times$ satisfying

- $\{R, +\}$ is an abelian group
- associative axiom : For $a, b, c \in R$,
  $a \times (b \times c) = (a \times b) \times c$
- distributive axioms: For $a, b, c \in R$,
  $a \times (b + c) = a \times b + a \times c$ and $(a + b) \times c = a \times c + b \times c$
- existence of 1: there is $e \in R$ such that for $a \in R$,
  $a \times e = e \times a = a$
- If $a \times b = b \times a$ for all $a, b \in R$, ring is called commutative

Class Surprise Quiz!

What is your favorite ring?

To qualify, your answer must be different–very different–from that given by a classmate!

More composition laws

Other composition laws take pairs [or triples,...] of sets: such as a function assigning to pairs of elements of **Y** and $\mathbb{X}$ an element of $\mathbb{X}$,

$$(a, b) \mapsto \mathbf{f}(a, b).$$

It is represented in a composition table

$\mathbf{f}$	$*$	b	$*$
$*$	$*$	$*$	$*$
a	$*$	$\mathbf{f}(a, b)$	$*$
$*$	$*$	$*$	$*$

We represent it also as $\mathbf{Y} \times \mathbb{X} \xrightarrow{\mathbf{f}} \mathbb{X}$

Typically we place requirements on $\mathbf{f}$, such as $\mathbf{f}(a, b + c) = \mathbf{f}(a, b) + \mathbf{f}(a, c)$

Modules

If R is a ring, a left R-module M is a set

- $\{M, +\}$ is an abelian group and equipped with a mapping $(R, M) \to M$, $(a, m) \to am$ such that
- associative axiom : For $a, b \in R$, $c \in M$, $a(bc) = (a \times b)c$
- distributive axiom: For $a \in R$, $b, c \in M$, $a(b + c) = ab + ac$
- If 1 is the identity of R, $1c = c$ for all $c \in M$

Submodules, quotient modules, homomorphisms

- If R is a ring and A and B are left R-modules, a group homomorphism $\mathbf{f} : A \to B$ is a R-homomorphism if

$$\mathbf{f}(ax) = a\mathbf{f}(x), \quad a \in R, \quad x \in A.$$

- A subgroup C of the R-module A is a submodule if the inclusion mapping $C \to A$ is a homomorphism. If C is a submodule, the quotient group A/C is an R-module
- If $\mathbf{f} : A \to B$ is a homomorphism of R-modules, $K = \ker(\mathbf{f}) = \{x \in A : \mathbf{f}(x) = 0\}$ is a submodule of A, and $E = \{\mathbf{f}(a) : a \in A\}$ is a submodule of B.
- There is a canonical isomorphism of R-modules $A/K \simeq E$

Direct sums and products

Let R be a ring and $\{M_\alpha : \alpha \in I\}$ be a family of modules.

- direct sum $M = \bigoplus_\alpha M_\alpha$ is the set of $(m_\alpha : \alpha \in I)$, almost all $m_\alpha = 0_\alpha$. Addition and multiplication by elements of R is component wise, for instance

$$(m_\alpha) + (n_\alpha) = (m_\alpha + n_\alpha)$$

- direct product $M = \prod_\alpha M_\alpha$ is the set of $(m_\alpha : \alpha \in I)$. Addition and multiplication by elements of R is component wise, for instance

$$a(m_\alpha) = (am_\alpha)$$

Generators of a module

- If A is an R-module, a subset $S \subset A$ is a set of generators of A if for $a \in A$ there are $s_1, \ldots, s_n$ in S and $r_i \in R$ such that
$$a = r_1 s_1 + \cdots + r_n s_n$$
- If S is finite, A is said to be finitely generated
- If $S = \{s\}$, A is said to be cyclic

Free modules

Let R be a ring and X a set. The free R-module with basis indexed by X:

$$F_X = \bigoplus_{x \in X} R_x, \quad R_x \simeq R$$

If $X = \{1, 2, \dots, n\}$,

$$R^n = \{(a_1, \dots, a_n), \quad a_i \in R\}$$

Set $e_1 = (1, 0, \dots, 0)$, ..., $e_n = (0, 0, \dots, 1)$,

$$(a_1, a_2, \dots, a_n) = a_1 e_1 + \dots + a_n e_n$$

Finitely generated module

Proposition

Let X be a set and A an R-module. For any (set) mapping $\varphi : X \longrightarrow A$ there is a (unique) module homomorphism

$$\mathbf{f} : F_X = \bigoplus_{x \in X} Re_x \longrightarrow A$$

such that $\mathbf{f}(e_x) = \varphi(x)$.

Proposition

An R-module A is finitely generated iff there is a surjection

$$\mathbf{f} : R^n \longrightarrow A,$$

for some $n \in \mathbb{N}$.

Outline

Chain Conditions

Let R be a ring and let M be a left (right) R-module and denote by X the set of R-submodules of M ordered by inclusion.

A chain of submodules is a sequence

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$$

or

$$B_1 \supseteq B_2 \supseteq \cdots \supseteq B_n \supseteq \cdots$$

The first is called ascending, the other descending.

Noetherian Module

Definition

M is a Noetherian (Artinian) module if every ascending (descending) chain of submodules is stationary, that is $A_n = A_{n+1} = \ldots$ from a certain point on.

R is a left (right) Noetherian(Artinian) ring if the ascending (descending) chains of left (right) ideals are stationary.

Example

$$\begin{bmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}$$

is a right (but not left) Noetherian ring.

$$\begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{bmatrix}$$

is a left (but not right) Artinian ring.

Example: Sides may matter

Here is an example (J. Dieudonné) of a left Noetherian that is not right Noetherian.

Let **A** be the ring generated by x and y, $\mathbb{Z}[x, y]$, such that $yx = 0$ and $yy = 0$, and let R be the subring $\mathbb{Z}[x]$. That is, R is the ring of polynomials in x over $\mathbb{Z}$ (therefore R is Noetherian). **A** is the R-module

$$\mathbf{A} \quad = \quad R + Ry$$

in particular **A** is a Noetherian left R-module, thus it is a left Noetherian ring.

Let I be the subgroup of **A** generated by $\{x^n y, n \geq 0\}$. Since $Ix = Iy = 0$, I is a right ideal and thus any system of right R-generators of I is also a system of $\mathbb{Z}$ generators. But I is not finitely generated over $\mathbb{Z}$

Maximal/Minimal Condition

Definition

M is an R-module with the Maximal Condition (Minimal Condition) if every subset S of X (set of submodules ordered by inclusion) contains a maximum submodule (minimum submodule).

Proposition

Let M be an R-module. Then

1. *M is Noetherian iff M has the Maximal Condition.*
2. *M is Artinian iff M has the Minimal Condition.*

Proof

Let S be a set of submodules of M. If S contains no maximal element, we can build an ascending chain

$$A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_n \subsetneq \cdots$$

contradicting the assumption that M is Noetherian. The converse has a similar proof.

Example: If $R = \mathbb{Z}$, $\mathbb{Z}$ is a Noetherian module, while for every prime number p, $\mathbb{Z}_{p^\infty}/\mathbb{Z}$ is Artinian.

Composition Series

Proposition

Let M be an R-module satisfying both chain conditions. Then there exists a chain of submodules

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} \subset M_n = M$$

such that each factor M_i/M_{i-1} is a simple module.

Such sequences are called composition series of length n. The existence of one such series is equivalent to M being both Noetherian and Artinian.

Theorem (Jordan-Holder)

All composition series of a module M have the same length (called the length of M and denoted $\lambda(M)$).

Noetherian Module

Proposition

M is a Noetherian R-module iff every submodule is finitely generated.

Proof.

Suppose M is Noetherian. Let us deny. Let A be a submodule of M and assume it is not finitely generated. It would permit the construction of an increasing sequence of submodules of A,

$$(a_1) \subset (a_1, a_2) \subset \cdots \subset (a_1, a_2, \ldots, a_n) \subset \cdots,$$

$a_{n+1} \in A \setminus (a_1, \ldots, a_n)$.
Conversely if $A_1 \subseteq A_2 \subseteq \cdots$ is an increasing sequence of submodules, let $B = \cup_{i \geq 1} A_i$ is a submodule and therefore $B = (b_1, \ldots, b_m)$. Each $b_i \in A_{n_i}$ for some n_i. If $n = \max\{n_i\}$, $A_n = A_{n+1} = \cdots$. □

SES

Proposition

Let R be a ring and

$$0 \to A \xrightarrow{\mathbf{f}} B \xrightarrow{\mathbf{g}} C \to 0$$

be a short exact sequence of R-modules (that is, $\mathbf{f}$ is 1-1, $\mathbf{g}$ is onto and Image $\mathbf{f} = \ker \mathbf{g}$). Then B is Noetherian (Artinian) iff A and C are Noetherian (Artinian).

Corollary

If R is a Noetherian (Artinian) ring, then any finitely generated R-module is Noetherian (Artinian).

Proof.

By the proposition, any f.g. free R-module $F = R \oplus \cdots \oplus R$ is Noetherian (Artinian). A f.g. R-module is a quotient of a f.g. free R-module. $\square$

Proof

Let $B_1 \subseteq B_2 \subseteq \cdots$ be an ascending sequence of submodules of B. Applying $\mathbf{g}$ to it gives an ascending sequence $\mathbf{g}(B_1) \subseteq \mathbf{g}(B_2) \subseteq \cdots$ of submodules of C.

There is also an ascending sequence of submodules of A by setting $A_i = \mathbf{f}^{-1}(B_i)$.
There is n such that both sequences are stationary from that point on: $\mathbf{g}(B_n) = \mathbf{g}(B_{n+1}) = \cdots$ and $\mathbf{f}^{-1}(B_n) = \mathbf{f}^{-1}(B_{n+1}) = \cdots$.

It follows easily that $B_n = B_{n+1} = \cdots$.

Outline

Assignment #6

Define the following composition laws ($\oplus$ and $\otimes$) on the set $\mathbb{Z}$:

- For $a, b \in \mathbb{Z}$, set $a \oplus b := a + b + 1$
- For $a, b \in \mathbb{Z}$, set $a \otimes b := ab + a + b = (a+1)(b+1) - 1$

Call the integers with these two operations $\mathbb{Z}$ (read red integers). With proofs, answer the questions:

1. Is $\mathbb{Z}$ a ring?
2. If $\mathbb{Z}$ is a ring, is it isomorphic to $\mathbb{Z}$?
3. Define similarly $\mathbb{Q}$: is it a field?
4. List all that goes wrong.
5. Which generalizations occur to you?

Class discussion

Let us prove the following characterization of Noetherian modules over commutative rings:

Definition

Let M be a module over the commutative ring R.The set I of elements $x \in R$ such that $xm = 0$ for all $m \in M$ is an ideal called the annihilator of M, $I = \operatorname{ann} M$.

Proposition

M is a Noetherian module if and only if M is finitely generated and $R/\operatorname{ann} M$ is a Noetherian ring.

Hints

If a module M is generated by $\{m_1, \ldots, m_n\}$ define the following mapping

$$\mathbf{f} : R \longrightarrow \underbrace{M \oplus \cdots \oplus M}_{\text{n copies}}, \quad \mathbf{f}(r) = (rm_1, \ldots, rm_n)$$

verify that

- $\mathbf{f}$ is a homomorphism, of kernel $\operatorname{ann} M$
- Form the appropriate embedding of $R/\operatorname{ann} M$ into the direct sum of the M's to argue one direction
- Use, for the other direction, that M is also a module over the ring $R/\operatorname{ann} M$

Quotient rings

Let I be a two-sided proper ideal of the R and denote by R/I the corresponding cosets $\{a+I : a \in R\}$.

The quotient ring R/I is defined by the operations:

$$\begin{aligned}(a+I)+(b+I) &= (a+b)+I \\ (a+I)\times(b+I) &= ab+I\end{aligned}$$

This is a source to many new rings

Examples

$$
\begin{aligned}
(2) \subset \mathbb{Z} \quad &\Rightarrow \quad \mathbb{Z}_2 = \mathbb{Z}/(2) \\
(x^2 + x + 1) \subset \mathbb{Z}_2[x] \quad &\Rightarrow \quad \mathbb{Z}_2[x]/(x^2 + x + 1) = \mathbf{F}_4 \\
(x^2 + 1) \subset \mathbb{R}[x] \quad &\Rightarrow \quad \mathbb{C} = \mathbb{R}[x]/(x^2 + 1) \\
(1 + 3i) \subset \mathbb{Z}[i] \quad &\Rightarrow \quad \mathbb{Z}_{10} = R = \mathbb{Z}[i]/(1 + 3i)
\end{aligned}
$$

$\mathbb{Z}[i]/(1+3i) \simeq \mathbb{Z}/(10)$

Consider the homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}[i] \to R = \mathbb{Z}[i]/(1+3i)$ induced by the embedding of $\mathbb{Z}$ in $\mathbb{Z}[i]$.We claim that φ is a surjection of kernel $10\mathbb{Z}$:

$$1+3i \equiv 0 \Rightarrow i(1+3i) \equiv 0 \Rightarrow i-3 \equiv 0 \Rightarrow i \equiv 3$$

$$a+bi \equiv a+3b \Rightarrow \varphi \text{ is surjection}$$

For n in kernel of φ,

$$\begin{aligned} n &= z(1+3i) = (a+bi)(1+31) \\ &= (a-3b) + \underbrace{(3a+b)i}_{=0} \quad \Rightarrow b = -3a \\ &= 10a \end{aligned}$$

Circle ring

Let $R = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$: the circle ring

- Consider the natural homomorphism

$$\mathbf{f} : \mathbb{R}[x, y] \longrightarrow \mathbb{R}[\cos t, \sin t], \quad \mathbf{f}(x) = \cos t, \mathbf{f}(y) = \sin t$$

  $\mathbb{R}[\cos t, \sin t]$ is the ring of trigonometric polynomials.
- $\mathbf{f}(x^2 + y^2 - 1) = 0$ so there is an induced surjection

$$\varphi : \mathbb{R}[x, y]/(x^2 + y^2 - 1) \to \mathbb{R}[\cos t, \sin t]$$

- φ is an isomorphism because: (i) $\mathbb{R}[\cos t, \sin t]$ is an infinite dimensional $\mathbb{R}$-vector space (why?); for any ideal L larger than $(x^2 + y^2 - 1)$, $\mathbb{R}[x, y]/L$ is a finite dimensional $\mathbb{R}$-vector space (why?).

- The circle ring $R = \mathbb{R}[\cos t, \sin t]$ contains as a subring $S = \mathbb{R}[\cos t]$. S is isomorphic to a polynomial ring over $\mathbb{R}$. As an S-module, R is generated by two elements

$$R = S \cdot 1 + S \cdot \sin t$$

- R as a $\mathbb{R}$-vector space has basis

$$\{\sin nt, \cos nt, \quad n \in \mathbb{Z}\}$$

$\mathbb{R}[x,y]/(xy)$

Exercise: Prove that

$$\mathbb{R}[x,y]/(xy) \simeq \{(p(x),q(y)) : p(0) = q(0))\}$$

Hint: Consider the homomorphism

$$\varphi : \mathbb{R}[x,y]/(xy) \to \mathbb{R}[x,y]/(y) \times \mathbb{R}[x,y]/(x)$$

$$\varphi(a + (xy)) = (a + (y), a + (x))$$

Check that φ is one-one and determine its image.

Integral domains

Let R be a commutative ring

- $u \in R$ is a unit if there is $v \in R$ such that $uv = 1$
- $a \in R$ is a zero divisor if there is $0 \neq b \in R$ such that $ab = 0$
- $a \in R$ is nilpotent if there is $n \in \mathbb{N}$ such that $a^n = 0$
- R is an integral domain if 0 is the only zero divisor, in other words, if $a, b \in R$ are not zero, then $ab \neq 0$.

Outline

Studying a commutative ring

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ \text{prime ideals of } R & & \text{morphisms } \varphi : R \to S \end{tikzcd}$$

Prime Ideals

Definition

Let R be a commutative ring. An ideal P of R is prime if $P \neq R$ and whenever $a \cdot b \in P$ then $a \in P$ or $b \in P$.

Equivalently:

- R/P is an integral domain
- If I and J are ideals and $I \cdot J \subset P$ then $I \subset P$ or $J \subset P$

Prime ideals arise in issues of factorization and very importantly:

Proposition

Let $\varphi : R \to S$ be a homomorphism of commutative ring. If S is an integral domain, then $P = \ker(\varphi)$ is a prime ideal. More generally, if S is an arbitrary commutative ring and Q is a prime ideal, then $P = \varphi^{-1}(Q)$ is a prime ideal of R.

Proof. Inspect the diagram

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & S \arrow[d] \\ R/P \arrow[r, hook] & S/Q \end{tikzcd}$$

Exercise

Consider the homomorphism of rings

$$\begin{aligned}\varphi : k[x,y,z] &\rightarrow k[t] \\ x &\rightarrow t^3 \\ y &\rightarrow t^4 \\ z &\rightarrow t^5\end{aligned}$$

Let P be the kernel of this morphism. Note that $x^3 - yz$, $y^2 - xz$ and $z^2 - x^2y$ lie in P.

Task: Prove that P is generated by these 3 polynomials.

Task: Describe the prime ideals of the ring

$$R = \mathbb{C}[x,y]/(y^2 - x(x-1)(x-2)).$$

Multiplicative Sets

Definition

A subset S of a commutative ring is multiplicative if $S \neq \emptyset$ and if $r, s \in S$ then $r \cdot s \in S$.

- If P is a prime ideal of R, $S = R \setminus P$ is a multiplicative set.
- If I is a proper ideal of R, then

$$S = \{1 + a : a \in I\}$$

is a multiplicative set.

Formation of Prime Ideals

Proposition

Let S be a multiplicative set and P an ideal maximum with respect $S \cap P = \emptyset$. Then P is a prime ideal.

Proof. Deny: let $a, b \notin P$, $ab \in P$.

Consider the ideals $P + Ra$ and $P + Rb$. They are both larger than P and therefore meet S:

$$x + pa, y + qb \in S, \quad x, y \in P$$

Multiplying we get

$$(x + pa)(y + qb) = xy + xqb + yqb + pqab \in S \cap P,$$

a contradiction.

Corollary

Every proper ideal I of a commutative ring is contained in a prime ideal.

Proof. Let $S = \{1\}$. Among all proper ideals $I \subseteq J$ pick one that is maximum with respect being disjoint relative to S (use Zorn's Lemma; no need if R is Noetherian).

Primary Ideal

Definition

Let R be a commutative ring. An ideal Q of R is primary if $Q \neq R$ and whenever $a \cdot b \in Q$ then $a \in Q$ or some power $b^n \in Q$.

Example: $Q = (x^2, y) \subset R = k[x, y]$, or $(p^n) \subset \mathbb{Z}$.
This is a far-reaching generalization of the notion of primary ideals of $\mathbb{Z}$

Radical of an Ideal

Definition

Let I be an ideal of the commutative ring R. The radical of I is the set

$$\sqrt{I} = \{x \in R : x^n \in I \quad \text{some } n = n(x)\}.$$

Proposition

$\sqrt{I}$ is an ideal.

Proof.

If $a, b \in \sqrt{I}$, $a^m \in I$, $b^n \in I$, then

$$(a+b)^{m+n-1} = \sum_{i+j=m+n-1} \binom{m+n-1}{i} a^i b^j \in I,$$

since $i \geq m$ or $j \geq n$.

Proposition

If I is a proper ideal of R,

$$\sqrt{I} = \bigcap P, \quad I \subseteq P \quad P \text{ prime ideal.}$$

Proof.

Deny it: Let $x \in \bigcap P \setminus \sqrt{I}$, that is for all n, $x^n \notin I$.

The set $\{x^n, n \in \mathbb{N}\}$ defines a multiplicative set S disjoint from I. By a previous proposition, there is a prime $P \supset I$ disjoint from S, a contradiction. $\square$

Outline

Assignment #7

A Boolean ring is a ring R such that $x^2 = x$ for all $x \in R$. For instance, an arbitrary direct product of copies of $\mathbb{Z}/(2)$. If R is a Boolean ring:

1. Prove that R is commutative and that for every prime ideal P, R/P is a field.
2. Prove that every finitely generated ideal I of R is principal (*Hint:* check that in a boolean ring, $a + b - ab$ is a multiple of both a and b).
3. If R is finite, show that R is a finite direct product of copies of $\mathbb{Z}/(2)$.

Idempotents

Proposition

Let R be a commutative ring and $0 \neq e \in R$ satisfy $e = e^2$. Then there is a decomposition R into the direct product of rings $R \simeq Re \times R(1-e)$.

Proof.

1. For any $x \in R$, $x = xe + x(1-e)$, so $Re + R(1-e) = R$. Furthermore if $a \in Re \cap R(1-e)$, then a is annihilated by $1-e$ and e, respectively. This means that $R = Re \oplus R(1-e)$ as modules.
2. Since $Re \cdot R(1-e) = 0$, we can view $R = Re \oplus R(1-e)$ as $R = Re \times R(1-e)$. Note that e is the identity in the ring Re, and $1-e$ in $R(1-e)$.

Outline

Emmy Noether (1882-1935)

http://upload.wikimedia.org/wikipedia/commons/e/e5/Noether.jpg

Irreducible Ideal/Module

Definition

The ideal I of the commutative ring R is irreducible if

$$I = J \cap L \Rightarrow I = J \quad \text{or} \quad I = L.$$

Primary Decomposition

Theorem (Emmy Noether)

Every proper ideal I of a Noetherian ring R has a finite decomposition

$$I = Q_1 \cap Q_2 \cap \cdots \cap Q_n,$$

with Q_i primary.

To prove her theorems, Emmy Noether often proved a special case and derive the more general assertion, or proved a more general assertion and specialize.

Irreducible decomposition

Definition

The ideal I of the commutative ring R is irreducible if

$$I = J \cap L \Rightarrow I = J \quad \text{or} \quad I = L.$$

Theorem (Emmy Noether)

Every proper ideal I of a Noetherian ring R has a finite decomposition

$$I = J_1 \cap J_2 \cap \cdots \cap J_n,$$

with J_i irreducible. Moreover, every irreducible ideal J of R is primary.

Famous Proof

Proof. Deny the existence of the decomposition of I as a finite intersection of irreducible ideals. Among all such ideals, denote by (keep the notation) I a maximum one.
I is not irreducible, so there is

$$I = J \cap L,$$

with J and L properly larger. But then each admits finite decompositions as intersection of irreducible ideals. Combining we get a contradiction.

Irreducible $\Rightarrow$ Primary

1. Deny that proper irreducible ideals of Noetherian rings are primary. Let I be maximum such: There is $a, b \in R$, $ab \in I$, $a \notin I$ and $b^n \notin I$ for all $n \in \mathbb{N}$.
2. Consider the chain
$$\{r \in R : br \in I\} = I : b \subseteq I : b^2 \subseteq \cdots \subseteq I : b^n \subseteq I : b^{n+1}$$
that becomes stationary at $I : b^n = I : b^{n+1}$.
3. Define $J = I : b^n$ and $L = (I, b^n)$. Both ideals are larger than I. We claim that $I = J \cap L$.
4. If $x \in J \cap L$, $x = u + rb^n$, $u \in I$. Then $b^n x = b^n u + rb^{2n} \in I$, so $rb^n \in I$ and therefore $x \in I$.

Irredundant Primary Decomposition

A refinement in the primary decomposition

$$I = Q_1 \cap Q_2 \cap \cdots \cap Q_n$$

arises as follows. Suppose two of the Q_i have the same radical, say $\sqrt{Q_1} = \sqrt{Q_2} = P$. Then it easy to check that $Q_1 \cap Q_2$ is also P-primary. So collecting the Q_i with the same radical:

Theorem (Emmy Noether)

Every proper ideal I of a Noetherian ring R has a finite decomposition

$$I = Q_1 \cap Q_2 \cap \cdots \cap Q_n,$$

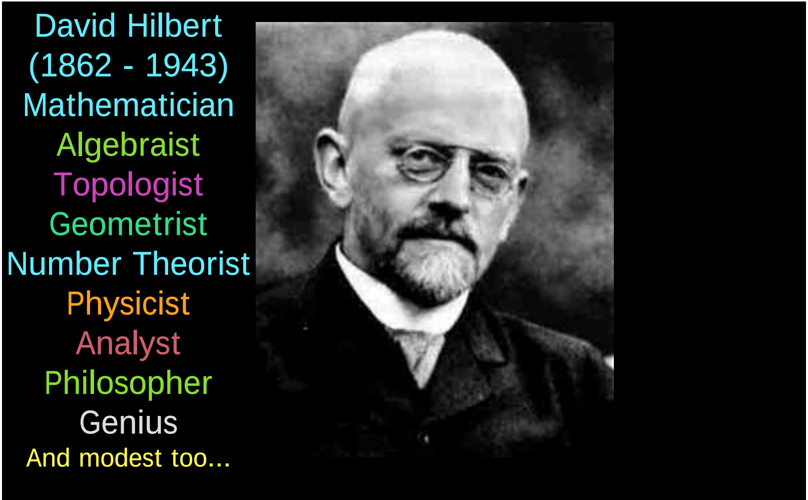
with Q_i primary ideals of distinct radicals. This decomposition is called irredundant.

It is known which Q_i are unique and which are not.

Outline

David Hilbert (1862-1943)

David Hilbert



"Physics is much too hard for physicists." - Hilbert, 1912

Hilbert Basis Theorem

Theorem (HBT)

If R is Noetherian then $R[x]$ is Noetherian.

1. If R is Noetherian and $x_1, \ldots, x_n$ is a set of independent indeterminates, then $R[x_1, \ldots, x_n]$ is Noetherian.
2. $\mathbb{Z}[x_1, \ldots, x_n]$ is Noetherian.
3. If k is a field, then $k[x_1, \ldots, x_n]$ is Noetherian.

Finitely Generated Algebras

If R is a commutative ring, a finitely generated R-algebra S is a homomorphic image of a ring of polynomials, $S = R[x_1, \ldots, x_n]/L$. If R is Noetherian, S is Noetherian as well. This is useful in many constructions.
If I is an R-ideal, the Rees algebra of I is the subring of $R[t]$ generated by all at, $a \in I$. It it denoted by $S = R[It]$. In general, subrings of Noetherian rings may not be Noetherian but Rees algebras are:

Exercise: If R is Noetherian, for every ideal I, $R[It]$ is Noetherian.

Proof of the HBT

Suppose the $R[x]$-ideal I is not finitely generated. Let $0 \neq f_1(x) \in I$ be a polynomial of smallest degree,

$$f_1(x) = a_1 x^{d_1} + \text{lower degree terms.}$$

Since $I \neq (f_1(x))$, let $f_2(x) \in I \setminus (f_1(x))$ of least degree. In this manner we get a sequence of polynomials

$$f_i(x) = a_i x^{d_i} + \text{lower degree terms,}$$

$$f_i(x) \in I \setminus (f_1(x), \ldots, f_{i-1}(x)), \quad d_1 \leq d_2 \leq d_3 \leq \cdots$$

Set $J = (a_1, a_2, \ldots,) = (a_1, a_2, \ldots, a_m) \subseteq R$

Let $f_{m+1}(x) = a_{m+1}x^{d_{m+1}} + \text{lower degree terms}$. Then

$$a_{m+1} = \sum_{i=1}^{m} s_i a_i, \quad s_i \in R.$$

Consider

$$\mathbf{g}(x) = f_{m+1} - \sum_{i=1}^{m} s_i x^{d_{m+1}-d_i} f_i(x).$$

$\mathbf{g}(x) \in I \setminus (f_1(x), \ldots, f_m(x))$, but $\deg \mathbf{g}(x) < \deg f_{m+1}(x)$, which is a contradiction.

Power Series Rings

Another construction over a ring R is that of the power series ring $R[[x]]$:

$$\mathbf{f}(x) = \sum_{n \geq 0} a_n x^n, \quad \mathbf{g}(x) = \sum_{n \geq 0} b_n x^n$$

with addition component wise and multiplication the Cauchy operation

$$\begin{aligned} \mathbf{f}(x)\mathbf{g}(x) = \mathbf{h}(x) &= \mathbf{h}(x) = \sum_{n \geq 0} c_n x^n \\ c_n &= \sum_{i+j=n} a_i b_{n-i} \end{aligned}$$

Theorem

If R is Noetherian then $R[[x]]$ is Noetherian.

Proposition

A commutative ring R is Noetherian iff every prime ideal is finitely generated.

Proof. If R is not Noetherian, there is an ideal I maximum with the property of not being finitely generated (Zorn's Lemma). We assume I is not prime, that is there exist $a, b \notin I$ such that $ab \in I$.

The ideals (I, a) and $I : a$ are both larger than I and therefore are finitely generated:

$$\begin{aligned} (I : a) &= (a_1, \dots, a_n) \\ (I, a) &= (b_1, \dots, b_m, a), \quad b_i \in I \end{aligned}$$

Claim: $I = (b_1, \dots, b_m, aa_1, \dots, aa_n)$

If $c \in I$,

$$c = \sum_{i=1}^{m} c_i b_i + ra, \quad r \in I : a$$

$R[[x]]$ is Noetherian

Proof. Let P be a prime ideal of $R[[x]]$. Set $\mathfrak{p} = P \cap R$. $\mathfrak{p}$ is a prime ideal of R and therefore it is finitely generated.

Denote by $\mathfrak{p}[[x]] = \mathfrak{p}R[[x]]$ the ideal of $R[[x]]$ generated by the elements of $\mathfrak{p}$. It consists of the power series with coefficients in $\mathfrak{p}$ and $R[[x]]/\mathfrak{p}[[x]]$ is the power series ring $R/\mathfrak{p}[[x]]$.

We have the embedding

$$P' = P/\mathfrak{p}[[x]] \hookrightarrow (R/\mathfrak{p})[[x]]$$

P' is a prime ideal of $R/\mathfrak{p}[[x]]$ and $P' \cap R/\mathfrak{p} = 0$. It will suffice to show that P' is finitely generated.

We have reduced the proof to the case of a prime ideal $P \subset R[[x]]$ and $P \cap R = (0)$.

If $x \in P$, $P = (x)$ and we are done.
For $\mathbf{f}(x) = a_0 + a_1 x + \cdots \in P$, let $J = (b_1, \ldots, b_m) \subset R$ be the ideal generated by all a_0,

$$\mathbf{f}_i = b_i + \text{higher terms} \in P.$$

Claim: $P = (\mathbf{f}_1, \ldots, \mathbf{f}_m)$.

From $a_0 = \sum_i s_i^{(0)} b_i$, we write

$$\mathbf{f}(x) - \sum_i s_i^{(0)} \mathbf{f}_i = x\mathbf{h} \quad \Rightarrow \mathbf{h} \in P.$$

We repeat with **h** and write

$$\mathbf{f}(x) = \sum_i s_i^{(0)} \mathbf{f}_i + x \sum_i s_i^{(1)} \mathbf{f}_i + x^2 \mathbf{g}, \quad \mathbf{g} \in P.$$

Iterating we obtain

$$\mathbf{f}(x) = \sum_i (s_i^{(0)} + s_i^{(1)} x + s_i^{(2)} x^2 + \cdots) \mathbf{f}_i.$$

Outline

Assignment #8

Do 2 problems.

1. Show that the kernel of the homomorphism ($\mathbf{K}$ is a field)

$$\varphi : \mathbf{K}[x, y, z] \longrightarrow \mathbf{K}[t],$$

defined by $\varphi(x) = t^3$, $\varphi(y) = t^4$ and $\varphi(z) = t^5$, is generated by the polynomials

$$x^3 - yz, y^2 - xz, z^2 - x^2y.$$

2. Let R be a Noetherian ring and let I be an R–ideal. Show that the number of prime ideals P minimal over I is finite. (*Hint*: primary decomposition helps.)
3. Describe all rings $\mathbb{Z} \subset R \subset \mathbb{Q}$ (*Hint:* For each R, consider the set of primes p of $\mathbb{Z}$ that blowup in R, that is, $pR = R$).
4. Let $\varphi : M \longrightarrow M$ be an endomorphism of a R-module. Prove that if M is Noetherian (resp. Artinian) and φ is surjective (resp. injective) then φ is an isomorphism.

Outline

Homework

1. Find the kernel of the homomorphism (**K** is a field)

$$\varphi : \mathbf{K}[x, y, z] \longrightarrow \mathbf{K}[t],$$

defined by $\varphi(x) = t^4$, $\varphi(y) = t^5$ and $\varphi(z) = t^7$. What do you think is true in general?

2. Show that $R = \mathbb{C}[x, y]/(y^2 - x(x-1)(x-2))$ is a Dedekind domain. [Show that $y^2 - x(x-1)(x-2)$ is irreducible, use the Nullstellensatz to describe the maximal ideals of R, and show that for each such ideal P, R_P is a discrete valuation domain.]
3. If R is a Dedekind domain, prove that for each nonzero ideal I, R/I is a principal ideal ring. Derive from this the fact that every ideal of R can be generated by 2 elements.
4. Show that an invertible ideal of a local integral domain is principal.

Outline

Modules of Fractions

Let R be a commutative ring, M an R-module and $S \subseteq R$ a multiplicative system.

On the set $M \times S$ define the following relation:

$$(a, r) \sim (b, s) \Leftrightarrow \exists t \in S : t(as - br) = 0$$

Why define it in this manner instead of the usual $as = br$?

Proposition

$\sim$ *is an equivalence relation.*

We focus on the properties of the set $S^{-1}M$ of equivalence classes. Actually, this is the initial step in the construction of a remarkable functor.

Properties

Proposition

Let R be a commutative ring, M an R-module and $S \subseteq R$ a multiplicative system. Denote the equivalence class of (a, r) in $S^{-1}M$ by $\overline{(a, r)}$ (or simply (a, r) or even a/r).

1. *The following operation is well-defined*

$$\overline{(a, r)} + \overline{(b, s)} = \overline{(sa + rb, rs)},$$

   and endows $S^{-1}M$ with a structure of abelian group.

2. *If $0 \notin S$, this construction applied to $R \times S$ gives rise to a ring structure on $S^{-1}R$ with multiplication $\overline{(x, r)} \cdot \overline{(y, s)} = \overline{(xy, rs)}$.*

3. *For $\overline{(x, r)} \in S^{-1}R$ and $\overline{(a, s)} \in S^{-1}M$, the operation $\overline{(x, r)} \cdot \overline{(a, s)} = \overline{(xa, rs)}$ defines an $S^{-1}R$-module structure on $S^{-1}M$.*

Module/Ring of Fractions

$S^{-1}R$ is called the ring of fractions of R relative to S. It is a refinement (due to Grell or Krull) of the classical formation of the field of fractions of an integral domain.
$S^{-1}M$ is called the module of fractions of M relative to S.

Another step:

Proposition

If $\varphi : M \to N$ is a homomorphism of R-modules, a homomorphism of $S^{-1}R$ modules $S^{-1}\varphi : S^{-1}M \to S^{-1}N$ is defined by

$$(S^{-1}\varphi)(a, s) = (\varphi(a), s).$$

Functorial Properties

This construction is a functor from the category of R-modules to the category of $S^{-1}R$-modules:

$$\begin{tikzcd} M \arrow[d, "\varphi"'] & \rightsquigarrow & S^{-1}M \arrow[d, "S^{-1}\varphi"] \\ N & \rightsquigarrow & S^{-1}N \end{tikzcd}$$

Proposition

If $\varphi : M \to N$ and $\psi : N \to P$ are R-homomorphisms of R-modules, then

1. $S^{-1}(\psi \circ \varphi) = S^{-1}\psi \circ S^{-1}\varphi$.
2. $S^{-1}(id_M) = id_{S^{-1}M}$.

Short Exact Sequences

Proposition

Let R be a ring, $S \subseteq R$ a multiplicative set and

$$0 \to A \xrightarrow{\mathbf{f}} B \xrightarrow{\mathbf{g}} C \to 0$$

a short exact sequence of R-modules. Then

$$0 \to S^{-1}A \xrightarrow{S^{-1}\mathbf{f}} S^{-1}B \xrightarrow{S^{-1}\mathbf{g}} S^{-1}C \to 0$$

is a short exact sequence of $S^{-1}R$-modules. In other words, $M \rightsquigarrow S^{-1}M$ is an exact functor.

The submodules of $S^{-1}M$

Proposition

Let L' be a $S^{-1}R$-submodule of $S^{-1}M$. Let

$$L = \{m \in M : \text{for some } s \in S \quad (m, s) \in L'.$$

Then L is a submodule of M and $S^{-1}L = L'$.

Corollary

If M is a Noetherian (Artinian) R-module, then $S^{-1}M$ is a Noetherian (Artinian) $S^{-1}R$-module.

The ideals of $S^{-1}R$

According to the above, the proper ideals of $S^{-1}R$ are of the form

$$S^{-1}I = \{a/s : a \in I \quad s \in S, \quad I \cap S = \emptyset.\}$$

In the special case of $S = R \setminus \mathfrak{p}$, for a prime ideal $\mathfrak{p}$, one uses the notation $M_{\mathfrak{p}}$ for the module of fractions and $R_{\mathfrak{p}}$ for the ring of fractions.

If $R = \mathbb{Z}$ and $\mathfrak{p} = (2)$, $\mathbb{Z}_{(2)}$ consists of all rational numbers m/n, with n odd. Its ideals are ordered. The largest proper ideal is $\mathfrak{m} = 2\mathbb{Z}_{(2)}$ and the others

$$\mathbb{Z}_{(2)} \supsetneq \mathfrak{m} \supsetneq \mathfrak{m}^2 \supsetneq \mathfrak{m}^3 \supsetneq \cdots \supsetneq (0)$$

Tool

Proposition

If R is a commutative ring and S is a multiplicative set, then for any two submodules A and B of M,

$$S^{-1}(A \cap B) = S^{-1}A \cap S^{-1}B.$$

Proof.

The intersection $A \cap B$ can be defined by the exact sequence

$$0 \to A \cap B \longrightarrow A \oplus B \xrightarrow{\varphi} A + B \to 0,$$

where $\varphi(a, b) = a - b$.

Now apply the fact that formation of modules of fractions is an exact functor. $\square$

Local Ring

Proposition

Let S be a multiplicative set of R. The ideal L of $S^{-1}R$ is prime iff $L = S^{-1}I$, for some prime I ideal of R with $I \cap S = \emptyset$.

Proof. Suppose I is as above. If $a/r \cdot b/s \in S^{-1}I$, $(ab, rs) \sim (c, t)$ for $c \in I$, $r, s, t \in S$. By definition, there is $u \in S$ such that $u(tab - rsc) = 0$. Since $S \cap I = \emptyset$, $tab - rsc \in I$ and therefore $tab \in I$. Thus $ab \in I$ and so $a \in I$ or $b \in I$. Therefore (a, r) or $(b, s) \in S^{-1}I$.

Corollary

The prime ideals of $R_{\mathfrak{p}}$ have the form $P = Q_{\mathfrak{p}}$, where Q is an ideal of R contained in $\mathfrak{p}$.

Local Ring

Definition

A commutative ring R is a local ring if it has a unique maximal ideal.

Example

If k is a field, $R = k[[x]]$, the ring of formal power series in x over k is a local ring. Its unique maximal ideal is $\mathfrak{m} = (x)$.

Definition

If R is a commutative ring and P a prime ideal, the ring of fractions R_P is a local ring called the localization of R at P.

The Prime Spectrum of a Ring

Definition

Let R be a commutative ring (with 1). The set of prime ideals of R is called the prime spectrum of R, and denoted $\mathrm{Spec}\,(R)$.

$\mathrm{Spec}\,(\mathbb{Z}) = \{(0), (2), (3), \ldots\}$, the ideals generated by the prime integers and 0.

Proposition

For each set $I \subset R$, set

$$V(I) = \{\mathfrak{p} \in \mathrm{Spec}\,(R) : I \subset \mathfrak{p}\}.$$

These subsets are the closed sets of a topology on $\mathrm{Spec}\,(R)$.

Note that $V(I) = V(I')$, where I' is the ideal of R generated by I.

Zariski Topology

Proof. This follows from the properties of the construction of the $V(I)$:

$$\begin{aligned} V(1) &= \emptyset \\ V(0) &= \mathrm{Spec}\,(R) \\ V(I \cap J) &= V(I) \cup V(J) \\ \bigcap_{\alpha} V(I_\alpha) &= V(\bigcup_{\alpha} I_\alpha). \end{aligned}$$

Example

Suppose $R_2, R_2, \ldots, R_n$ are commutative rings and $R = R_1 \times R_2 \times \cdots \times R_n$ is their direct product. Observe:

1. If $1 = e_1 + e_2 + \cdots + e_n$, $e_i \in R_i$, then $R_i = Re_i$ and $e_i e_j = 0$ if $i \neq j$
2. Because of $e_i e_j = 0$ for $i \neq j$, if P is a prime ideal of R and some $e_i \notin P$ then the other $e_j \in P$. This shows $P = R_1 \times \cdots \times P_i \times \cdots \times R_n$, where P_i is a prime ideal of R_i, $R/P = R_i/P_i$
3. $\mathrm{Spec}(R) = \mathrm{Spec}(R_1) \cup \cdots \cup \mathrm{Spec}(R_n)$
4. In particular, if $R_1 = R_2 = \cdots = R_n = \mathbf{K}$, $\mathbf{K}$ a field, the $\mathrm{Spec}(R)$ is a set of n points with the discrete topology.

Irreducible Representation

Proposition

Let I be an ideal of the Noetherian ring R and let

$$I = Q_1 \cap Q_2 \cap \cdots \cap Q_n,$$

be a primary representation. Then

$$V(I) = V(P'_1) \cup V(P'_2) \cup \cdots \cup V(P'_m),$$

where the P'_j are the minimal primes amongst the $\sqrt{Q_i}$, is the unique irreducible representation of $V(I)$.

Morphisms

Proposition

If R is a commutative ring, $\mathrm{Spec}\,(R)$ is quasi-compact. (Not necessarilly Hausdorff.)

Proof.

Let $\{D(I_\alpha)\}$ be an open cover of X

$$X = \bigcup_\alpha D(I_\alpha) = \sum_\alpha I_\alpha = D(1).$$

This means that there is a finite sum

$$\sum_1^n I_{\alpha_i} = R, \quad \text{and therefore } X = \bigcup_{i=1}^n D(I_{\alpha_i}).$$

□

Proposition

If $\varphi : R \to S$ is a homomorphism of commutative rings ($\varphi(1_R) = 1_S$), then the mapping

$$\Phi : \mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R),$$

given by $\Phi(Q) = \varphi^{-1}(Q)$, is continuous.

Proof.

If $D(I)$ is an open set of $\mathrm{Spec}\,(R)$, $\varphi^{-1}(D(I)) = D(IS)$. $\square$

Outline

Assignment #9

Do 1 problem.
For the ring $R = \mathbb{Z}[\mathbf{T}]$

1. Describe (with proofs) its prime ideals, that is the points of $\mathrm{Spec}\,(R)$.
2. Describe (with proofs) its maximal ideals, that is the closed points of $\mathrm{Spec}\,(R)$.
3. Let $\mathbb{X}$ be a compact, Hausdorff space and denote by **A** the ring of real continuous functions on $\mathbb{X}$.
    - If M is a maximal ideal of **A** prove that there is a point $p \in \mathbb{X}$ such that $M = \{\mathbf{f}(\mathbf{x}) \in \mathbf{A} : \mathbf{f}(p) = 0\}$.
    - Prove that there is a homeomorphism of topological spaces $\mathbb{X} \approx \mathrm{MaxSpec}(\mathbf{A})$.

Outline

Integral Extensions

Let $R \hookrightarrow S$ be commutative rings.

Definition

$s \in S$ is integral over R if there is an equation

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0, \quad a_i \in R.$$

Proposition

$s \in S$ is integral over R if and only if the subring $R[s]$ of S generated by s is a finitely generated R-module.

Would like to prove [as done first by Weierstrass] that if s_1 and s_2 in S are integral over R then

- $s_1 + s_2$ is integral over R;
- $s_1 s_2$ is integral over R.

The key to their proof is the fact that both $s_1 + s_2$ and $s_1 s_2$ are elements of the subring $R[s_1, s_2]$ which is finitely generated as an R-module

$$R[s_1, s_2] = \sum_{i,j} R s_1^i s_2^j,$$

where i and j are bounded by the degrees of the equations satisfied by s_1 and s_2.

Integrality Criterion

Proposition

Let M be a finitely generated R-module and $S = R[u]$ a ring such that $uM \subset M$. If M is a faithful S-module then u is integral over R.

Proof. Let $x_1, \ldots, x_n$ be a set of R-generators of M. we have a set of relations with $a_{ij} \in R$

$$\begin{aligned} ux_1 &= a_{11}x_1 + \cdots + a_{1n}x_n \\ &\vdots \\ ux_n &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned}$$

Cayley-Hamilton

That is

$$\begin{aligned} 0 &= (a_{11} - u)x_1 + \cdots + a_{1n}x_n \\ &\vdots \\ 0 &= a_{n1}x_1 + \cdots + (a_{nn} - u)x_n \end{aligned}$$

Which we rewrite in matrix form

$$\begin{bmatrix} a_{11} - u & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} - u \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{A}[\mathbf{x}] = O.$$

Thus

$$(\text{adj } \mathbf{A})\mathbf{A}[\mathbf{x}] = \det \mathbf{A} \cdot [\mathbf{x}] = O.$$

This means that $\det \mathbf{A}$ annihilates each generator x_i of M and therefore $\det \mathbf{A} = 0$.

But

$$\det \mathbf{A} = \pm u^n + \text{lower powers of } u \text{ with coefficients in } R$$

This shows that u is integral over R.

Principle of Specialization

Why are we allowed to write $\operatorname{adj} \mathbf{A} \cdot \mathbf{A} = \det \mathbf{A} \cdot \mathbf{I}$ when the entries of $\mathbf{A}$ lie in a commutative ring?

If $T = \mathbb{Z}[x_{ij},\ 1 \leq i, j \leq n]$ is a ring of polynomials in the indeterminates x_{ij}, and use them as the entries of a matrix $\mathbf{B}$, certainly the formula $\operatorname{adj}\ \mathbf{B} \cdot \mathbf{B} = \det \mathbf{B} \cdot \mathbf{I}$ makes sense since T lies in a field.

Now define a ring homomorphism $\phi : T \to R$, with $\phi(x_{ij})$ the corresponding entry in $\mathbf{A}$, to get the desired equality.

In our application, $M = R[s_1, s_2]$ and u is either $s_1 + s_2$ or $s_1 s_2$, and certainly M is faithful since $1 \in M$.

Corollary

If $R \hookrightarrow S$ are commutative rings, and $s_1, s_2, \ldots, s_n$ are integral over R, then any element of $R[s_1, \ldots, s_n]$ is integral over R. Moreover, if T is the set of elements of S integral over R, T is a subring. It is called the integral closure of R in S.

Definition

If $T = S$, S is called an integral extension of R.

Transitivity

Proposition

If $R \hookrightarrow S_1 \hookrightarrow S_2$ are commutative rings with S_1 integral over R and S_2 integral over S_1, then S_2 is integral over R.

Proof. Let $u \in S_2$ be integral over S_1

$$u^n + s_{n-1}u^{n-1} + \cdots + s_1 u + s_0 = 0, \quad s_i \in S_1.$$

It suffices to observe that

$$M = R[u, s_{n-1}, \ldots, s_1, s_0]$$

is a finitely generated R-module.

Surjections

Another use of the Cayley-Hamilton theorem is the following property of surjective epimorphims of modules:

Theorem

Let R be a commutative ring and M a finitely generated R. If $\varphi : M \to M$ is a surjective R-module homomorphism, then φ is an isomorphism.

Proof. We first turn M into a module over the ring of polynomials $S = R[t]$ by setting $t \cdot m = \varphi(m)$ for $m \in M$.

The assumption means that $tM = M$. Using the proof of Cayley-Hamilton, we have

$$\begin{bmatrix} ta_{11}-1 & \cdots & ta_{1n} \\ \vdots & \ddots & \vdots \\ ta_{n1} & \cdots & ta_{nn}-1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{A}[\mathbf{x}] = O.$$

Which implies that $\det \mathbf{A}$ annihilates M. Since

$$\det \mathbf{A} = \pm 1 + t\mathbf{f}(t),$$

it is clear that $t \cdot m \neq 0$ for $m \neq 0$, that is φ is one-to-one.

Jacobson Radical

Definition

Let R be a commutative ring. Its Jacobson radical is the intersection $\bigcap Q$ of all maximal (proper) ideals.

Example: If R is a local ring, its Jacobson radical is its unique maximal ideal $\mathfrak{m}$.

If $R = \mathbb{Z}$, or $R = k[t]$, polynomial ring over the field k, then (0) is the Jacobson radical: from the infinity of prime elements.

Proposition

The Jacobson radical J of R is the set

$$J' = \{a \in R : 1 + ra \quad \text{is invertible for all } r \in R\}.$$

Proof. If $a \in J$, then $1 + ra$ cannot be contained in any proper maximal ideal, that is it must be invertible.
Conversely, if $a \in J'$, suppose a does not belong to the maximal ideal Q. Therefore

$$(a, Q) = R$$

which means there is an equation $ra + q = 1$, $q \in Q$, and q would be invertible.

Nakayama Lemma

Theorem (Nakayama Lemma)

Let M be a finitely generated R module and J its Jacobson radical. If

$$M = JM,$$

then $M = 0$.

Proof. If M is cyclic, this is clear: $M = (x)$ implies $x = ux$ for some $u \in J$, so that $(1 - u)x = 0$, which implies $x = 0$ since $1 - u$ is invertible.

We are going to argue by induction on the minimal number of generators of M. Suppose $M = (x_1, \ldots, x_n)$. By assumption $x_1 \in JM$, that is we can write

$$x_1 = u_1 x_1 + u_2 x_2 + \cdots + u_n x_n, \quad u_i \in J.$$

Which we rewrite as

$$(1 - u_1)x_1 = u_2x_2 + \cdots + u_nx_n$$

This shows that $x_1 \in J(x_2, \ldots, x_n)$, and therefore $M = (x_2, \ldots, x_n)$.

Corollary

Let M be a finitely generated R module and N a submodule. If $M = N + JM$ then $M = N$.

Proof.

Apply the Nakayama Lemma to the quotient module M/N

$$M/N = N + JM/N = J(M/N).$$

□

.

Scholium

Let R be a commutative ring and M a finitely generated R-module. If for some ideal I, $IM = M$, then $(1 + a)M = 0$ for some $a \in I$.

Proof.

If $M = (x_1, \ldots, x_n)$, from the proof of Cayley-Hamilton, there are $a_{ij} \in I$

$$\begin{bmatrix} a_{11} - 1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{A}[\mathbf{x}] = O.$$

Which implies that $\det \mathbf{A}$ annihilates M. Since $\det \mathbf{A} = \pm 1 + a, \quad a \in I$, done $\square$

Corollary

Let R be a commutative ring and I a finitely generated ideal. Then $I = I^2$ if and only if I is generated by an idempotent, that is $I = Re$, $e^2 = e$.

Proof.

If $(1 + a)I = 0$, $I \subset (a)$ and $a^2 = a$. $\square$

Outline

Integral Morphisms

Let $\varphi : R \to S$ an injective homomorphism of commutative rings.

Theorem (Lying-Over Theorem)

If S is integral over R then for each $\mathfrak{p} \in \operatorname{Spec}(R)$ there is $P \in \operatorname{Spec}(S)$ such that $\mathfrak{p} = P \cap R$, that is the morphism

$$\operatorname{Spec}(S) \to \operatorname{Spec}(R)$$

is surjective.

Proposition

If S is integral over R and T is a multiplicative set of R, then $T^{-1}S$ is integral over $T^{-1}R$.

Proof.

Let $s/t \in T^{-1}S$. s satisfies an equation

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0, \quad a_i \in R.$$

Then

$$(s/t)^n + a_{n-1}/t(s/t)^{n-1} + \cdots + a_1/t^{n-1}s/t + a_0/t^n = 0,$$

$a_i/t^{n-i} \in T^{-1}R$. $\square$

Proof of Lying-Over

Suppose $\mathfrak{p} \in \operatorname{Spec}(R)$. Consider the integral extension $R_\mathfrak{p} \hookrightarrow S_\mathfrak{p}$.

The maximal ideal of $R_\mathfrak{p}$ is $\mathfrak{m} = \mathfrak{p}R_\mathfrak{p}$.

Claim: $\mathfrak{m}S_\mathfrak{p} \neq S_\mathfrak{p}$.

Otherwise we would have

$$\begin{aligned} 1 &\in \mathfrak{m}S\mathfrak{p} \\ 1 &= \sum_{i=1}^{n} a_i s_i / t_i, \quad a_i \in \mathfrak{m},\ s_i \in S,\ t_i \in R \setminus \mathfrak{p} \end{aligned}$$

1. Set $S' = R_{\mathfrak{p}}[s_1, \ldots, s_n]$.
2. S' is a finitely generated $R_{\mathfrak{p}}$-module with $S' = \mathfrak{m}S'$. By Nakayama Lemma, $S' = 0$.
3. Since $\mathfrak{m}S_{\mathfrak{p}} \neq S_{\mathfrak{p}}$, it is contained in a prime ideal P' of $S_{\mathfrak{p}}$. In particular, $P' \cap R_{\mathfrak{p}} = \mathfrak{m}$.
4. Since $P' = P_{\mathfrak{p}}$ for some $P \in \mathrm{Spec}\,(S)$, it is clear that $P \cap R = \mathfrak{p}$, as desired.

Going-Up Theorem

Theorem

Let $R \hookrightarrow S$ be an integral extension of commutative rings. Let $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ be prime ideals of R and suppose P_1 is a prime ideal of S such that $P_1 \cap R = \mathfrak{p}_1$. Then there is a prime ideal $P_1 \subsetneq P_2$ of S such that $P_2 \cap R = \mathfrak{p}_2$.

Proof. Consider the diagram

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d] & S \arrow[d] \\ R/\mathfrak{p}_1 \arrow[r, hook] & S/P_1 \end{tikzcd}$$

Now apply the Lying-Over theorem to the integral extension

$$R/\mathfrak{p}_1 \hookrightarrow S/P_1.$$

Going-Down Theorem

? Is there

Theorem (?Going-Down Theorem)

Let $R \hookrightarrow S$ be an integral extension of commutative rings. Let $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ be prime ideals of R and suppose P_2 is a prime ideal of S such that $P_2 \cap R = \mathfrak{p}_2$. Then there is a prime ideal $P_1 \subsetneq P_2$ of S such that $P_1 \cap R = \mathfrak{p}_1$.

Yes, but needs additional assumptions. Proof uses some basic Galois theory.

Outline

Assignment #10

Let $R \hookrightarrow S$ be an integral extension. Prove the following assertions:

1. If R and S are integral domains and one of them is a field, then the other is also a field.
2. Equivalently: Let $P \in \operatorname{Spec}(S)$ and $\mathfrak{p} \in \operatorname{Spec}(R)$ and $P \cap R = \mathfrak{p}$. Then P is maximal iff $\mathfrak{p}$ is maximal.

Outline

TakeHome #1

Do 5 problems.

- Describe [with proof] a method to construct a regular pentagon with ruler and compass.
- Show that if $n \geq 3$, then $x^{2^n} + x + 1$ is reducible over $\mathbb{Z}_2$.
- Describe (with proofs) the maximal ideals of $R = \mathbb{Z}[\mathbf{T}]$, that is the closed points of $\operatorname{Spec}(R)$. **Achtung:** Pay attention to polynomials such as $a\mathbf{T} - 1$.
- Let $R = k[x_1, \ldots, x_n, \ldots]$, the ring of polynomials in a countable set of indeterminates over the field k. Prove that every ideal of R admits a countable number of generators.
- Find the kernel of the homomorphism ($\mathbf{K}$ is a field)

$$\varphi : \mathbf{K}[x, y, z] \longrightarrow \mathbf{K}[t],$$

defined by $\varphi(x) = t^4$, $\varphi(y) = t^5$ and $\varphi(z) = t^7$.
- $\varphi : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is a one-one group homomorphim, prove it is onto. (You may want to look at the action on the primary