Math 451: Abstract Algebra I

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Set 5: Rings

Fall 2009

Outline

Rings

A ring R is a set with two composition laws, called ‘addition’ and ‘multiplication’, say $+$ and $\times$: $\forall a, b \in R$ have compositions $a + b$ and $a \times b$. (The second composition is also written $a \cdot b$, or simply ab.)

- $(R, +)$ is an abelian group
- $(R, \times)$: multiplication is associative, and distributive over $+$, that is $\forall a, b, c \in R$,

$$(ab)c = a(bc), \quad ab = ba, \quad a(b + c) = ab + ac$$

- existence of identity: $\exists e \in R$ such that

$$\forall a \in R \quad e \times a = a \times e = a$$

- If $ab = ba$ for all $a, b \in R$, the ring is called commutative

There is a unique identity element e, usually we denote it by 1:

$$e = ee' = e'e = e'$$

Some terminology in studying a commutative ring

Let R be a commutative ring

- $u \in R$ is a unit if there is $v \in R$ such that $uv = 1$
- $a \in R$ is a zero divisor if there is $0 \neq b \in R$ such that $ab = 0$: $\overline{2} \cdot \overline{3} = 0$ in $\mathbb{Z}_6$.
- $a \in R$ is nilpotent if there is $n \in \mathbb{N}$ such that $a^n = 0$: $\overline{2}^3 = 0$ in $\mathbb{Z}_8$.
- R is an integral domain if 0 is the only zero divisor, in other words, if $a, b \in R$ are not zero, then $ab \neq 0$.

Field

A field $\mathbf{F}$ is a set with two composition laws, called 'addition' and 'multiplication', say $+$ and $\times$: $\forall a, b \in \mathbf{F}$ have compositions $a + b$ and $a \times b$. (The second composition is also written $a \cdot b$, or simply ab.)

- $(\mathbf{F}, +)$ is an abelian group
- $(\mathbf{F}, \times)$: multiplication is associative, commutative and distributive over $+$, that is $\forall a, b, c \in \mathbf{F}$,

$$(ab)c = a(bc), \quad ab = ba, \quad a(b + c) = ab + ac$$

- existence of identity $\exists e \in \mathbf{F}$ such that

$$\forall a \in \mathbf{F} \quad a \times e = a$$

- existence of inverses For every $a \neq 0$, there is $b \in \mathbf{F}$

$$a \times b = e.$$

There is a unique element e, usually we denote it by 1. For $a \neq 0$, the element b such that $ab = 1$ is unique; it is often denoted by $1/a$ or a^{-1}.

We can now define scalars: the elements of a field.

Fields are ubiquotous:

- $\mathbb{R}$: real numbers
- The integers $\mathbb{Z}$ is not a field (not all integers have inverses), but $\mathbb{Q}$, the rational numbers is a field.
- $\mathbb{C}$: complex numbers, $z = a + bi$, $i = \sqrt{-1}$, with compositions

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) \times (c + di) = (ac - bd) + (ad + bc)i$$

The arithmetic here requires a bit more care:

If $a + bi \neq 0$,

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$

Exercise: Number fields

Let **F** be the set of all real numbers of the form

$$z = a + b\sqrt{2}, \quad a, b \in \mathbb{Q}$$

prove that **F** is a field.

Query: How to prove a subset **F** of the field $\mathbb{R}$ is a field?
Suffices to check that **F** is closed under addition, product and inverse of nonzero element.
For instance, if $a + b\sqrt{2} \neq 0$,

$$\frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} \in \mathbf{F}$$

Another noteworthy example is $\mathbb{F}_2$, the set made up by two elements $\{0, 1\}$ (or (even, odd))with addition defined by the table

$+$	0	1
0	0	1
1	1	0

$1 + 1 = 0!$

and multiplication by

$\times$	0	1
0	0	0
1	0	1

Exercise 1: Prove that in any field $\mathbf{F}$ the rule minus times minus is plus holds, that is for any $a, b \in \mathbf{F}$,

$$-(-a) = a, \quad (-a)(-b) = ab.$$

Solution: The first assertion folllows from

$a + (-a) = (-a) + a = O$.

Because of the above, we must show that $(-a)(-b)$ is the negative of $-(ab)$. We first claim $(-a)b = -(ab)$. Note

$$(-a)b + ab = ((-a) + a)b = Ob = O.$$

$$(-a)(-b) - (ab) = (-a)(-b) + (-a)b = (-a)((-b) + b) = (-a)O = O.$$

A field is the mathematical structure of choice to do arithmetic. Given a field $\mathbf{F}$, fractions can defined as follows: If $a, b \in \mathbf{F}, \quad b \neq 0,$

$$\frac{a}{b} := ab^{-1}.$$

The usual calculus of fractions then follows, for instance

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Rings of Functions

Let $\mathbf{R}$ be a ring, S a nonempty set and $\mathcal{S}$ the set of all functions $\mathbf{f} : S \to \mathbf{R}$.

Proposition

We endow $\mathcal{R}$ with a ring structure by defining two operations: For all $s \in S$,

$$\begin{aligned} (\mathbf{f}+\mathbf{g})(s) &:= \mathbf{f}(s)+\mathbf{g}(s) \\ (\mathbf{f}\cdot\mathbf{g})(s) &:= \mathbf{f}(s)\cdot\mathbf{g}(s) \end{aligned}$$

Proof. It is clear that $\mathcal{R}$ inherits all the ring axioms from $\mathbf{R}$.

- If $1 \in \mathbf{R}$, the function $\mathbf{I}(s) = 1$ is the identity of $\mathcal{R}$.
- If $\mathbf{R}$ is commutative, $\mathcal{R}$ is also commutative.
- Major examples: If $S = \mathbb{R}$, and $\mathbf{f}$ are continuous.

Rings of Matrices

Let $R = M_n(\mathbb{R})$ be the set of all $n \times n$ matrices (n fixed), with the ordinary matrix addition and multiplication.

R is a ring, but it is not commutative if $n > 1$.

Subrings

Definition

A subring of a ring R is a subset S that satisfies:

1. S is a subgroup of R^+;
2. $1_R \in S$;
3. If $a, b \in S$, then $ab \in S$. (This product is the product of R.)

Example

$\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ is a tower of rings/subrings. Later, when we have more examples of rings, we will give various methods to construct subrings.

Outline

Rational Numbers

At the outset of our journey are the **natural** numbers

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

Its 'modern' construction [e.g. Peano's] is a paradigm of beauty. It is enlarged by the **integers**

$$\mathbb{N} \subset \mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

and the **rational** numbers

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} = \left\{\frac{m}{n}, \quad m, n \in \mathbb{Z}, n \neq 0\right\}$$

These sets exhibit different **structures**: of a monoid, of a ring and of a field, respectively.

Peano

The construction by Peano of the set $\mathbb{N}$ is grounded on two ingredients: The set $\mathbb{N}$ contains a particular element 1.

- **[Successor Function]** There is a function $s : \mathbb{N} \to \mathbb{N}$ that is injective, and for every $n \in \mathbb{N}$ $s(n) \neq 1$.
- **[Induction Axiom]** If the subset $S \subset \mathbb{N}$ has the properties

$$1 \in S \quad \& \quad \text{whenever} \quad n \in S \Rightarrow s(n) \in S$$

then $S = \mathbb{N}$

Given these definitions, we can define several operations/compositions and structures on $\mathbb{N}$:

- $a + b$:=?

$$\begin{aligned} a + 1 &:= s(a) \\ a + s(n) &:= s(a + n) \end{aligned}$$

- $a \times b$:=?

$$\begin{aligned} a \times 1 &:= a \\ a \times s(n) &:= a \times n + a \end{aligned}$$

Ordering

Out of these notions, addition and multiplication are defined in $\mathbb{N}$, and then extended to $\mathbb{Z}$ and $\mathbb{Q}$. An interesting consequence that arises is a notion of **order**: $\forall a, b \in \mathbb{Q}$, exactly one of the following holds

$$a < b, \quad a > b, \quad a = b$$

It has the properties: If $a > b$ then

$$\begin{aligned} \forall c \quad &\Rightarrow \quad a + c > b + c \\ \forall c > 0 \quad &\Rightarrow \quad ac > bc \end{aligned}$$

Significance: This leads to metric properties: lengths, angles, etc.

Peano and Mathematical Induction

http://upload.wikimedia.org/wikipedia/commons/3/3a/Giuseppe_Peano.jpg

Induction

The set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of **natural numbers** arises logically from the following construction of Peano.

$\mathbb{Z}$ and Peano's Axioms

- $\mathbb{N}$ contains a particular element 1.
- *Successor function:* There is an injective [one-one] function $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$, for each $n \in \mathbb{N}$, $\sigma(n) \neq 1$. [Another notation: $\sigma(n) = n'$]
- *Induction axiom:* Suppose that $S \subset \mathbb{N}$ satisfies
  1. $1 \in S$;
  2. if $n \in S$ then $\sigma(n) \in S$. Then $S = \mathbb{N}$.

The second axiom means 3 things [there are 5 axioms in all]: (1) every natural number has a successor; (2) no two natural numbers have the same successor; (3) 1 is not the successor of any natural number.

Defining Operations $+$ and $\times$

Operations

- *Addition:*

$$m + 1 = m', \quad m + n' = (m + n)'$$

- *Multiplication:*

$$m \cdot 1 = m, \quad m \cdot n' = m \cdot n + m$$

With these operations, $\mathbb{N}$ satisfies:

- Associativity properties: For all x, y and z in $\mathbb{N}$,

$$\begin{aligned} x + (y + z) &= (x + y) + z. \\ x(yz) &= (xy)z. \end{aligned}$$

- Commutativity properties: For all x and y in $\mathbb{N}$,

$$\begin{aligned} x + y &= y + x. \\ xy &= yx. \end{aligned}$$

- Distributivity properties: For all x, y and z in $\mathbb{N}$,

$$\begin{aligned} x(y + z) &= xy + xz. \\ (y + z)x &= yx + zx. \end{aligned}$$

- Order properties: For all x, y and z in $\mathbb{N}$, $x < y$ if there is $w \in \mathbb{N}$ such that $x + w = y$. Several properties arise: e.g. If $x < y$ then $\forall z \in \mathbb{N}$ $x + z < y + z$.

$\mathbb{N}$ can extended by 0 and 'negatives': $\mathbb{Z}$. Operations also. Then all the ordinary properties of addition and multiplication are verified:

Let us illustrate with:

Proof of the associative law of addition for $\mathbb{N}$:

$$(a+b)+n = a+(b+n) \quad \forall a,b,n \in \mathbb{N}$$

From the definitions check $n = 1$:

$$(a+b)+1 = (a+b)' = a+b' = a+(b+1)$$

Assume axiom holds for n and let us check for n' (*induction hypothesis*):

$$
\begin{aligned}
(a+b)+n' &= (a+b)+(n+1) \text{ (definition)} \\
&= ((a+b)+n)+1 \text{ (case } n=1\text{)} \\
&= (a+(b+n))+1 \text{ (ind. hypothesis)} \\
&= a+((b+n)+1) \text{ (case } n=1\text{)} \\
&= a+(b+(n+1)) \text{ (case } n=1\text{)} \\
&= a+(b+n') \text{ (definition)}
\end{aligned}
$$

Principle of Mathematical Induction

Let us state Peano's 5th Axiom again:

Definition (PMI)

If S is a subset of $\mathbb{N}$ and

1. $1 \in S$,
2. for all $n \in \mathbb{N}$, if $n \in S$, then $n + 1 \in S$,

then $S = \mathbb{N}$.

A set with Property (2) is called an **inductive set.** Examples, besides $\mathbb{N}$ are $\emptyset$, $S = \{x : x \in \mathbb{N}, x \geq 10\}$.$\mathbb{N}$ is the only inductive set containing 1: This is **PMI.**

The **PMI** is used to define mathematical objects and in proofs galore.

We are discussing the Principle of Mathematical Induction (**PMI** for short). It is a mechanism to study (i.e. prove) certain open sentences $P(n)$ that depend on $n \in \mathbb{N}$ when we seek to verify that it is true for all values.

The method is rooted in the following property of the natural numbers $\mathbb{N}$:

If S is a subset of $\mathbb{N}$ and

1. $1 \in S$,
2. for all $n \in \mathbb{N}$, if $n \in S$, then $n + 1 \in S$,

then $S = \mathbb{N}$.

Verifying $P(n)$

To verify whether $S = \{n : P(n)\}$ is equal to $\mathbb{N}$, we follow the template:

1. (Base step) $P(1)$ is true;
2. (Inductive step) If for some n, $P(n)$ is true then $P(n+1)$ is also true.

PMI guarantees that $S = \mathbb{N}$.

Sequences

Definition

A sequence is a function **f** whose domain is $\mathbb{N}$.

It can be represented as

$$\{\mathbf{f}(1), \mathbf{f}(2), \mathbf{f}(3), \ldots\}$$

$$\{\mathbf{f}(0), \mathbf{f}(1), \mathbf{f}(2), \mathbf{f}(3), \ldots\}$$

or

$$\{\mathbf{f}(n), \ldots, \quad n \geq n_0\}$$

We will first examine sequences of real numbers, $\mathbf{f} : \mathbb{N} \to \mathbb{R}$.

Sequences with values in a ring

Let **R** be a ring and $\mathcal{R}$ the set [actually a ring] of all sequences $\mathbf{f} : \mathbb{N} \to \mathbf{R}$. The operations are:

$$\begin{aligned}(a_1, a_2, a_3, \ldots) + (b_1, b_2, b_3, \ldots) &= (a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots)\\ (a_1, a_2, a_3, \ldots) \times (b_1, b_2, b_3, \ldots) &= (a_1 \cdot b_1, a_2 \cdot b_2, a_3 \cdot b_3, \ldots)\end{aligned}$$

This ring, sometimes denoted by $\mathbf{R}^{\mathbb{N}}$, is a direct product of copies of **R**.

Note that we have also the operation

$$r(a_1, a_2, a_3, \ldots) = (ra_1, ra_2, ra_3, \ldots)$$

Rings of Polynomials

Let us endow the set of sequences above with a different multiplication. For convenience we label the sequence as:

$$(a_0, a_1, a_2, a_3, \ldots), \quad a_i \in \mathbf{R}$$

$$(a_0, a_1, a_2, a_3, \ldots) \times (b_0, b_1, b_2, b_3, \ldots) \quad = \quad (c_0, c_1, c_2, c_3, \ldots)$$

$$\begin{aligned} c_0 &= a_0 b_0 \\ c_1 &= a_0 b_1 + a_1 b_0 \\ &\vdots \\ c_n &= \sum_{i+j=n} a_i b_j = a_0 b_n + \cdots + a_n b_0 \end{aligned}$$

Special Sequences

$$\begin{aligned} \mathbf{I} &= (1, 0, 0, 0, \ldots) \\ x &= (0, 1, 0, 0, \ldots) \end{aligned}$$

$$\begin{aligned} x &= (0, 1, 0, 0, \ldots) \\ x^2 &= (0, 0, 1, 0, \ldots) \\ x^3 &= (0, 0, 0, 1, \ldots) \end{aligned}$$

And most importantly

$$(r_0, r_1, r_2, r_3, \ldots) = r_0\mathbf{I} + r_1 x + r_2 x^2 + r_3 x^3 + \cdots$$

Polynomials

Proposition

With the composition above:

1. *The set of all sequences with values in* $\mathbf{R}$ *is a ring, denoted* $\mathbf{R}[[x]]$.
2. *The subset of all sequences* $\mathbf{f}$ *such that* $\mathbf{f}(n) = 0$ *for all* $n \gg 0$ *is also a ring, called the ring of polynomials of* $\mathbf{R}$*, and is denoted by* $\mathbf{R}[x]$.

As abelian groups:

1. $R[[x]] \simeq R^{\mathbb{N}}$
2. $R[x] \simeq R^{\oplus \mathbb{N}}$

Rings of Polynomials

Rings of polynomials in n indeterminates, $n > 1$, can be built on a similar construction: Let $\mathbf{R}$ be a ring

- Set $\mathbf{N} = \{0, 1, 2, \ldots\}$ and $\mathbf{M} = \mathbb{N}^n$ be the set $\alpha = (\alpha_1, \ldots, \alpha_n)$. We refer to $\deg \alpha = \alpha_1 + \cdots + \alpha_n$ as the total degree of α (referred to as a multi-index.
- Let $\mathcal{P}(n)$ the set of functions

$$\mathbf{f} : \mathbf{M} \to \mathbf{R}$$

- Addition in $\mathcal{P}(n)$ is defined by $(\mathbf{f} + \mathbf{g})(\alpha) = \mathbf{f}(\alpha) + \mathbf{g}(\alpha)$

- Multiplication in $\mathcal{P}(n)$ is defined by the convolution rule: Note that for each $\gamma \in \mathbf{M}$ there are only finitely many pairs (α, β) such that
$$\gamma = \alpha + \beta$$
- Define multiplication by
$$(\mathbf{f} \cdot \mathbf{g})(\gamma) = \sum_{\alpha+\beta=\gamma} \mathbf{f}(\alpha) \cdot \mathbf{g}(\beta)$$

Proposition

$\mathcal{P}(n)$ is a ring with these operations.

$\mathcal{P}(n)$

- The elements of $\mathcal{P}(n)$ are called polynomials in n indeterminates
- For a given multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, the function $\mathbf{f}$ such that $\mathbf{f}(\alpha) = 1$ and $\mathbf{f}(\beta) = 0$ for $\beta \neq \alpha$, is written

$$\mathbf{f} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

  or simply $\mathbf{x}^\alpha$. These functions are called monomials.
- Every $\mathbf{f}$ can be written as a finite sum

$$\mathbf{f} = \sum_\alpha c_\alpha \mathbf{x}^\alpha,$$

  where c_α is a constant function.
- Typically $\mathbf{f}$ is a sum of several terms. It is called a binomial, trinomial etc if .... If $\mathbf{f}$ has few terms it is called a fewnomial...

$R[x, y]$

The ring $\mathcal{P}(2)$ is noteworthy.

- The set of functions $\mathbf{f} : \mathbf{M} \to R$ such that $\mathbf{f}(m) = 0$ for almost all $m \in \mathbf{M}$ that we used to get $\mathcal{P}(2)$ can be realized another way.
- Let $\mathbf{F} : \mathbb{N} \to R[x]$ which is zero for almost all $r \in \mathbb{N}$. For each $r \in \mathbb{N}$, $\mathbf{F}(r) \in R[x]$ means that $\mathbf{F}(r) : \mathbb{N} \to R$ which is zero for almost all $s \in \mathbb{N}$, that is

$$\mathbf{F}(r)(s) = 0$$

  for almost all $(r, s) \in \mathbb{N}^2$. These are the functions used to define $\mathcal{P}(2)$.
- This shows that $\mathcal{P}(2) = R[x, y]$. More precisely, we must still verify that the two products coincide–which is easy.

Outline

Homomorphisms

Definition

A *homomorphism* $\varphi : R \to R'$ from one ring to another is a map which is compatible with the laws of composiiton and which carries 1 to 1, that is, a map such that

$$\varphi(a+b) = \varphi(a) + \varphi(b), \quad \varphi(ab) = \varphi(a)\varphi(b), \quad \varphi(1_R) = 1_{R'},$$

for all $a, b \in R$. An *isomorphism* of rings is bijective homomorphism. If there is an isomorphism $R \to R'$, the two rings are said to be *isomorphic*.

Example

Let $R = \mathbb{C}$. complex conjugation, $a + bi \to a - bi$ is an isomorphism of $\mathbb{C}$.

Matrix Rings

Let $R = M_n(\mathbb{R})$ be the ring of $n \times n$ real matrices, and let $\mathbf{A}$ be an invertible matrix. Define

$$\varphi : R \to R, \quad \varphi(\mathbf{X}) = \mathbf{AXA}^{-1}$$

$$\begin{aligned}
\varphi(\mathbf{I}) &= \mathbf{AIA}^{-1} = \mathbf{I} \\
\varphi(\mathbf{X}+\mathbf{Y}) &= \mathbf{A}(\mathbf{X}+\mathbf{Y})\mathbf{A}^{-1} = \mathbf{AXA}^{-1} + \mathbf{AYA}^{-1} = \varphi(\mathbf{X}) + \varphi(\mathbf{Y}) \\
\varphi(\mathbf{XY}) &= \mathbf{A}(\mathbf{XY})\mathbf{A}^{-1} = \mathbf{AXA}^{-1}\mathbf{AYA}^{-1} = \varphi(\mathbf{X})\varphi(\mathbf{Y})
\end{aligned}$$

Thus conjugation by $\mathbf{A}$ is an isomorphism of R.

The Substitution Principle

Proposition

Let $\varphi : R \to R'$ be a ring homomorphism.

(a) *Given an element $\alpha \in R'$, there is a unique homomorphism $\Phi : R[x] \to R'$ which agrees with the map φ on constant polynomials and which sends $x \rightsquigarrow \alpha$.*

(b) *More generally, given elements $\alpha_1, \ldots, \alpha_n \in R'$, there is a unique homomorphism $\Phi : R[x_1, \ldots, x_n] \to R'$ from the polynomial ring in n variables to R', which agrees with φ on constant polynomials and which sends $x_\nu \rightsquigarrow \alpha_\nu$, for $\nu = 1, \ldots, n$.*

Proof. If Φ exists,

$$\Phi(a_n x^n + \cdots + a_0) = \Phi(a_n)\Phi(x^n) + \cdots + \Phi(a_0) = \varphi(a_n)\alpha^n + \cdots + \varphi(a_0)$$

Thus Φ is uniquely defined by φ and $\Phi(x) = \alpha$.

To prove the existence, we define Φ by the formula above, and check that

$$\Phi(f(x) + g(x)) = \Phi(f(x)) + \Phi(g(x)), \quad \Phi(f(x)g(x)) = \Phi(f(x))\Phi(g(x))$$

Having done this so many times in Calculus, we believe.

Corollary

Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ denote sets of variables. There is a unique isomorphism $R[x, y] \to R[x][y]$ which is the identity on R and which sends the variables to themselves.

Proposition

Let $\mathcal{R}$ denote the ring of continuous real-valued functions on $\mathbb{R}^n$. The map $\varphi : \mathbb{R}[x_1, \dots, x_n] \to \mathcal{R}$ sending a polynomial to its associated polynomial function is an injective homomorphism.

Proposition

There is exactly one homomorphism

$$\varphi : \mathbb{Z} \to R$$

from the ring of integers to an arbitary ring R. It is the map defined by $\varphi(n) = 1_R + \cdots + 1_R$ *(n times) if* $n > 0$*, and* $\varphi(-n) = -\varphi(n)$.

Ideals

The property of the kernel of a ring homomorphism – that it is closed under multiplication by arbitrary elements of the ring – is abstracted in the concept of an *ideal*.

Definition

An *ideal* I of a ring R is a subset of R with these properties :

(i) I is a subgroup of R^+ ;

(ii) If $a \in I$ and $r \in R$, then $ra \in I$.

Example

Let R be a commutative ring and $x \in R$. The set of multiples of x, $Rx = \{ra; r \in R\}$, is an ideal. It is called a principal, or one-generated ideal.

Example

If R is a ring and $S = \{a_1, \ldots, a_n\}$ is a set of elements of R, the set of all combinations

$$r_1 a_1 + \cdots + r_n a_n, \quad r_i \in R$$

is an ideal. It is called the ideal generated, or spanned, by S.

If R is not commutative, there are other notions of ideals:

- I is a left ideal if I is a subgroup of R^+, and for every $a \in I$, $r \in R$, $ra \in I$.
- I is a right ideal if I is a subgroup of R^+, and for every $a \in I$, $r \in R$, $ar \in I$.
- I is a two-sided ideal if I is a subgroup of R^+, and for every $a \in I$, $r, s \in R$, $ras \in I$.

Ideals of Fields

Proposition

(a) *Let F be a field. The only ideals of F are the zero ideal and the unit ideal.*
(b) *Conversely, if a ring R has exactly two ideals, then R is a field.*

Proof.

(a) Let I be a nonzero ideal. If $0 \neq a \in I$, since F is a field, $a^{-1} \in F \Rightarrow 1 = a^{-1}a \in I$. Thus $I = R$.
(b) If $0 \neq a$, Ra is a nonzero ideal, so $Ra = R$, which means there $r \in R$ such that $ra = 1$.

Corollary

Let F be a field and let R' be a nonzero ring. Every homomorphism $\varphi : F \to R'$ is injective.

Proof.

Let I be $\ker \varphi$. Since $\varphi(1_F) = 1_R$, φ is not the null mapping, and thus its kernel $\neq F$. But the only other ideal of F is (0). $\square$

The ideals of $\mathbb{Z}$

Proposition

Every ideal in the ring $\mathbb{Z}$ of integers is a principal ideal.

Proof.

Every ideal I of $\mathbb{Z}$ is a subgroup of $\mathbb{Z}^+$. But we have already seen that the subgroups of $\mathbb{Z}$ are cyclic, that is $I = \mathbb{Z}a$, for some integer a. Note $\mathbb{Z}a$ is also closed multiplication by elements of $\mathbb{Z}$. $\square$

Long Division Algorithm

Proposition

Let R be a ring and let f, g be polynomials in $R[x]$. Assume that the leading coefficient of f is a unit in R. (This is true, for instance, if f is a monic polynomial.) Then there are polynomials $q, r \in R[x]$ such that

$$g(x) = f(x)q(x) + r(x),$$

and such that the degree of the remainder r is less than the degree of f or else $r = 0$.

Proof. We may assume that $\deg g(x) \geq \deg f(x)$, as otherwise there is nothing to prove. We are going to induction on $\deg g(x)$ assuming that the assertion is true for polynomials of lesser degree.

$$
\begin{aligned}
g(x) &= b_m x^m + \text{lower degree} \\
f(x) &= a_n x^n + \text{lower degree}
\end{aligned}
$$

By assumption $u = a_n$ is invertible. Note that

$$
h(x) = g(x) - b_m u^{-1} x^{m-n} f(x)
$$

satisfies $\deg h(x) < \deg g(x)$.

By induction we have

$$h(x) = f(x)q'(x) + r(x), \quad \deg r(x) < \deg f(x)$$

and therefore

$$g(x) = f(x)(q'(x) + b_m u^{-1} x^{m-n}) + r(x), \quad \deg r(x) < \deg f(x)$$

Corollary

Let $g(x)$ be a monic polynomial in $R[x]$, and let α be an element of R such that $g(\alpha) = 0$. Then $x - \alpha$ divides g in $R[x]$.

Euclidean Ring

Proposition

Let F be a field. Every ideal in the ring $F[x]$ of polynomials in a single variable x is a principal ideal.

Proof. Let I be an ideal of $F[x]$. If $I = (0)$ there is nothing to prove.

If $I \neq (0)$, let $f(x)$ be a nonzero polynomial of least degree. We claim that every element $g(x)$ of I is a multiple of $f(x)$. If $g(x) = 0$, there is nothing to do, so assume $g(x) \neq 0$. Since the leading coefficient of $f(x)$ is invertible, by the Long Division Algorithm there are polynomials $q(x)$ and $r(x)$ such that

$$g(x) = f(x)q(x) + r(x), \quad \deg r(x) < \deg f(x)$$

But $r(x) = g(x) - f(x)q(x)$ is an element of I, so must be 0 by the choice of $f(x)$.

Corollary

Let F be a field, and let f and g be polynomials which are not both zero. There is a unique monic polynomial $d(x)$ called the ***greatest common divisor*** *of f and g, with the following properties:*

1. *d generates the ideal (f, g) of $F[x]$ generated by the two polynomials f, g.*
2. *d divides f and g.*
3. *If h is any divisor of f and g, then h divides d.*
4. *There are polynomials $p, q \in F[x]$ such that $d = pf + qg$.*

Recall: The ideal (f, g) is made up of all combinations

$$a(x)f(x) + b(x)g(x)$$

Radical of an Ideal

Definition

Let I be an ideal of the commutative ring R. The radical of I is the set

$$\sqrt{I} = \{x \in R : x^n \in I \quad \text{some } n = n(x)\}.$$

Proposition

$\sqrt{I}$ is an ideal.

Proof.

If $a, b \in \sqrt{I}$, $a^m \in I$, $b^n \in I$, then

$$(a+b)^{m+n-1} = \sum_{i+j=m+n-1} \binom{m+n-1}{i} a^i b^j \in I,$$

since $i \geq m$ or $j \geq n$.

Principal Ideal Ring

Definition

A ring $\mathbf{R}$ is a principal ideal ring if every ideal I is generated by one element, $I = \{ra : r \in \mathbf{R}\}$.

- $\mathbb{Z}$ and $\mathbf{F}[x]$ where $\mathbf{F}$ is a field are principal ideal rings.
- $\mathbf{R} = \mathbf{F}[x, y]$ is not: The ideal I generated by x, y cannot be generated by 1 element.

Idempotents

Let $\mathbf{R} = \mathbb{Z}_6$ and consider the element $z = \overline{3}$. Note $z^2 = \overline{9} = \overline{3} = z$. These elements are called:

Definition

The element $e \in \mathbf{R}$ is called idempotent if $e^2 = e$.

Definition

$\mathbf{R}$ is a Boolean ring if $z^2 = z$ for all $z \in \mathbf{R}$.

Proposition

If $\mathbf{R}$ is a Boolean ring, then

1. $2z = 0$ *for* $z \in \mathbf{R}$*;*
2. *If* $a, b \in \mathbf{R}$*, then* a, b *are multiples of* $a + b - ab$*.*

Class proof

Example: Boolean ring

Example

For a non-empty set $\mathbf{X}$ let $\mathbf{R}$ the set of all functions $\mathbf{f} : \mathbf{X} \to \mathbb{Z}_2$.

- $(\mathbf{f} + \mathbf{g})(s) = \mathbf{f}(s) + \mathbf{g}(s)$, and
- $(\mathbf{f} \cdot \mathbf{g})(s) = \mathbf{f}(s) \cdot \mathbf{g}(s)$, define a ring structure on $\mathbf{R}$.
- $\mathbf{f}^2(s) = \mathbf{f}(s) \cdot \mathbf{f}(s) = \mathbf{f}(s)$, so $\mathbf{R}$ is Boolean.

Outline

Quotient rings

The most effective method to build new rings is the following:

Let I be a two-sided proper ideal of the **R** and denote by $\overline{\mathbf{R}} = \mathbf{R}/I$ the corresponding cosets $\{a + I : a \in R\}$. It defines on $\overline{R}$ an abelian group structure called the quotient ring R/I:

$$(a + I) + (b + I) \quad = \quad (a + b) + I$$

We claim that this operation and

$$(a+I) \times (b+I) \quad = \quad ab+I$$

defines a ring structure. Let us verify that if $a'+I=a+I$ and $b+I=b'+I$, then $ab+I=a'b'+I$: Since $a'=a+r$, $b'=b+s$, with $r,s \in I$

$$a'b'=(a+r)(b+s)=ab+(rb+sa+rs)$$

and thus $a'b'$ and ab live in the same coset.

The axioms of associativity and distributivity are easily verified.

This is a source to many new rings

$\mathbb{Z}_n$

Example

Let $R = \mathbb{Z}$ and $I = \mathbb{Z}n$. Then R/I is the ring of integers modulo n.

Examples: Quotient rings

$$
\begin{array}{rcl}
(2) \subset \mathbb{Z} & \Rightarrow & \mathbb{Z}_2 = \mathbb{Z}/(2) \\
(x^2 + x + 1) \subset \mathbb{Z}_2[x] & \Rightarrow & \mathbb{Z}_2[x]/(x^2 + x + 1) = \mathbf{F}_4 \\
(x^2 + 1) \subset \mathbb{R}[x] & \Rightarrow & \mathbb{C} = \mathbb{R}[x]/(x^2 + 1) \\
(1 + 3i) \subset \mathbb{Z}[i] & \Rightarrow & \mathbb{Z}_{10} = R = \mathbb{Z}[i]/(1 + 3i)
\end{array}
$$

Will check out some of these soon.

Theorem

Let I be an ideal of a ring R.

(a) *There is a unique ring structure on the set of cosets $\overline{R} = R/I$ such that the canonical map $\pi : R \to \overline{R}$ sending $a \rightsquigarrow \overline{a} = a + I$ is a homomorphism.*

(b) *The kernel of π is I.*

Mapping property of quotient rings

Proposition

Let $f : R \to R'$ be a ring homomorphism with kernel I and let J be an ideal which is contained in I. Denote the residue ring R/J by $\overline{R}$.

(a) *There is a unique homomorphism $\overline{f} : \overline{R} \to R'$ such that $\overline{f}\pi = f$:*

$$\begin{tikzcd} R \arrow[rr, "f"] \arrow[dr, "\pi"'] & & R' \arrow[dl, "\overline{f}"] \\ & \overline{R} = R/J & \end{tikzcd}$$

(b) *(First Isomorphism Theorem) If $J = I$, then $\overline{f}$ maps $\overline{R}$ isomorphically to the image of f.*

Correspondence Theorem

Proposition

Let $\overline{R} = R/J$, and let π denote the canonical map $R \to \overline{R}$.

(a) *There is a bijective correspondence between the set of ideals of R which contain J and the set of all ideals of $\overline{R}$, given by*

$$I \rightsquigarrow \pi(I), \quad \text{and} \quad \pi^{-1}(I) \rightsquigarrow \bar{I}.$$

(b) *If $I \subset R$ corresponds to $\bar{I} \subset \overline{R}$, then R/I and $\overline{R}/\bar{I}$ are isomorphic rings.*

$\mathbb{Z}[i]/(1+3i) \simeq \mathbb{Z}/(10)$

Proposition

The ring $\mathbb{Z}[i]/(1+3i)$ is isomorphic to the ring $\mathbb{Z}/10\mathbb{Z}$ of integers modulo 10.

Proof. Consider the homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}[i] \to R = \mathbb{Z}[i]/(1+3i)$ induced by the embedding of $\mathbb{Z}$ in $\mathbb{Z}[i]$.We claim that φ is a surjection of kernel $10\mathbb{Z}$:

$$1+3i \equiv 0 \Rightarrow i(1+3i) \equiv 0 \Rightarrow i-3 \equiv 0 \Rightarrow i \equiv 3$$
$$a+bi \equiv a+3b \Rightarrow \varphi \text{ is surjection}$$

For n in kernel of φ,

$$\begin{aligned} n &= z(1+3i) = (a+bi)(1+31) \\ &= (a-3b) + \underbrace{(3a+b)i}_{=0} \quad \Rightarrow b = -3a \quad \Rightarrow n = 10a \end{aligned}$$

The Circle Ring

Proposition

$\mathbb{R}[x, y]/(x^2 + y^2 - 1) \simeq \mathbb{R}[\cos t, \sin t]$.

The ring $R = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$: known as the circle ring

- Consider the natural homomorphism

  $$\mathbf{f} : \mathbb{R}[x, y] \longrightarrow \mathbb{R}[\cos t, \sin t], \quad \mathbf{f}(x) = \cos t, \mathbf{f}(y) = \sin t$$

  $\mathbb{R}[\cos t, \sin t]$ is the ring of trigonometric polynomials.
- $\mathbf{f}(x^2 + y^2 - 1) = 0$ so there is an induced surjection

  $$\varphi : \mathbb{R}[x, y]/(x^2 + y^2 - 1) \to \mathbb{R}[\cos t, \sin t]$$

- φ is an isomorphism because: (i) $\mathbb{R}[\cos t, \sin t]$ is an infinite dimensional $\mathbb{R}$-vector space (why?); for any ideal L larger than $(x^2 + y^2 - 1)$, $\mathbb{R}[x, y]/L$ is a finite dimensional $\mathbb{R}$-vector space (why?).

$\mathbb{R}[x, y]/(xy)$

Proposition

The ring $\mathbb{R}[x, y]/(xy)$ is isomorphic to the subring of the product ring $\mathbb{R}[x] \times \mathbb{R}[y]$ consisting of the pairs $(p(x), q(y))$ such that $p(0) = q(0)$.

Proof. Let us sketch the proof, leaving the details to reader:

$$\mathbb{R}[x, y]/(xy) \simeq \{(p(x), q(y)) : p(0) = q(0))\}$$

Consider the homomorphism

$$\varphi : \mathbb{R}[x, y]/(xy) \to \mathbb{R}[x, y]/(y) \times \mathbb{R}[x, y]/(x)$$
$$\varphi(a + (xy)) = (a + (y), a + (x))$$

Check that φ is one-one and determine its image.

Outline

Integral Domains and Rings of Fractions

Definition

An integral domain **R** is a nonzero ring having no zero divisors. That is, if $ab = 0$, then $a = 0$ or $b = 0$.

Example

Any subring **R** of a field **F** is an integral doimain.

Properties

Proposition

1. *If $\mathbf{R}$ is an integral domain then the polynomial ring $\mathbf{R}[x]$ is also an integral domain.*
2. *An integral domain with finitely many elements is a field.*

Proof. Class proof.

Embedding

Theorem

Let $\mathbf{R}$ *be an integral domain. There exists an embedding of* $\mathbf{R}$ *into a field, meaning an injective homomorphism* $\varphi : \mathbf{R} \to \mathbf{F}$*, where* $\mathbf{F}$ *is a field.*

Proof. We are going to build fractions with the elements of $\mathbf{R}$.

- Let S be the set of all ordered pairs (a, b), $a, b \in \mathbf{R}$, $b \neq 0$. Define the following relation on S:

$$(a, b) \simeq (c, d) \Leftrightarrow ad = bc$$

- Claim: $\simeq$ is an equivalence relation.
  reflexive: $(a, b) \simeq (a, b)$ clear
  symmetric: $(a, b) \simeq (c, d) \Leftrightarrow (c, d) \simeq (a, b)$
  transitive: $(a, b) \simeq (c, d) \simeq (e, f) \Rightarrow$

$$ad = bc, cf = de \Rightarrow adf = bcfbcf = bde \Rightarrow af = be$$

Field of fractions

Let **F** be the set of equivalence classes. We denote the equivalence of (a, b) by a/b.

- We define a field structure on **F** by the rules:

$$(a/b)(c/d) = ac/bd, \quad a/b + c/d = \frac{ad + bc}{cd}$$

- It must be verified that these definitions do not depend on the representative taken, for instance, if $a/b = a'/b'$, then $(a/b)(c/d) = (a/b')(c/d)$. We believe!
- With these rules, **F** is a field. For instance, if a/b is such that $a \neq 0$, then $(a/b)^{-1} = (b/a)$.
- Finally, define $\varphi : \mathbf{R} \to \mathbf{F}$ by the rule $\varphi(a) = a/1$. It is easy to verify that φ is an injective homomorphism.

Examples

- What are fractions in $\mathbb{Q}$?
- $\mathbb{Z} \to \mathbb{Q}$
- $\mathbb{R}[x] \to \mathbb{R}(x)$: $\frac{p(x)}{q(x)}$

Class Exercise

Proposition

Let $\mathbf{R}$ *be an integral domain, with field of fractions* $\mathbf{F}$*, and let* $\varphi : \mathbf{R} \to \mathbf{K}$ *be an injective homomorphism of* $\mathbf{R}$ *to the field* $\mathbf{K}$*. Then the rule*

$$\Phi(a/b) = \varphi(a)\varphi(b)^{-1}$$

defines the unique extension of φ *to a homomorphism* $\Phi : \mathbf{F} \to \mathbf{K}$*.*

Outline

Homework #10

1. If $\mathbf{R}$ is a Boolean ring, prove that every finitely generated ideal I is generated by one element.
2. If $\mathbf{R}$ is a finite Boolean ring, $|\mathbf{R}| = 2^n$, for some integer n. *Hint:* For each $e \in \mathbf{R}$, show that $\mathbf{R} = \mathbf{R}e \times \mathbf{R}(1 - e)$. Note that $\mathbf{R}e$ is a Boolean ring with identity e.
3. Prove that if $\mathbf{R}$ is a finite integral domain then:
   - $\mathbf{R}$ is a field;
   - $\mathbf{R}$ contains a subfield $\mathbb{Z}_p$, for some prime p;
   - $|\mathbf{R}| = p^n$
4. Let $\mathbf{R}_1, \mathbf{R}_2$ be two rings. Describe the ideals of $\mathbf{R}_1 \times \mathbf{R}_2$ in terms of the ideals of $\mathbf{R}_1$ and $\mathbf{R}_2$.

Outline

Maximal Ideals

Definition

An ideal M is maximal if $M \neq \mathbf{R}$ but M is not contained in any ideals other than M or $\mathbf{R}$.

Proposition

1. *An ideal M of a ring $\mathbf{R}$ is maximal iff $\overline{\mathbf{R}} = \mathbf{R}/M$ is a field.*
2. *The zero ideal of $\mathbf{R}$ is maximal iff $\mathbf{R}$ is a field.*

Examples

Proposition

The maximal ideals of $\mathbb{Z}$ *are the ideals* (p), *where* p *is a nonzero prime number.*

Proposition

The maximal ideals of the ring $\mathbb{C}[x]$ *of complex polynomials are the ideals* $(\mathbf{f}(x))$ *where* $\mathbf{f}(x) = x - \mathbf{c}$, *were* $\mathbf{c} \in \mathbb{C}$.

Proof.

Let M be a maximal ideal; clearly $M \neq (0)$. We know that $\mathbb{C}[x]$ is a principal ideal ring that every ideal is generated by a single polynomial, $M = (\mathbf{f}(x))$. If $\deg(\mathbf{f}(x)) > 1$, and $\mathbf{c}$ is a root, $\mathbf{f}(x) = (x - \mathbf{c})\mathbf{g}(x)$.
It follows that $M \subset (x - \mathbf{c})$. Since M is maximal, $M = (x - \mathbf{c})$. □

Example

Let $\mathbf{R} = \mathbb{R}[x, y]$, the ring of polynomials in two indeterminates over $\mathbb{R}$. Define a homomorphism

$$\varphi : \mathbf{R} \to \mathbb{C}, \quad x \to i, y \to i$$

Let M be the kernel. Note that $x - y \to 0$ and $x^2 + 1 \to 0$, and $r \to r$ if $r \in \mathbb{R}$

Note that φ is surjective, so $\mathbf{R}/M \simeq \mathbb{C}$. Therefore M is maximal.
Claim: $M = (x - y, x^2 + 1)$.

Example from Analysis

Let **R** be the ring of real continuous functions on the interval $\mathbf{I} = [0, 1]$. For each $\mathbf{a} \in \mathbf{I}$, the evaluation $\mathbf{f}(x) \to \mathbf{f}(\mathbf{a})$ defines a surjective homomorphism

$$\varphi : \mathbf{R} \to \mathbb{R}$$

The kernel is $M = \{\mathbf{f}(x) : \mathbf{f}(\mathbf{a}) = 0\}$. Since $\mathbf{R}/M \simeq \mathbb{R}$, M is a maximal ideal.

Now we are going to use hard analysis to prove the converse. We are going to use the fact that the interval **I** is compact: any covering

$$\mathbf{I} \subset \bigcup (a_i, b_i)$$

has a finite subcover.

Example

Theorem

For maximal ideal M of the ring $\mathbf{R}$ of continous functions on $\mathbf{I} = [0, 1]$ there is $\mathbf{a} \in \mathbf{I}$ such that $M = \{\mathbf{f}(x) : \mathbf{f}(\mathbf{a}) = 0\}$.

Proof. Deny it. This means that for each $\mathbf{a} \in \mathbf{I}$ there is $\mathbf{f}(x) \in M$ such that $\mathbf{f}(\mathbf{a}) \neq 0$. Since $\mathbf{f}(x)$ is continuous with $\mathbf{f}(\mathbf{a}) \neq 0$, in a small interval (c, d) about $\mathbf{a}$, $\mathbf{f}(x) \neq 0$ for $x \in (c, d)$.

This gives rise to a covering

$$\mathbf{I} \subset \bigcup_{i=1}^{n} (c_i, d_i)$$

by such intervals (actually a finite collection) and functions $\mathbf{f}_i(x) \in M$ nonvanishing on (c_i, d_i).

Consider the function

$$\mathbf{f}(x) = \sum_{i=1}^{n} \mathbf{f}_i(x)^2$$

$\mathbf{f} \in M$ and does not vanish anywhere in $\mathbf{I}$. This implies that $1/\mathbf{f}(x) \in \mathbf{R}$, and therefore $1 = (1/\mathbf{f}(x))\mathbf{f}(x) \in M$, a contradiction.

Prime Ideals

Definition

Let R be a commutative ring. An ideal P of R is prime if $P \neq R$ and whenever $a \cdot b \in P$ then $a \in P$ or $b \in P$.

Equivalently:

- R/P is an integral domain
- If I and J are ideals and $I \cdot J \subset P$ then $I \subset P$ or $J \subset P$

Prime ideals and homomorphisms

$$
\begin{tikzcd}
 & R \arrow[dl] \arrow[dr] & \\
\text{prime ideals of } R & & \text{morphisms } \varphi : R \to S
\end{tikzcd}
$$

Prime ideals arise in issues of factorization and very importantly:

Proposition

Let $\varphi : R \to S$ be a homomorphism of commutative ring. If S is an integral domain, then $P = \ker(\varphi)$ is a prime ideal. More generally, if S is an arbitrary commutative ring and Q is a prime ideal, then $P = \varphi^{-1}(Q)$ is a prime ideal of R.

Proof. Inspect the diagram

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow & & \downarrow \\ R/P & \hookrightarrow & S/Q \end{array}$$

Exercise

Consider the homomorphism of rings

$$\begin{aligned} \varphi : k[x, y, z] &\rightarrow k[t] \\ x &\rightarrow t^3 \\ y &\rightarrow t^4 \\ z &\rightarrow t^5 \end{aligned}$$

Let P be the kernel of this morphism. Note that $x^3 - yz$, $y^2 - xz$ and $z^2 - x^2y$ lie in P.

Task: Prove that P is generated by these 3 polynomials.

Task: Describe the prime ideals of the ring

$$R = \mathbb{C}[x, y]/(y^2 - x(x-1)(x-2)).$$

Significance: Prime and Maximal Ideals

These ideals give rise to new interesting rings:

- Prime ideals are significant because: R/P is a domain
- Maximal ideals are significant because: R/P is a field
- In particular maximal ideals are prime

Prospecting for prime ideals

Let $\mathbf{R}$ be a ring. Given a proper ideal I, how to add something to it an still get a proper ideal?

- If $a \notin I$, add a to I, which means form all $ra + s$, $r \in \mathbf{R}$, $s \in I$.
- This ideal, (a, I), may be improper, $(a, I) = \mathbf{R}$, that is we have a term $ra + s = 1$. Hard to predict.

A theorem for believers

Theorem

Let $\mathbf{R}$ *be a ring. Every ideal* I *of* $\mathbf{R}$ *which is not the unit ideal is contained in a maximal ideal.*

How we are going to do this?

Proof. [?]

- Let I be an ideal. If I is maximal, we are done.
- If not, there is a larger proper ideal $I \subset I_1$. If I_1 is maximal,...
- In this manner we get a chain of proper ideals $I \subset I_1 \subset \cdots \subset I_n \subset$
- Observation: $\bigcup_n I_n$ is a proper ideal–obviously closed under addition, multiplication and 1 is not in the union. What else can we do?

Zorn Lemma

This is an extra axiom which when added to the more common common axioms of mathematics asserts:

Any subset **Y** of a partially ordered set **X** such the chains of elements of **Y** have a supremum has maximal elements

Outline

Emmy Noether (1882-1935)

http://upload.wikimedia.org/wikipedia/commons/e/e5/Noether.jpg

Noetherian Rings

Definition

R is a Noetherian if every ascending chain of ideals is stationary, that is $A_n = A_{n+1} = \dots$ from a certain point on.

Definition

The ring **R** has the Maximal Condition if every subset S of the X (set of ideals ordered by inclusion) contains a maximum submodule

Example

Let $\mathbf{R} = \mathbb{Z}$: a chain of ideals

$$(a_1) \subset (a_2) \subset \cdots \subset (a_n)$$

means a sequenc of integers $a_2 | a_1$, $a_3 | a_2$, ..., each dividing the preceding, in a process that must stop.
The same argument applies of the ring $\mathbf{R} = \mathbf{F}[x]$, where $\mathbf{F}$ is a field.

Proposition

R *is a Noetherian ring iff* **R** *has the Maximal Condition.*

Proof. Let S be a set of ideals of **R**. If S contains no maximal element, we can build an ascending chain

$$A_1 \subsetneqq A_2 \subsetneqq \cdots \subsetneqq A_n \subsetneqq \cdots$$

contradicting the assumption that **R** is Noetherian. The converse has a similar proof.

Proposition

R *is Noetherian iff every ideal is finitely generated.*

Proof. Suppose **R** is Noetherian. Let us deny. Let A be an ideal of **R** and assume it is not finitely generated. It would permit the construction of an increasing sequence of submodules of A,

$$(a_1) \subset (a_1, a_2) \subset \cdots \subset (a_1, a_2, \ldots, a_n) \subset \cdots,$$

$a_{n+1} \in A \setminus (a_1, \ldots, a_n)$.
Conversely if $A_1 \subseteq A_2 \subseteq \cdots$ is an increasing sequence of ideals, let $B = \cup_{i \geq 1} A_i$ is an ideal and therefore $B = (b_1, \ldots, b_m)$. Each $b_i \in A_{n_i}$ for some n_i. If $n = \max\{n_i\}$, $A_n = A_{n+1} = \cdots$.

Hilbert Basis Theorem

Theorem (HBT)

If R is Noetherian then $R[x]$ is Noetherian.

1. If R is Noetherian and $x_1, \ldots, x_n$ is a set of independent indeterminates, then $R[x_1, \ldots, x_n]$ is Noetherian.
2. $\mathbb{Z}[x_1, \ldots, x_n]$ is Noetherian.
3. If k is a field, then $k[x_1, \ldots, x_n]$ is Noetherian.

Proof of the HBT

Suppose the $R[x]$-ideal I is not finitely generated. Let $0 \neq f_1(x) \in I$ be a polynomial of smallest degree,

$$f_1(x) = a_1 x^{d_1} + \text{lower degree terms.}$$

Since $I \neq (f_1(x))$, let $f_2(x) \in I \setminus (f_1(x))$ of least degree. In this manner we get a sequence of polynomials

$$f_i(x) = a_i x^{d_i} + \text{lower degree terms,}$$

$$f_i(x) \in I \setminus (f_1(x), \ldots, f_{i-1}(x)), \quad d_1 \leq d_2 \leq d_3 \leq \cdots$$

Set $J = (a_1, a_2, \ldots,) = (a_1, a_2, \ldots, a_m) \subseteq R$

Let $f_{m+1}(x) = a_{m+1}x^{d_{m+1}} + \text{lower degree terms}$. Then

$$a_{m+1} = \sum_{i=1}^{m} s_i a_i, \quad s_i \in R.$$

Consider

$$\mathbf{g}(x) = f_{m+1} - \sum_{i=1}^{m} s_i x^{d_{m+1}-d_i} f_i(x).$$

$\mathbf{g}(x) \in I \setminus (f_1(x), \ldots, f_m(x))$, but $\deg \mathbf{g}(x) < \deg f_{m+1}(x)$, which is a contradiction.

Examples

- $\mathbb{Z}$ is Noetherian, so is $\mathbf{R} = \mathbb{Z}[x_1, \ldots, x_n]$
- A field $\mathbf{F}$ is Noetherian, so is $\mathbf{R} = \mathbf{F}[x_1, \ldots, x_n]$
- $\mathbf{A}$ is Noetherian, so is $\mathbf{R} = \mathbf{A}[x_1, \ldots, x_n]/I$

Power Series Rings

Another construction over a ring R is that of the power series ring $R[[x]]$:

$$\mathbf{f}(x) = \sum_{n\geq 0} a_n x^n, \quad \mathbf{g}(x) = \sum_{n\geq 0} b_n x^n$$

with addition component wise and multiplication the Cauchy operation

$$\begin{aligned} \mathbf{f}(x)\mathbf{g}(x) = \mathbf{h}(x) &= \mathbf{h}(x) = \sum_{n\geq 0} c_n x^n \\ c_n &= \sum_{i+j=n} a_i b_{n-i} \end{aligned}$$

Theorem

If R is Noetherian then $R[[x]]$ is Noetherian.

Proposition

A commutative ring R is Noetherian iff every prime ideal is finitely generated.

Proof. If R is not Noetherian, there is an ideal I maximum with the property of not being finitely generated (Zorn's Lemma). We assume I is not prime, that is there exist $a, b \notin I$ such that $ab \in I$.

The ideals (I, a) and $I : a$ are both larger than I and therefore are finitely generated:

$$\begin{aligned}(I : a) &= (a_1, \ldots, a_n) \\ (I, a) &= (b_1, \ldots, b_m, a), \quad b_i \in I\end{aligned}$$

Claim: $I = (b_1, \ldots, b_m, aa_1, \ldots, aa_n)$

If $c \in I$,

$$c = \sum_{i=1}^{m} c_i b_i + ra, \quad r \in I : a$$

$R[[x]]$ is Noetherian

Proof. Let P be a prime ideal of $R[[x]]$. Set $\mathfrak{p} = P \cap R$. $\mathfrak{p}$ is a prime ideal of R and therefore it is finitely generated.

Denote by $\mathfrak{p}[[x]] = \mathfrak{p}R[[x]]$ the ideal of $R[[x]]$ generated by the elements of $\mathfrak{p}$. It consists of the power series with coefficients in $\mathfrak{p}$ and $R[[x]]/\mathfrak{p}[[x]]$ is the power series ring $R/\mathfrak{p}[[x]]$.

We have the embedding

$$P' = P/\mathfrak{p}[[x]] \hookrightarrow (R/\mathfrak{p})[[x]]$$

P' is a prime ideal of $R/\mathfrak{p}[[x]]$ and $P' \cap R/\mathfrak{p} = 0$. It will suffice to show that P' is finitely generated.

We have reduced the proof to the case of a prime ideal $P \subset R[[x]]$ and $P \cap R = (0)$.

If $x \in P$, $P = (x)$ and we are done.
For $\mathbf{f}(x) = a_0 + a_1 x + \cdots \in P$, let $J = (b_1, \ldots, b_m) \subset R$ be the ideal generated by all a_0,

$$\mathbf{f}_i = b_i + \text{higher terms} \in P.$$

Claim: $P = (\mathbf{f}_1, \ldots, \mathbf{f}_m)$.

From $a_0 = \sum_i s_i^{(0)} b_i$, we write

$$\mathbf{f}(x) - \sum_i s_i^{(0)} \mathbf{f}_i = x\mathbf{h} \quad \Rightarrow \mathbf{h} \in P.$$

We repeat with **h** and write

$$\mathbf{f}(x) = \sum_i s_i^{(0)} \mathbf{f}_i + x \sum_i s_i^{(1)} \mathbf{f}_i + x^2 \mathbf{g}, \quad \mathbf{g} \in P.$$

Iterating we obtain

$$\mathbf{f}(x) = \sum_i (s_i^{(0)} + s_i^{(1)} x + s_i^{(2)} x^2 + \cdots) \mathbf{f}_i.$$

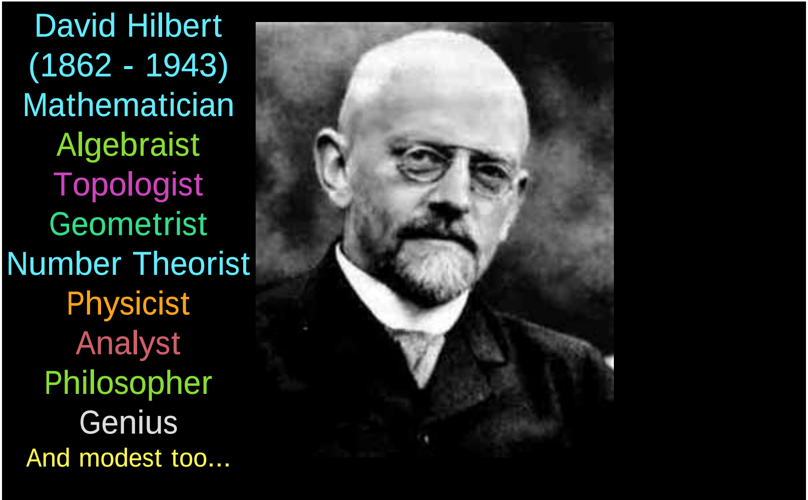
Outline

What is Algebraic Geometry?

Needs lots of space [it is, in fact, about Space] to describe all it is about.

David Hilbert (1862-1943)

David Hilbert



"Physics is much too hard for physicists." - Hilbert, 1912

Do polynomials have roots?

Let $\mathbf{f}(\mathbf{x}) = \mathbf{f}(x_1, \ldots, x_n)$ be a nonconstant polynomial of $R = \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_n]$, $n > 1$.

Fact: There is $\mathbf{c} \in \mathbb{C}^n$ such that $\mathbf{f}(\mathbf{c}) = 0$.
The answer is easy when

$$\mathbf{f}(x_1, \ldots, x_n) = x_n^d + \mathbf{g}(x_1, \ldots, x_n),$$

where $\mathbf{g}(\mathbf{x})$ is a polynomial of degree $< d$ in the variable x_n.
For example: Discuss

$$x^6 + yx^5 + y^8 + 1$$

More generally, let $\mathbf{f}_1(\mathbf{x}), \ldots, \mathbf{f}_m(\mathbf{x})$ be a set of elements of $R = \mathbb{C}[\mathbf{x}]$.

Question: What are the obstructions to finding $\mathbf{c} \in \mathbb{C}^n$ such that

$$\mathbf{f}_1(\mathbf{c}) = \mathbf{f}_2(\mathbf{c}) = \cdots = \mathbf{f}_m(\mathbf{c}) = 0 \text{ ?}$$

Obviously one is: there exist $\mathbf{g}_1(\mathbf{x}), \ldots, \mathbf{g}_m(\mathbf{x})$ such that

$$\mathbf{g}_1(\mathbf{x})\mathbf{f}_1(\mathbf{x}) + \cdots + \mathbf{g}_m(\mathbf{x})\mathbf{f}_m(\mathbf{x}) = 1$$

What else?

Volunteer!

- Sketch the graph of the equation

$$y^2 = x(x-1)(x-2)$$

- Can you see a group in the graph?

Hilbert Nullstellensatz

Let k be a field and denote by $\overline{k}$ its algebraic closure. (What are these? Like $\mathbb{R}$ and $\mathbb{C}$) We stay with $\mathbb{C}$.

The Hilbert Nullstellensatz is about qualitative results on systems of polynomial equations.

Let $\mathbf{f}_i(x_1, \ldots, x_n) \in R = k[x_1, \ldots, x_n]$, $1 \leq i \leq m$, be a set of polynomials.

Definition

The algebraic variety defined by the $\mathbf{f}_i$ is the set of zeros

$$V(\mathbf{f}_1, \ldots, \mathbf{f}_m) = \{\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{C}^n : \mathbf{f}_i(\mathbf{c}) = 0, \quad 1 \leq i \leq m\}.$$

A hypersurface is a variety defined by a single equation $V(\mathbf{f})$. If I is the ideal generated by the $\mathbf{f}_i$, then the variety defined by I is $V(I) = V(\mathbf{f}_1, \ldots, \mathbf{f}_m)$.

Notes about $\mathbb{C}$

- $\mathbb{C}$ is a two-dimensional vector space over $\mathbb{R}$
- If $\mathbb{C} \subset \mathbf{F}$ is a field that is of finite dimension over $\mathbb{C}$, obviously it is of (double) finite dimension over $\mathbb{R}$
- This means that if $u \in \mathbf{F}$, the vector subspace spanned by the powers of u,

  $$1, u, u^2, \ldots,$$

  is finite dimensional over $\mathbb{R}$ and thus there must be a polyonomial $\mathbf{f}(x) \in \mathbb{R}[x]$ such that $\mathbf{f}(u) = 0$. This will imply $u \in \mathbb{C}$–that is $\mathbb{C}$ is algebraically closed

The field extensions of $\mathbb{C}$,

$$\mathbb{C} \to \mathbf{F}$$

have the property

- If $u \in \mathbf{F}$ satisfies an equation

$$\mathbf{f}(u) = 0,$$

  $u \in \mathbb{C}$

- Otherwise u said to be transcendental over $\mathbb{C}$. This is the case for every nonconstant

$$u = \frac{\mathbf{f}(x)}{\mathbf{g}(x)} \in \mathbb{C}(x)$$

Hilbert Nullstellensatz

Theorem

If the ideal $I \subset R = \mathbb{C}[x_1, \ldots, x_n]$ is proper, i.e. $I \neq R$, then $V(I) \neq \emptyset$–that is, if $I \neq R$, there is $\mathbf{c}$ such that $\mathbf{f}(\mathbf{c}) = 0$ for all $\mathbf{f} \in I$.

Proof. We make two reductions.

1. Let $\mathfrak{m}$ be a maximal ideal of R containing I. Since $V(\mathfrak{m}) \subset V(I)$, ETA that I is maximal.
2. Indeed, if $\mathbf{c} \in \mathbb{C}^n$ is such that $\mathbf{f}(\mathbf{c}) = 0$ for all $\mathbf{f}(\mathbf{x}) \in \mathfrak{m}$, then $\mathbf{g}(\mathbf{c}) = 0$ for all $\mathbf{g} \in I \subset \mathfrak{m}$.

Nullstellensatz

After these reductions the assertion is:

Theorem

If M is a maximal ideal of $R = \mathbb{C}[x_1, \ldots, x_n]$, then there is

$$\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{C}^n$$

such that

$$\mathbf{f}(\mathbf{c}) = 0 \quad \forall \mathbf{f}(\mathbf{x}) \in M.$$

Special case: $\mathbb{C}$

Consider the field $\mathbf{F} = \mathbb{C}[x_1, \dots, x_n]/M$.

Proposition

It is ETS that $\mathbf{F}$ *is isomorphic to* $\mathbb{C}$*.*

Proof. Indeed, if $\mathbf{F} \simeq \mathbb{C}$, for each indeterminate x_i its equivalence class in $\mathbb{C}[x_1, \dots, x_n]/M$ contains some element c_i of $\mathbb{C}$, that is $x_i - c_i \in M$. this means that

$$(x_1 - c_1, \dots, x_n - c_n) \subset M.$$

But $(x_1 - c_1, \dots, x_n - c_n)$ is also a maximal ideal, therefore it is equal to M. Clearly every polynomial of M vanishes at $\mathbf{c} = (c_1, \dots, c_n)$. $\square$

Proof of $\mathbb{C} = \mathbb{C}[x_1, \ldots, x_n]/M$

1. ETS that the extension $\mathbb{C} \to \mathbf{F} = \mathbb{C}[x_1, \ldots, x_n]/M$ is algebraic.
2. Observe that $[\mathbf{F} : \mathbb{C}]$, the dimension of $\mathbf{F}$ as a vector space over $\mathbb{C}$, is countable, $\mathbf{F}$ being a homomorphic image of the countably generated vector space $\mathbb{C}[x_1, \ldots, x_n]$.
3. If $\mathbf{F}$ is not algebraic over $\mathbb{C}$, suppose $t \in \mathbf{F}$ is transcendental over $\mathbb{C}$.
4. Consider the uncountable set $\{1/(t - c), c \in \mathbb{C}\}$.

Since they cannot be linearly independent, there are distinct c_i, $1 \leq i \leq m$ and nonzero $r_i \in \mathbb{C}$ such that

$$r_1 \frac{1}{t - c_1} + \cdots + r_m \frac{1}{t - c_m} = 0.$$

Clearing denominators gives the equality of two polynomials of $\mathbb{C}[t]$:

$$r_1(t - c_2)(t - c_3)\cdots(t - c_m) = (t - c_1)\mathbf{g}(t),$$

which is a contradiction as the c_i are distinct.

Comaximal ideals

Definition

Two ideals I and J of a ring $\mathbf{R}$ are comaximal if

$$I + J = \mathbf{R}.$$

Example

$\mathbf{R} = \mathbb{Z}$, $I = (6)$, $J = (35)$, then $I + J = \mathbb{Z}$.

Partition of the Unity

If **R** is a commutative ring, a partition of the unity is an special decomposition of the form

$$R = J_1 + \cdots + J_n, \quad J_i \text{ ideals of } R$$

Suppose $I_1, \ldots, I_n$ is a set of a ideals that is pairwise co-maximal, meaning $I_i + I_j = R$, for $i \neq j$. This obviously is a partition of the unity.

Another arises from it [check!] if we set $J_i = \prod_{j \neq i} I_j$

$$R = J_1 + \cdots + J_n, \quad J_i \text{ ideals of } R$$

Chinese Remainder Theorem

Theorem

If I_i, $i \leq n$, is a family of ideals that is pairwise co-maximal, then for $I = I_1 \cap I_2 \cap \cdots \cap I_n$ there is an isomorphism

$$R/I \quad \approx \quad R/I_1 \times \cdots \times R/I_n.$$

Proof. Set $J_i = \prod_{j \neq i} I_j$. Note that $I_i + J_i = R$. Since $J_1 + \cdots + J_n = R$, there is an equation

$$1 \quad = \quad a_1 + \cdots + a_n, \quad a_i \in J_i$$

Note that for each i, $a_i \cong 1 \mod I_i$. Define a mapping $\mathbf{h}$ from R to $R/I_1 \times \cdots \times R/I_n$, by $\mathbf{h}(x) = (\overline{xa_1}, \ldots, \overline{xa_n})$. We claim that $\mathbf{h}$ is a surjective homomorphism of kernel I.

Proof Cont'd

1. Since $a_i \cong 1 \mod I_i$,

$$\mathbf{h}(x) = (\overline{xa_1}, \ldots, \overline{xa_n}) = (\overline{x}_1, \ldots, \overline{x}_n)$$

   which is clearly a homomorphism.
2. The kernel consists of the x such that $\overline{x}_i = 0$ for each i, that is $x \in I_i$ for each i–that is, $x \in I$.
3. To prove $\mathbf{h}$ surjective, for $u = (\overline{x_1}, \ldots, \overline{x_n})$, setting

$$x = x_1 a_1 + \cdots + x_n a_n$$

   gives $\mathbf{h}(x) = u$.

Example

How ancient astronomers calculated 1^o: That is, how to divide the circle by 360.

- $360 = 8 \times 9 \times 5$: primary decomposition.
- The numbers 72, 40 and 45 have no common factor, so form a partition of the 1:

$$1 = 5 \times 45 - 2 \times 72 - 2 \times 40$$

$$\frac{1}{360} = \frac{5}{8} - \frac{2}{5} - \frac{2}{9}$$

Outline

GCD of polynomials

If $f(x)$ and $g(x)$ are polynomials in $\mathbf{F}[x]$, the **greatest common divisor** is the monic polynomial of highest degree $h(x)$ that divides $f(x)$ and $g(x)$

$$\gcd(f(x), g(x)) = h(x)$$

For example,

$$\gcd((x-1)^3(x-2)^2, (x-1)(x-2)^4) = (x-1)(x-2)^2.$$

An elementary, but very useful fact, is that long division provides an effective method to find gcds.

Proposition

A polynomial $f(x) \in \mathbb{R}[x]$ of degree $f(x) \geq 1$ has multiple roots if and only if $\gcd(f(x), f'(x)) \neq 1$.

Thus, while it is hard to find the roots of a polynomial $f(x)$, it is easy to determine whether it has multiple roots!
The explanation is very simple: If $f(x)$ has a root of algebraic multiplicity m,

$$f(x) = (x-a)^m g(x), \quad g(a) \neq 0,$$

its derivative

$$f'(x) = m(x-a)^{m-1} g(x) + (x-a)^m g'(x)$$

has a as a root with multiplicity $m-1$. This implies that $(x-a)^{m-1}$ is a common factor of $f(x)$ and $f'(x)$, and therefore will be a factor of $\gcd(f(x), f'(x))$.

1. If $\gcd(f(x), f'(x)) = 1$, then $f(x)$ has no repeated (complex) roots.
2. Suppose $f(x)$ is the characteristic polynomial of a 3-by-3 complex matrix **A**, and we must decide whether it is diagonalizable.What to do?
    1. If $\gcd(f(x), f'(x)) = 1$, by the discussion above the roots are distinct, and we are done: **A** is diagonalizable.
    2. If there is a double root a and a single root b, $\gcd(f(x), f'(x)) = (x - a)$. We check the dimension of the eigenspace E_a, if $\dim E_a = 2$, ok, otherwise not diagonalizable.
    3. If a is a triple root, $\gcd(f(x), f'(x)) = (x - a)^2$. Again we check whether $\dim E_a = 3$.

Long division

Recall the long division algorithm for polynomials in $\mathbf{F}[x]$: If $f(x), g(x) \neq 0$ are polynomials, there exist polynomials $q(x), r(x)$ such that

$$f(x) = q(x)g(x) + r(x), \quad r(x) = 0 \text{ or } \deg r(x) < \deg g(x)$$

Look at a consequence:

$$\gcd(f(x), g(x)) = \gcd(g(x), r(x))$$

since any polynomial $p(x)$ that divides (both) $f(x), g(x)$ will divide $g(x), r(x)$, and conversely. Note that the data of $g(x), r(x)$ has lower degrees, so we can turn this into an algorithm:

gcd algorithm

Starting at

$$f(x) = q(x)g(x) + r(x),$$

1. Iterating, if $r(x) \neq 0$ and we divide $g(x) = q_1(x)r(x) + r_1(x)$, then any polynomial $p(x)$ that divides (both) $f(x), g(x)$ will divide $r(x), r_1(x)$, and conversely.
2. Since $\deg g(x) > \deg r(x) > \deg r_1(x) > \cdots$, ultimately we shall have $r_{n-1}(x) = q_{n-1}(x)r_n(x), \quad r_n(x) \neq O$.
3. $r_n(x)$ is (a) largest degree polynomial that divides both $f(x)$ and $g(x)$, and any such polynomial will divide $r_n(x)$.

Theorem

If $r_n(x)$ is the last nonzero remainder in the sequence of long divisions, then $r_n(x)$ divides $f(x)$ and $g(x)$. Moreover, there exist polynomials $a(x), b(x)$ such that

$$r_n(x) = a(x)f(x) + b(x)g(x).$$

$r_n(x)$ is called the (a) **GCD** *of $f(x)$ and $g(x)$.*

Proof: For simplicity suppose $n = 2$, so we have the divisions

$$f = qg + r, \quad g = q_1 r + r_1, \quad r = q_2 r_1 + r_2, \quad , r_1 = q_3 r_2$$

$$\begin{aligned} r_2 &= r - q_2 r_1 = r - q_2(g - q_1 r) = r(1 + q_2 q_1) - q_2 g \\ &= (f - qg)(1 + q_2 q_1) - q_2 g \end{aligned}$$

Now we collect the coefficient of f–it will be $a(x)$–and of g–it will be $b(x)$: $\gcd(f, g) = a(x)f(x) + b(x)g(x)$

We are now going to apply these observations to the characteristic polynomial $p(x) = \det(\mathbf{A} - x\mathbf{I})$ of a matrix $\mathbf{A}$, whose eigenvalues λ_i exist in the field $\mathbf{F}$. Note for $\mathbf{F} = \mathbb{C}$, this is the case for all matrices.

Underlying the following discussion is the assumption that

$$p(x) = \pm \prod_{i=1}^{m} (x - \lambda_i)^{m_i}.$$

1. If $f(x) = (x - \lambda)^m$, $g(x) = (x - \mu)^n$ and $\lambda \neq \mu$ are different scalars, then $\gcd(f(x), g(x)) = 1$, this means that there is a (decomposition) $1 = a(x)f(x) + b(x)g(x)$.
2. Consider now the case of the 3 polynomials,

$$f(x) = (x-\lambda_1)^m(x-\lambda_2)^n, g(x) = (x-\lambda_1)^m(x-\lambda_3)^p, h(x) = (x-\lambda_2)^n(x-\lambda$$

where $\lambda_1, \lambda_2, \lambda_3$ are distinct. Note that

$$\begin{aligned} \gcd(f,g) &= (x-\lambda_1)^m \\ \gcd(f,h) &= (x-\lambda_2)^n \\ \gcd(g,h) &= (x-\lambda_3)^p \\ \gcd(f,g,h) &= \gcd((x-\lambda_1)^m, h) = 1 \end{aligned}$$

3. These equations, will imply that we have an equality

$$1 = a(x)f(x) + b(x)g(x) + c(x)h(x).$$

Suppose the characteristic polynomial of $\mathbf{T}$ has a decomposition

$$\det(x\mathbf{I} - \mathbf{T}) = (x-a)^m(x-b)^n(x-c)^p.$$

The polynomials $\mathbf{f}(x) = (x-b)^n(x-c)^p$, $\mathbf{g}(x) = (x-a)^m(x-c)^p$, $\mathbf{h}(x) = (x-a)^m(x-b)^n$, have $\gcd = 1$ as they have no common divisor. According to the observation above, we have an equality

$$1 = A(x)\mathbf{f}(x) + B(x)\mathbf{g}(x) + C(x)\mathbf{h}(x)$$

Evaluating $x \to \mathbf{T}$ gives the equality

$$\mathbf{I} = A(\mathbf{T})\mathbf{f}(\mathbf{T}) + B(\mathbf{T})\mathbf{g}(\mathbf{T}) + C(\mathbf{T})\mathbf{h}(\mathbf{T})$$

Applying to an arbitrary vector **v** we have

$$\begin{aligned}\mathbf{v} = \mathbf{I}(\mathbf{v}) &= \underbrace{A(\mathbf{T})(\mathbf{T}-b\mathbf{I})^n(\mathbf{T}-c\mathbf{I})^p(\mathbf{v})}_{v_1} + \underbrace{B(\mathbf{T})(\mathbf{T}-a\mathbf{I})^m(\mathbf{T}-c\mathbf{I})^p(\mathbf{v})}_{v_2} \\ &+ \underbrace{C(\mathbf{T})(\mathbf{T}-a\mathbf{I})^m(\mathbf{T}-b\mathbf{I})^n(\mathbf{v})}_{v_3}\end{aligned}$$

$$\mathbf{v} = v_1 + v_2 + v_3$$

$$(\mathbf{T}-a\mathbf{I})^m(v_1) = A(\mathbf{T})(\mathbf{T}-a\mathbf{I})^m(v_1) = A(\mathbf{T})(\mathbf{T}-a\mathbf{I})^m(\mathbf{T}-b\mathbf{I})^n(\mathbf{T}-c\mathbf{I})^p(v) = 0$$

by Cayley-Hamilton. This says that every vector **v** is a sum of vectors in K_a, K_b and K_c. It is also easy to see that v_1, v_2, v_3 are linearly independent.

Chinese Remainder Theorem

Theorem

Let $f_1(x), \ldots, f_m(x)$ *be polynomials of* $\mathbf{F}[x]$*. If* $g(x) = gcd(f_1(x), \ldots, f_m(x))$ *there are polynomials* $a_i(x)$ *such that*

$$g(x) = a_1(x)f_1(x) + \cdots + a_m(x)f_m(x).$$

Let $\mathbf{T}$ be a linear operator on the finite-dimensional vector space $\mathbf{V}$. Suppose its characteristic polynomial $\det(\mathbf{T} - x\mathbf{I})$ splits:

$$f(x) = \pm \prod_{i=1}^{m} (x - \lambda_i)^{n_i}, \quad \text{distinct } \lambda_i.$$

For each i, setting $f_i(x) = \frac{f(x)}{(x-\lambda_i)^{n_i}}$, gives us a collection $f_1(x), \ldots, f_m(x)$ of gcd $= 1$: In

$$1 = a_1(x)f_1(x) + \cdots + a_m(x)f_m(x)$$

$$\mathbf{I} = a_1(\mathbf{T})f_1(\mathbf{T}) + \cdots + a_m(\mathbf{T})f_m(\mathbf{T})$$

Now we are going to make several observations about this decomposition.

1. The range of $f_i(\mathbf{T})$ is contained in the generalized eigenspace K_{λ_i}:If $u = f_i(\mathbf{T})(v)$,

   $$(\mathbf{T} - \lambda_i)^{n_i} f_i(\mathbf{T})(v) = f(\mathbf{T})(v) = 0,$$

   since by the Cayley-Hamilton theorem $f(\mathbf{T}) = 0$.
2. For every $v \in \mathbf{V}$

   $$v = \mathbf{I}(v) = \overbrace{a_1(\mathbf{T})f_1(\mathbf{T})(v)}^{\in K_{\lambda_1}} + \cdots + \overbrace{a_m(\mathbf{T})f_m(\mathbf{T})(v)}^{\in K_{\lambda_m}}$$

Generalized eigenvectors and eigenspaces

- If $\mathbf{T}$ is a linear operator of the vector space $\mathbf{V}$ and λ is a scalar, a nonzero vector $v \in \mathbf{V}$ is a **generalized eigenvector** of $\mathbf{T}$ if $(\mathbf{T} - \lambda\mathbf{I})^p(v) = O$ for some positive integer p. We denote this set, together with the vector O, by K_λ. K_λ is usually bigger than the eigenspace E_λ.
- In fact,

$$\mathbf{V} = \bigoplus_i K_{\lambda_i},$$

  in particular, $\mathbf{V}$ has a basis made up of generalized eigenvectors.

This representation says that every vector $v \in \mathbf{V}$ can be written as

$$v = v_1 + \cdots + v_m, \quad v_i \in K_{\lambda_i}$$

Since we already proved that $\dim K_{\lambda_i} \leq n_i$, the algebraic multiplicity of λ_i, this equality proves equality of the dimensions. It can be written as

$$\mathbf{V} = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_m},$$

and the matrix representation of $\mathbf{T}$ has the block format (after picking bases of the K_{λ_i}'s)

$$[\mathbf{T}] = \begin{bmatrix} [\mathbf{T}]_1 & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & [\mathbf{T}]_m \end{bmatrix}$$

What this does is to allow us to assume that the characteristic polynomial of $\mathbf{T}$ has the form $(x-\lambda)^n$. We will argue that such linear operator have a matrix representation made up of Jordan blocks with the same λ. Let us look at one such $p \times p$ block

$$\mathbf{A} = [v_1|\cdots|v_p] = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

$$\underbrace{\mathbf{A}(v_1) = \lambda v_1}_{\text{eigenvector}}, \quad \mathbf{A}(v_2) = v_1 + \lambda v_2, \cdots, \mathbf{A}(v_p) = v_{p-1} + \lambda v_p$$

If we write these equations in the reverse order, we get

$$\begin{aligned}
(\mathbf{A}-\lambda\mathbf{I})(v_p) &= v_{p-1}\\
(\mathbf{A}-\lambda\mathbf{I})^2(v_p) &= v_{p-2}\\
&\vdots\\
(\mathbf{A}-\lambda\mathbf{I})^{p-1}(v_p) &= v_1\\
(\mathbf{A}-\lambda\mathbf{I})^{p}(v_p) &= 0
\end{aligned}$$

Starting on v_p and applying $\mathbf{U} = \mathbf{A} - \lambda\mathbf{I}$ repeatedly we get all the vectors of the basis

$$v_p \to v_{p-1} \to \cdots \to v_2 \to v_1 \to O$$

We will say that v_p is the generator of the basis, and that $\gamma = \{v_1, v_2, \ldots, v_p\}$ is a cycle of generalized eigenvectors, v_1 is the initial and v_p the end vectors: They form a so-called **dot diagram**

Proposition

Let $\mathbf{T}$ *be a linear operator on the vector space* $\mathbf{V}$*. For some scalar* λ *and some integer p, suppose v is a nonzero vector such that*

$$(\mathbf{T}-\lambda\mathbf{I})^p(v) = O, \quad (\mathbf{T}-\lambda\mathbf{I})^{p-1}(v) \neq O.$$

Then the p vectors $(\mathbf{T}-\lambda\mathbf{I})^{p-1}(v), \ldots, (\mathbf{T}-\lambda\mathbf{I})(v), v$ *are linearly independent. They span a* $\mathbf{T}$*-invariant subspace* $\mathbf{W}$ *and the matrix representation of* $[\mathbf{T}]_{\mathbf{W}}$ *with respect to this basis is a Jordan block.*

Proof: Let us denote these vectors by $v_1, \ldots, v_p = v$, respectively. Suppose we have a linear relation $c_1 v_1 + \cdots + c_p v_p = O$. Let us prove all $c_i = 0$. Let us argue just one case as the general case is similar. Suppose $c_p \neq 0$. Apply the operator $(\mathbf{T}-\lambda\mathbf{I})^{p-1}$ to the relation to obtain

$$v_i = (\mathbf{T} - \lambda \mathbf{I})^{p-i}(v)$$

$$c_1 (\mathbf{T} - \lambda \mathbf{I})^{p-1}(v_1) + \cdots + c_p \underbrace{(\mathbf{T} - \lambda \mathbf{I})^{p-1}(v_p)}_{=v_1} = O$$

Note that all terms vanish, except for the last. This contradicts $c_p \neq 0$.

The subspace **W** clearly satisfies $\mathbf{T}(\mathbf{W}) \subset \mathbf{W}$. Finally, note that

$$\begin{aligned} \mathbf{T}(v_i) &= \mathbf{T}(\mathbf{T} - \lambda \mathbf{I})^{p-i}(v) \\ &= (\mathbf{T} - \lambda \mathbf{I})^{p-i+1}(v) + \lambda (\mathbf{T} - \lambda \mathbf{I})^{p-i}(v) = v_{i-1} + \lambda v_i, \end{aligned}$$

which shows that the matrix representation is

$$\begin{bmatrix} \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & \lambda \end{bmatrix}$$

We come now to the crux of the problem: Given a linear operator **T** whose characteristic polynomial is $\pm(x-\lambda)^n$, to prove that there is a matrix representation made up of λ-Jordan blocks (same λ)

$$\begin{bmatrix} \boxed{\mathbf{J}_1} & O & O \\ O & \boxed{\mathbf{J}_2} & O \\ O & O & \boxed{\mathbf{J}_3} \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$$

We are going to prove the existence of such representation and the uniqueness of the number and sizes of the blocks.

Situation:

$\mathbf{T} : K_\lambda \to K_\lambda$, $\dim K_\lambda = n$, characteristic polynomial of $\mathbf{T}$ is $(x-\lambda)^n$. The eigenspace is $E_\lambda \subset K_\lambda$.

Goal: We will show that K_λ has a basis

$$\mathcal{B} = \bigcup_{i=1}^{m} \gamma_i$$

where each γ_i is a cycle of generalized eigenvectors. The Jordan representation comes from the corresponding matrix representation. For example, if $K_\lambda = E_\lambda$, then a basis of E_λ

gives the cycles, all of length 1, and the matrix representation is just $\lambda \mathbf{I}_n$.

1. We are going to argue by induction on $n = \dim K_\lambda$. If $n = 1$ (or, more generally, $K_\lambda = E_\lambda$), there is nothing to prove.
2. Let $\mathbf{Z}$ be the range of $\mathbf{T} - \lambda\mathbf{I}$. For simplicity of notation call this map $\mathbf{U} : K_\lambda \to K_\lambda$. Note that E_λ is the nullspace of $\mathbf{U}$, and therefore $\dim E_\lambda + \dim \mathbf{Z} = n$, by the dimension formula.
3. Since $\dim \mathbf{Z} < n$ and the characteristic polynomial of the restriction of $\mathbf{T}$ to $\mathbf{Z}$ divides $(x - \lambda)^n$, the induction hypothesis guarantees a basis for $\mathbf{Z}$:

$$\gamma' : w, (\mathbf{T} - \lambda\mathbf{I})(w), \dots, (\mathbf{T} - \lambda\mathbf{I})^{p-1}(w)$$

$$\mathcal{B}' = \bigcup_{i=1}^{r} \gamma'_i$$

where each γ'_i is a cycle of generalized eigenvectors of $\mathbf{Z}$. Let us consider one of these cycles γ':

$$\gamma_i' : w, (\mathbf{T} - \lambda\mathbf{I})(w), \ldots, (\mathbf{T} - \lambda\mathbf{I})^{p-1}(w)$$

But w belongs to the range of $(\mathbf{T} - \lambda\mathbf{I})$, that is $w = (\mathbf{T} - \lambda\mathbf{I})(v)$, for some $v \in \mathbf{V}$. This gives a cycle of $\mathbf{V}$ itself:

$$\gamma_i : v, (\mathbf{T} - \lambda\mathbf{I})(v), \ldots, (\mathbf{T} - \lambda\mathbf{I})^{p}(v)$$

In this manner, for every γ_i' of $\mathbf{Z}$ we get a longer cycle (by 1 more vector) of $\mathbf{V}$.

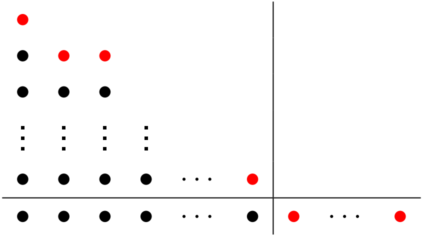
We recall that vector at the end of the list are the only eigenvectors and that

$$\bigcup_{i=1}^{r} \gamma_i$$

contains just r independent eigenvectors, the same set as the basis $\mathcal{B}'$ of $\mathbf{Z}$. If these eigenvectors are $u_1, \ldots, u_r$, add (if necessary) $u_{r+1}, \ldots, u_s$ to form a basis of the eigenspace E_λ. Each of these u_i defines a new cycle γ_i of length 1, $i > r$.

Dot Diagrams and Enlarged Cycles

•: vectors in the set $\mathcal{B}'$
•: vectors added.



$\mathbf{T} - \lambda\mathbf{I}$ maps each dot to dot under. Last row is a basis of E_λ: it is mapped to O

Proposition (Very technical, I apologize)

The vectors in the set

$$\mathcal{B} = \bigcup_{i=1}^{s} \gamma_i$$

form a basis of $\mathbf{V}$.

Proof: First let us count the number of elements of added to pass from the basis $\mathcal{B}'$ of $\mathbf{Z}$ to the set $\mathcal{B}$ of $\mathbf{V}$:

$$r \text{ (1 for each of the } r \text{ cycles in } \mathcal{B}') + (s - r) = s = \dim E_\lambda$$

$$\text{Therefore} \quad \text{cardinality of } \mathcal{B}' + s = \dim \mathbf{Z} + s = n = \dim \mathbf{V}$$

To prove $\mathcal{B}$ is a basis, ETS that it spans $\mathbf{V}$, as they have already the right number of elements for a basis.

Let $u \in \mathbf{V}$ and consider $(\mathbf{T} - \lambda\mathbf{I})(u) \in \mathbf{Z}$. Since every vector in $\mathcal{B}'$ is the image under $\mathbf{T} - \lambda\mathbf{I}$ of some vector in $\mathcal{B}$, we can write

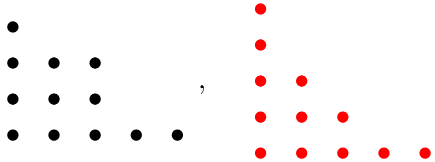
$$(\mathbf{T} - \lambda\mathbf{I})(v) = \text{Linear combination of } (\mathbf{T} - \lambda\mathbf{I})(v_i), \quad v_i \in \mathcal{B}.$$

This implies that

$$(\mathbf{T} - \lambda\mathbf{I})\underbrace{(v - \text{Linear combination of } v_i)}_{=w} = O$$

Thus $w \in E_\lambda$. Since $\mathcal{B}$ contains a basis of E_λ, this implies v lies in the span of $\mathcal{B}$.

To illustrate the uniqueness of Jordan decomposition, suppose **T** gives rise to two different cycle decomposition for K_λ:



Observe that many things match: dim $K_\lambda = 12$ [number of dots, red or black], dim $E_\lambda = 5$ (number of piles, columns). Now we are going to observe things that are off:

$$(\mathbf{T} - \lambda\mathbf{I})^4(\text{any } \bullet) = 0, \quad (\mathbf{T} - \lambda\mathbf{I})^4(\text{top } \bullet) \neq 0$$

This illustrate the argument: The number of dots at level ℓ is the dimension of the subspace of the vectors v of $\mathbf{V}$ such that

$$(\mathbf{T} - \lambda \mathbf{I})^{\ell}(v) = 0$$

Outline

Diagonalization and Minimal Polynomials

Let S be the ring of $n \times n$ matrices and $\mathbf{A} \in S$. We look at $\mathbf{A}$ as a linear transformation $\mathbf{A} : \mathbf{F}^n \to \mathbf{F}^n$. S is a ring which as a $\mathbf{F}$-vector space has dimension n^2.
Consider the ring homomorphism defined by the evaluation

$$\varphi : R = \mathbf{F}[x] \to S, \quad \varphi(x) = \mathbf{A}$$

Proposition

$\ker \varphi \neq (0)$.

Proof.

φ cannot be injective since it maps the infinite dimensional vector space $\mathbf{F}[x]$ into the finite dimensional vector space S. □

Minimal Polynomial

By the theorem about the ideals of $\mathbf{F}[x]$, $\ker(\varphi) = (m(x))$. For convenience we pick $m(x)$ as monic.
Thus, given a square matrix $\mathbf{A}$, there are polynomials $\mathbf{f}(x)$ such that

$$\mathbf{f}(\mathbf{A}) = 0.$$

The best known is $\mathbf{f}(x) = \det(\mathbf{A} - x\mathbf{I})$, the characteristic polynomial: by Cayley-Hamilton:

$$\mathbf{f}(\mathbf{A}) = 0.$$

What else?

Definition

Let $\mathbf{A}$ be a n-by-n matrix. The minimal polynomial of $\mathbf{A}$ is the monic polynomial $m(x) = x^m + c_{m-1}x^{m-1} + \cdots + c_0$ of least degree such that

$$m(\mathbf{A}) = \mathbf{A}^m + c_{m-1}\mathbf{A}^{m-1} + \cdots + c_0\mathbf{I} = O.$$

1. If $\mathbf{A} = \mathbf{I}_n$, then $m(x) = x - 1$.
2. If $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $m(x) = x^2$.
3. In the case of [the Jordan block] $\mathbf{J} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$, $m(x) = (x - \lambda)^3$. For a block of size n, $m(x) = (x - \lambda)^n$.

$$\mathbf{J} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \quad \mathbf{U} = \mathbf{J} - \lambda\mathbf{I} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{U}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{U}^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{U}^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$m(x) = (x - \lambda)^4$$

Observe the right drift of the diagonal of 1's until it leaves the matrix!

Corollary

The minimal polynomial $m(x)$ *of* $\mathbf{A}$ *divides the characteristic polynomial* $p(x) = \det(\mathbf{A} - x\mathbf{I})$ *of* $\mathbf{A}$*. In particular* $\deg m(x) \leq n$*.*

Diagonalization

Theorem

A *is diagonalizable if and only if its minimal polynomial $m(x)$ has no repeated root.*

Proof. In the forward direction, the assertion is clear: If **A** is made up of diagonal blocks

$$\mathbf{A} = \begin{bmatrix} \lambda_1 \mathbf{I}_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 \mathbf{I}_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_r \mathbf{I}_r \end{bmatrix},$$

with λ_i distinct, its minimal polynomial is

$$m(x) = \prod_{i=1}^{r} (x - \lambda_i)$$

For the converse, suppose the characteristic polynomial of $\mathbf{T}$ has a decomposition

$$\det(x\mathbf{I} - \mathbf{T}) = (x - a)^m (x - b)^n (x - c)^p.$$

The polynomials $\mathbf{f}(x) = (x - b)^n (x - c)^p$, $\mathbf{g}(x) = (x - a)^m (x - c)^p$, $\mathbf{h}(x) = (x - a)^m (x - b)^n$, their gcd $= 1$ as they have no common divisor. According to earlier observations, above we have an equality

$$1 = A(x)\mathbf{f}(x) + B(x)\mathbf{g}(x) + C(x)\mathbf{h}(x)$$

Evaluating $x \to \mathbf{T}$ gives the equality

$$\mathbf{I} = A(\mathbf{T})\mathbf{f}(\mathbf{T}) + B(\mathbf{T})\mathbf{g}(\mathbf{T}) + C(\mathbf{T})\mathbf{h}(\mathbf{T})$$

Applying to an arbitrary vector **v** we have

$$\mathbf{v} = \mathbf{I}(\mathbf{v}) = \underbrace{A(\mathbf{T})(\mathbf{T}-b\mathbf{I})^n(\mathbf{T}-c\mathbf{I})^p(\mathbf{v})}_{v_1} + \underbrace{B(\mathbf{T})(\mathbf{T}-a\mathbf{I})^m(\mathbf{T}-c\mathbf{I})^p(\mathbf{v})}_{v_2}$$
$$+ \underbrace{C(\mathbf{T})(\mathbf{T}-a\mathbf{I})^m(\mathbf{T}-b\mathbf{I})^n(\mathbf{v})}_{v_3}$$

$$\mathbf{v} = v_1 + v_2 + v_3$$

$$(\mathbf{T}-a\mathbf{I})^m(v_1) = A(\mathbf{T})(\mathbf{T}-a\mathbf{I})^m(v_1) = A(\mathbf{T})(\mathbf{T}-a\mathbf{I})^m(\mathbf{T}-b\mathbf{I})^n(\mathbf{T}-c\mathbf{I})^p(v) = 0$$

by Cayley-Hamilton. This says that every vector **v** is a sum of vectors in K_a, K_b and K_c. It is also easy to see that v_1, v_2, v_3 are linearly independent.

Now we are going to make several observations about this decomposition.

1. The range of $f_i(\mathbf{T})$ is contained in the generalized eigenspace K_{λ_i}:If $u = f_i(\mathbf{T})(v)$,

$$(\mathbf{T} - \lambda_i)^{n_i} f_i(\mathbf{T})(v) = f(\mathbf{T})(v) = 0,$$

since by the Cayley-Hamilton theorem $f(\mathbf{T}) = 0$.

2. For every $v \in \mathbf{V}$

$$v = \mathbf{I}(v) = \overbrace{a_1(\mathbf{T})f_1(\mathbf{T})(v)}^{\in K_{\lambda_1}} + \cdots + \overbrace{a_m(\mathbf{T})f_m(\mathbf{T})(v)}^{\in K_{\lambda_m}}$$

Generalized eigenvectors and eigenspaces

- If $\mathbf{T}$ is a linear operator of the vector space $\mathbf{V}$ and λ is a scalar, a nonzero vector $v \in \mathbf{V}$ is a **generalized eigenvector** of $\mathbf{T}$ if $(\mathbf{T} - \lambda\mathbf{I})^p(v) = O$ for some positive integer p. We denote this set, together with the vector O, by K_λ. K_λ is usually bigger than the eigenspace E_λ.
- In fact,

$$\mathbf{V} = \bigoplus_i K_{\lambda_i},$$

  in particular, $\mathbf{V}$ has a basis made up of generalized eigenvectors.

This representation says that every vector $v \in \mathbf{V}$ can be written as

$$v = v_1 + \cdots + v_m, \quad v_i \in K_{\lambda_i}$$

Since we already proved that $\dim K_{\lambda_i} \leq n_i$, the algebraic multiplicity of λ_i, this equality proves equality of the dimensions. It can be written as

$$\mathbf{V} = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_m},$$

and the matrix representation of $\mathbf{T}$ has the block format (after picking bases of the K_{λ_i}'s)

$$[\mathbf{T}] = \begin{bmatrix} [\mathbf{T}]_1 & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & [\mathbf{T}]_m \end{bmatrix}$$

Conclusion:

- This block decomposition says that the minimal polynomial $f(x)$ of $\mathbf{T}$ is the product of the minimal polynomials of the restrictions on K_{λ_i}

$$f(x) = p_1(x) \cdots p_m(x)$$

- If some $\mathbf{T}_i$ is not diagonalizable, its minimal polynomial has a factor $(x - a)^2$, and $f(x)$ will have some multiple root.

Group Representations

Theorem

Let $\mathbf{G}$ *be a finite subgroup of* $GL_n(\mathbb{C})$*. Then any element* $\mathbf{A} \in \mathbf{G}$ *is diagonalizable.*

Proof.

- Since $\mathbf{G}$ is finite, $\mathbf{A}$ has finite order, that is $\mathbf{A}^r = \mathbf{I}$ for some integer r.
- This implies that $x^r - 1$ lies in the ideal $(m(x))$ generated by the minimal polynomial of $\mathbf{A}$, and therefore $x^r - 1 = m(x)p(x)$.
- It follows that every root of $m(x)$ is a root of $x^r - 1$. But the roots of $x^r - 1$ are distinct (the derivative is rx^{r-1}, whose roots are zero). Therefore the roots of $m(x)$ are distinct.

Corollary

If $\mathbf{G}$ is a finite subgroup of $GL_n(\mathbb{C})$, then the order of every element $\mathbf{A} \in \mathbf{G}$ is at most n.

Outline

Homework #11

Do 5 Problems.

1. Prove that the kernel of the homomorphism $\varphi : \mathbb{C}[x, y] \to \mathbb{C}[t]$ defined by $x \mapsto t^2$, $y \mapsto t^3$ is the principal ideal generated by $x^3 - y^2$.
2. The nilradical N of a ring **R** is the set of nilpotent elements. Prove that N is an ideal. Find N when $\mathbf{R} = \mathbb{Z}_{72}$.
3. Prove that $\mathbb{Z}[i]/(i + 2)$ is isomorphic to $\mathbb{Z}/(m)$ for some m. Determine m.
4. Determine the maximal ideals of $\mathbb{R}[x]/(x^2 - 3x + 2)$.
5. Prove that the ring $\mathbb{Z}_2[x]/(x^3 + x + 1)$ is a field but $\mathbb{Z}_3[x]/[x^3 + x + 1)$ is not.
6. Find an isomorphic direct product of cyclic groups for the group:

- V is generated by the elements x, y, z;
- These elements satisfy the relations $7x + 5y + 2z = 0$, $3x + 3y = 0$, $13x + 11y + 2z = 0$.