Review

Math 350–Section #01 Final Exam

Name: ____________________

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1. (10 pts) Let A be an orthogonal (real) matrix.

(a) Show that $\det(A) = \pm 1$. Given an example that is not the identity matrix.

(b) If A is 2×2, and $\det(A) = -1$, prove that A is diagonalizable.

Answer:

Orthogonal : real square matrix

$$A \cdot A^t = I$$

$$\det(A A^T) = \det(A) \cdot \det(A^T) = (\det(A))^2 = 1$$

$$\det(A^T) = \det(A)$$

$$\therefore \det A = \pm 1$$

Example : $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$

$\det A = -1$

$$\det(A - tI) = t^2 - 1 = (t-1)(t+1)$$

distinct eigen

$\therefore$ diag'ble

Quiz # 7

A orthogonal 3×3, then A is similar to

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$

Proof. The eigenvalues are $\lambda, \bar{\lambda}$ and $|\lambda| = 1$

so there is real eigenvalue ($\lambda = \pm 1$).

$Av = \lambda v$

But a property of normal oper is: If $Av = \lambda v \Rightarrow A^* v = \bar{\lambda} v$

So v is an eigenvector of A^*

Let $W = v^\perp$: $v \cdot w = 0$

$\Rightarrow v \cdot Aw = A^* v \cdot w = \bar{\lambda} v \cdot w = 0$

So W is invariant under A:

$A(W) \subset W$

So the block decomposition of A

$$\begin{bmatrix} [A]_W & 0 \\ 0 & \pm 1 \end{bmatrix}$$

But the restriction of an orthogonal oper is orthogonal

(why? ortho means

$\|u\| = \|Au\|$ for all vectors

so if A is restricted to a subspace, it will still preserve length.

2. (10 pts) Given the matrix

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 5 \end{bmatrix}$$

(a) Find its characteristic polynomial.
(b) Find its eigenvalues.
(c) Explain why symmetric real matrices are diagonalizable. [It will be graded on the mathematical quality of the explanation.]

Answer:

Look up: A symmetric are normal

But an important point is: why the eigenvalues are real: Look it up

3. (10 pts) Let T be the linear transformation of $V = M_{2\times 2}(\mathbb{C})$

$$T(A) = 2A + 3A^t.$$

(a) Find a matrix representation of T (it will be a 4×4 matrix).
(b) Describe its eigenspaces.

Answer: Done in Hourly # 2

e.g.

eigenvectors

$$T(A) = 2A + 3A^t = \lambda A$$

$$\Rightarrow \quad 3A^t = (\lambda - 2)A$$

Two ways to solve

1)

$$(3A^t)^t = (\lambda-2)A^t$$

$$3A = (\lambda - 2)A^t$$

$$= \frac{(\lambda-2)^2}{3} A$$

$$(\lambda - 2)^2 = 9 \qquad \lambda - 2 = \pm 3$$

$$\lambda = 5, -1.$$

so: eigenspaces are

- sym mats
- skew syms

2) Try

A : sym

$$3A = (\lambda - 2)A \quad \therefore \quad \lambda = 5$$

A : skew-sym

$$-3A = (\lambda - 2)A \quad \lambda = -1$$

4. (10 pts) (a) Given that the 3×3 matrix $A = [c_1|c_2|c_3]$ has determinant 5, find the determinant of the matrix

$$B = [c_2 + c_3|c_3 + c_1|c_1 + c_2 + c_3].$$

(b) Given that the 5×5 matrix $A = [c_1|c_2|c_3|c_4|c_5]$ has determinant d, find the determinant of the matrix

$$B = [c_2 + c_3|c_3 + c_4|c_4 + c_5|c_5 + c_1|c_1 + c_2].$$

Answer:

$$B = \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{D} \cdot A$$

$$\Rightarrow \det B = \det D \cdot \det A = \det A = 5$$

(b) Same trick

5. (10 pts) (a) Find an orthogonal matrix S such that $S^{-1}AS$ is diagonal, where

$$A = \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix}.$$

(b) Use Part (a) to decide whether the equation $10x^2 + 6xy + 2y^2 = 18$ represents an ellipse, a parabola or a hyperbola.

Answer:

Done many time

6. (10 pts) Let V be an inner product space.

(a) If W is a subspace of V, what is the orthogonal complement $W^\perp$? Prove that $W^\perp$ is a subspace.

(b) If $T = p_W$ is the projection mapping defined by W, prove that W and $W^\perp$ are the eigenspaces of T and that T is diagonalizable.

Answer:

(a) Look

$T: V \to V$

If $w_1, \ldots, w_m$ is an ortho basis of W

$$T(v) = \sum (v \cdot w_i) w_i$$

Note (prove) $T^2 = T$

If $v \in W$, $T(v) = v = 1 \cdot v$

If $v \in W^\perp$ $T(v) = 0 = 0 \cdot v$

so 1 & 0 are eigenvalues

W & $W^\perp$ are corresponding eigenspaces

and $\therefore$ Have basis of eigenvectors

$T \Rightarrow \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$

7. (10 pts) (a) What is the Cayley-Hamilton Theorem?

(b) Verify it for the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(c) Prove the (full) theorem.

Answer:

Look it up

8. (10 pts) Given the matrix

$$A = \begin{bmatrix} 1 & -1 & -2 & 2 \\ -4 & 2 & 3 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 3 \\ 2 & -1 & -2 & 1 \end{bmatrix}$$

(a) Find its reduced echelon form R of A.
(b) Argue that $R = E_m \cdots E_1 A$, where the E_i are elementary matrices.
(c) What are the rank and the nullity of A. Explain why these two numbers always add to the number of columns.
(d) Argue that the rows of R with pivots are linearly independent.
(e) Argue that the columns of A with pivots are linearly independent.

Answer:

9. [10 pts] Let V be the vector space of all continuous real functions on $[-1,1]$. For $f(t), g(t) \in V$, define the product

$$\langle f(t), g(t) \rangle = \int_{-1}^{1} f(t)g(t)dt.$$

(a) Prove that this defines an inner product on V.

(b) Find an orthonormal basis for the subspace spanned by e^t, e^{2t}.

Answer:

10. [10 pts] About Jordan canonical forms:

(a) What are they and explain some of its uses.

(b) What are the ideas that were introduced to enable the decomposition.

(c) Find [give all the steps] the Jordan decomposition of the complex matrix
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}.$$

Answer:

Questions to Think

- Why the eigenvalues of real symm matrices are real?
- Why the determinant of real skew symm matrices are ≥ 0
- Exponential of a matrix

Syllabus

- Up to 7.1
- Mainly topics from Homework #1
- About 12 questions

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- Diagonalization of Normal opers
- Jordan decomposition

were the main results: you will be asked about one of them