Math 350: Linear Algebra

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Set 5

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- Multilinear Functions
- The Guises of the Determinant
- Computation Rules
- Applications

Outline

1. **Multilinear Algebra**
2. Determinants
3. Determinant of a Product
4. Applications
5. Homework
6. HomeQuiz #5

Multilinear functions

What is this? We have been studying linear functions on vector spaces

$$\mathbf{T} : \mathbf{V} \to \mathbf{W},$$

$$\mathbf{T}(au + bv) = a\mathbf{T}(u) + b\mathbf{T}(v).$$

A bilinear function is an extension of the product operation

$$(\mathbf{x}, \mathbf{y}) \to \mathbf{x}\mathbf{y}.$$

Note that it is additive in 'each variable', e.g.

$$\mathbf{x}(\mathbf{y}_1 + \mathbf{y}_2) = \mathbf{x}\mathbf{y}_1 + \mathbf{x}\mathbf{y}_2$$

$$(\mathbf{x}_1 + \mathbf{x}_2)\mathbf{y} = \mathbf{x}_1\mathbf{y} + \mathbf{x}_2\mathbf{y}$$

We want to examine functions like these whose sources and targets are vector spaces. For example, the function **B** is bilinear if

$$\mathbf{B} : \mathbf{V} \times \mathbf{V} \to \mathbf{W},$$

is linear in each variable

$$\mathbf{B}(u_1 + u_2, v) = \mathbf{B}(u_1, v) + \mathbf{B}(u_2, v), \quad \mathbf{B}(au, v) = a\mathbf{B}(u, v)$$

$$\mathbf{B}(u, v_1 + v_2) = \mathbf{B}(u, v_1) + \mathbf{B}(u, v_2), \quad \mathbf{B}(u, av) = a\mathbf{B}(u, v)$$

You can define trilinear, and generally multilinear in the same manner: $\mathbf{B}(v_1, v_2, \ldots, v_n)$, linear in each variable.

Let us begin with a beautiful example: Let $\mathbf{V} = \mathbf{F}^2$ be a plane. For every pair of vectors $u = (a, b), v = (c, d)$, define

$$\mathbf{B}(u, v) = ad - bc.$$

You can check easily that $\mathbf{B}$ is a bilinear function from $\mathbf{F}^2$ into $\mathbf{F}$. For example, $\mathbf{B}(u, v_1 + v_2) = \mathbf{B}(u, v_1) + \mathbf{B}(u, v_2)$.

This particular function is called **the 2-by-2 determinant**: $\det(u, v)$ It has many uses in Mathematics.

Another example, on this same space, is

$$\mathbf{C}(u, v) = ac + bd.$$

This one is called a **dot or scalar product.**

$\mathbf{B}(u, v)$ and $\mathbf{C}(u, v)$ read different info about the pair of vectors u, v as we shall see.

Another well-known bilinear transformation $\mathbf{F}^3 \times \mathbf{F}^3 \to \mathbf{F}^3$ is the following: For $u = (a, b, c)$, $v = (d, e, f)$,

$$(u, v) \to u \wedge v = (bf - ce, -af + cd, ae - bd)$$

This function is called the **exterior**, or **vector** product of $\mathbf{F}^3$.

When $\mathbf{F} = \mathbb{R}$, it has many useful properties geometric used in Physics [in Mechanics, Electricity, Magnetism]. Partly this arises because

$$u \wedge v \perp u \quad \& \perp v$$

and its magnitude says something about the parallelogram defined by u and v.

There are two main classes of multilinear functions. Say **B** is n-linear, that is it has n input cells and is linear in each separately: $\mathbf{B}(v_1, \ldots, v_n)$.
B is **symmetric**: If you exchange the contents of two cells

$$\mathbf{B}(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = \mathbf{B}(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n)$$

causes no change. Like the dot product above.

B is **skew-symmetric** or **alternating**: If

$$\mathbf{B}(v_1, \ldots, v_i = v, \ldots, v_j = v, \ldots, v_n) = 0$$

whenever two cells have the same content. Like the determinant above.

Let $\mathbf{M}_n(\mathbf{F})$ be the vector space of all $n \times n$ matrices over the field $\mathbf{F}$. Consider the **trace** function on $\mathbf{A} \in \mathbf{M}_n(\mathbf{F})$, $\mathbf{A} = [a_{ij}]$:

$$\mathbf{trace}([a_{ij}]) = \sum_{i=1}^{n} a_{ii}$$

Now define the function

$$\mathbf{T}(\mathbf{A}, \mathbf{B}) = \mathbf{trace}(\mathbf{AB})$$

T is clearly a bilinear function. It is a good exercise (do it) to show that

$$\mathbf{trace}(\mathbf{AB}) = \mathbf{trace}(\mathbf{BA})$$

so **T** is **symmetric**

Here is a variation that will appear later

$$\mathbf{T}(\mathbf{A}, \mathbf{B}) = \mathbf{trace}(\mathbf{A}\mathbf{B}^t),$$

where $\mathbf{B}^t$ denotes the **transpose** of **B.**

Question: On the same space $\mathbf{M}_n(\mathbf{F})$, define

$$\textbf{total}([a_{ij}]) = \sum_{i,j} a_{ij}$$

It is clear that

$$\mathbf{S}(\mathbf{A}, \mathbf{B}) = \textbf{total}(\mathbf{AB})$$

is a bilinear function.

Is it **symmetric**?

Proposition

If **B** *is an alternating multilinear function, then*

$$\mathbf{B}(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = -\mathbf{B}(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n),$$

that is, switching two variables changes the sign of the function.

Proof.

For convenience we assume $\mathbf{B}(u, v)$ has two variables. We must show that $\mathbf{B}(v, u) = -\mathbf{B}(u, v)$. By definition, we have

$$\begin{aligned} \mathbf{B}(u+v, u+v) &= 0, \quad \text{which we expand} \\ &= \mathbf{B}(u, u) + \mathbf{B}(u, v) + \mathbf{B}(v, u) + \mathbf{B}(v, v) \end{aligned}$$

Notice that the first and fourth summands are zero. Thus $\mathbf{B}(u, v) + \mathbf{B}(v, u) = 0$, as desired. □

Here are some additional properties.

Proposition

The set **M** *of all n–linear functions on the vector space* **V** *with values in* **W** *is a vector space. The subsets* **S** *and* **K** *of symmetric and alternating functions are subspaces.*

Proof.

If $\mathbf{B}_1$ and $\mathbf{B}_2$ are (say) symmetric bilinear functions,

$$(c_1\mathbf{B}_1+c_2\mathbf{B}_2)(u,v) = c_1\mathbf{B}_1(u,v)+c_2\mathbf{B}_2(u,v) = c_1\mathbf{B}_1(v,u)+c_2\mathbf{B}_2(v,u),$$

which shows that any linear combination of $\mathbf{B}_1$ and $\mathbf{B}_2$ is symmetric. The argument is similar for alternating functions. □

If $\mathbf{B}$ is bilinear and $2 \neq 0$, we could do as in an early exercise:

$$\mathbf{B}(u, v) = \frac{\mathbf{B}(u, v) + \mathbf{B}(v, u)}{2} + \frac{\mathbf{B}(u, v) - \mathbf{B}(v, u)}{2}$$

that shows that every bilinear function is a [unique] sum of a symmetric and an alternating bilinear function.

It is very easy to create multilinear functions, at least general functions and symmetric ones. Here are a couple of approaches:

- Let $\mathbf{f}_1, \mathbf{f}_2$ and $\mathbf{f}_3$ be linear functions on $\mathbf{V} = \mathbf{F}^3$. Now define

  $$\mathbf{T} : \mathbf{V}^3 \to \mathbf{F}, \quad \mathbf{T}(v_1, v_2, v_3) := \mathbf{f}_1(v_1)\mathbf{f}_2(v_2)\mathbf{f}_3(v_3).$$

  $\mathbf{T}$ is clearly trilinear

- Let $\mathbf{T}$ be a trilinear function on $\mathbf{F}^3$. We get a symmetric function $\mathbf{S}$ by 'mixing up' [symmetrizing] $\mathbf{T}$:

  $$\begin{aligned} \mathbf{S}(v_1, v_2, v_3) &:= \mathbf{T}(v_1, v_2, v_3) + \mathbf{T}(v_2, v_1, v_3) + \mathbf{T}(v_1, v_3, v_2) \\ &+ \mathbf{T}(v_3, v_1, v_2) + \mathbf{T}(v_2, v_3, v_1) + \mathbf{T}(v_3, v_2, v_1) \end{aligned}$$

  If $\mathbf{T}$ is already symmetric, $\mathbf{S} = 6\mathbf{T}$.

Let us begin to see what makes the **determinant** important:

Proposition

The vector space $\mathbf{K}$ *of all skew-symmetric bilinear functions on* $\mathbf{F}^2$ *with values in* $\mathbf{F}$ *has a basis which is the 2-by-2 determinant function.*

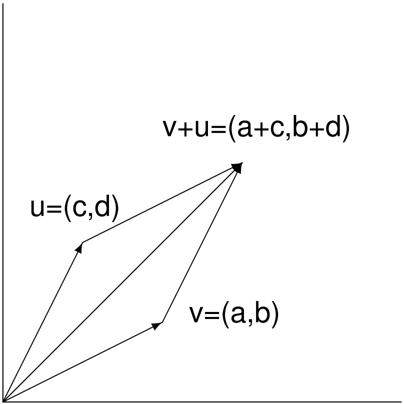
Proof.

1. Let $e_1 = (1, 0)$, $e_2 = (0, 1)$ be the standard basis of $\mathbf{F}^2$.
2. Given any two vectors $u, v \in \mathbf{F}^2$, we can write $u = ae_1 + be_2$, $v = ce_1 + de_2$.
3. If $\mathbf{B} \in \mathbf{K}$, expand $\mathbf{B}(u, v) = \mathbf{B}(ae_1 + be_2, ce_1 + de_2)$:

$$ac\mathbf{B}(e_1, e_1) + ad\mathbf{B}(e_1, e_2) + bc\mathbf{B}(e_2, e_1) + bd\mathbf{B}(e_2, e_2)$$

4. Note that the first and fourth terms are zero and $\mathbf{B}(e_1, e_2) = -\mathbf{B}(e_2, e_1)$. It gives
5. $\mathbf{B}(u, v) = (ad - bc)\mathbf{B}(e_1, e_2) = \mathbf{B}(e_1, e_2)\det(u, v)$

□



Area of parallelogram defined by u and v is $\det(v, u) = ad - bc$

Exercise 1: Prove that the space of all symmetric bilinear functions of $\mathbf{F}^2$ has dimension 3. Note that the space of linear functions

$$\mathbf{T} : \mathbf{F}^2 \times \mathbf{F}^2 \to \mathbf{F}$$

has dimension 4. [This is the dual space of $\mathbf{F}^2 \times \mathbf{F}^2 = \mathbf{F}^4$]. Since bilinear functions are **linear**, the space of symmetric bilinear functions is a subspace and therefore has dimension at most 4. You must show that it has a basis of 3 functions.

Exercise 2:

If **V** is a vector space of dimension n, and **S** and **K** are the spaces of symmetric and skew-symmetric bilinear functions, prove that

$$\dim \mathbf{S} = \binom{n+1}{2}$$
$$\dim \mathbf{K} = \binom{n}{2}$$

Out of order remarks

A quick way to get new multilinear functions from old ones is the following:

If $\mathbf{B} : \mathbf{V} \times \mathbf{V} \to \mathbf{W}$ is a bilinear transformation, and $\mathbf{T} : \mathbf{W} \to \mathbf{Z}$ is a linear transformation, the composite

$$\mathbf{T} \circ \mathbf{B} : \mathbf{V} \times \mathbf{V} \to \mathbf{Z}$$

$$\mathbf{T} \circ \mathbf{B}(u, v) = \mathbf{T}(\mathbf{B}(u, v))$$

is a bilinear transformation.

The most famous bilinear (multi also) is called the **tensor product**,

$$\mathbf{B} : \mathbf{V} \times \mathbf{V} \to \mathbf{V} \otimes \mathbf{V},$$

$$(u, v) \to u \otimes v$$

Mention [but don't write!] some crazy things about this function.

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3-by-3 determinants

Let us explore 'bigger' multilinear functions, like for instance 3-linear ones on $\mathbf{F}^3$. This means that the input is an ordered triple (v_1, v_2, v_3) of vectors. If we pick a basis $\{e_1, e_2, e_3\}$, each of the vectors can be represented in row or column format and the triple can be represented as a matrix

$$[v_1 \mid v_2 \mid v_3] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The point: A 3-linear function $\mathbf{M}$ on $\mathbf{F}^3$ is really a function on 3-by-3 matrices:

$$\mathbf{M} : \mathbf{A} \to \mathbf{M}(\mathbf{A}).$$

Proposition

The vector space **K** *of all skew-symmetric 3-linear functions on* $\mathbf{F}^3$ *with values in* **F** *has a basis which is the 3-by-3 determinant function.*

Proof.

1. Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ be the standard basis of $\mathbf{F}^3$.
2. Given any three vectors $v_1, v_2, v_3 \in \mathbf{F}^3$, we can write $v_i = a_{1i}e_1 + a_{2i}e_2 + a_{3i}e_3$.
3. If $\mathbf{M} \in \mathbf{K}$, expand $\mathbf{M}(v_1, v_2, v_3)$: Note that in all there are 27 terms [fortunately most are zero] of the form
$$\mathbf{M}(a_{j1}e_j, a_{k2}e_k, a_{\ell 3}e_\ell) = a_{j1}a_{k2}a_{\ell 3}\mathbf{M}(e_j, e_k, e_\ell)$$
4. Note that $\mathbf{M}(e_j, e_k, e_\ell) = 0$ when two of the e_j, e_k, e_ℓ are equal.

1. This leaves 6 possible nonzero terms, the coefficients of the scalar $\mathbf{M}(e_1, e_2, e_3)$. They are

$$\begin{aligned}\det(v_1, v_2, v_3) &= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) \\ &- (a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31})\end{aligned}$$

2. Thus

$$\mathbf{M}(v_1, v_2, v_3) = \mathbf{M}(e_1, e_2, e_3) \cdot \det(v_1, v_2, v_3).$$

3. This shows that $\mathbf{M}$ is a multiple of det, so $\dim \mathbf{K} \leq 1$
4. This still requires to check that det is 3-linear and skew-symmetric.

To track the correct sign for the products will require some analysis. In special cases, there are simple rules: To find

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

repeat the first two columns

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

and form the products of the **lines**

$$\begin{matrix} a_{11} & a_{12} & a_{13} & & \\ & a_{22} & a_{23} & a_{21} & \\ & & a_{33} & a_{31} & a_{32} \end{matrix} \qquad \begin{matrix} & & a_{13} & a_{11} & a_{12} \\ & a_{22} & a_{23} & a_{21} & \\ a_{31} & a_{32} & a_{33} & & \end{matrix}$$

Adding the 6 terms, the first 3 are positive, the others negative, gives the determinant.

Questions

1. How to define ‘larger’ determinant functions?
2. What are their properties and the rules of computation?
3. Applications?

To answer the first question, we look at what we got in evaluating det **A**,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The answer was a sum of terms

$$\pm a_{1j} a_{2k} a_{3\ell},$$

with j, k, ℓ distinct, preceded by a $\pm$ sign. The sign is determined as follows: Compare

$$\{1, 2, 3\} \leftrightarrow \{j, k, \ell\}$$

and count the number of **transpositions** required to sort the second list into the first.This number is called the **parity** of the ordered list: even $\rightarrow$ 1, odd $\rightarrow -1$.

For example

$$\{2,3,1\} \to \{1,3,2\} \to \{1,2,3\}$$

took 2 transpositions so its is an **even** permutation. This mean that in the determinant formula $a_{12}a_{23}a_{31}$ appears with $+$.
Quick question: What is the parity of $\{2,3,4,5,6,1\}$?

This would be one path to define n-by-n determinants

$$\det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Add all products

$$\text{signature} \;\; a_{1j_1} a_{2j_2} \cdots a_{nj_n},$$

where $\{j_1, j_2, \ldots, j_n\}$ is a permutation of $\{1, 2, \ldots, n\}$, where its **signature** is $+1$ if the permutation is even, or -1 if it is odd.
This is a very explicit formula but it is long, it has n! [n factorial] terms, a function that grows very fast. [For $n = 100$ our universe has not enough atoms to code the determinant formula, one atom per term!]. If you forgot the signature, and set they all $+1$, you get another function, the **permanent** of the matrix.

Let us try a recursive construction: Given a n-by-n matrix $\mathbf{A}$, For each cell (i, j) consider the submatrix $\mathbf{A}_{ij}$ obtained by deleting the row i and the column j of $\mathbf{A}$. $\mathbf{A}_{ij}$ is an $(n-1)$-by-$(n-1)$ matrix. We will assume that we already have a working definition for determinants in this size, that is $\det \mathbf{A}_{ij}$ is known [it is called the (i, j)-minor]. We also say that the sign, or signature, of the cell (i, j) is $(-1)^{i+j}$. Let us display this data in two arrays [3×3 case for simplicity]:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Finally define the (i, j)-cofactor:

$$c_{ij} = (-1)^{i+j} \det \mathbf{A}_{ij}$$

If **A** is a 2-by-2 matrix,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

the matrix of cofactors is

$$\mathbf{B} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}.$$

Just for the future, observe what you get by multiplying **A** by $\mathbf{B}^t$:

$$\begin{aligned} \mathbf{AB}^t &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix} = \det(\mathbf{A})\mathbf{I}_2 \end{aligned}$$

Curious!

Determinant of a matrix

Definition

The determinant of the $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is the scalar

$$\det \mathbf{A} = a_{11}c_{11} + \cdots + a_{1n}c_{1n} = \sum_{i=1}^{n} a_{1i}c_{1i}.$$

cofactors expansion along row 1:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

Let us see how this works: Given $\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 5 \\ 2 & 6 & 1 \end{bmatrix}$ the matrix of minors and the matrix of cofactors are

$$\begin{bmatrix} -30 & -6 & 24 \\ -17 & -4 & 10 \\ 5 & -2 & -4 \end{bmatrix} \quad \begin{bmatrix} -30 & +6 & 24 \\ * & * & * \\ * & * & * \end{bmatrix} \quad \begin{bmatrix} -30 & +6 & 24 \\ 17 & -4 & -10 \\ 5 & 2 & -4 \end{bmatrix}$$

$$\det \mathbf{A} = 2 \times (-30) + 1 \times 6 + 3 \times 24 = 18$$

Here are two important calculations:

$$\det(\mathbf{I}_n) = \det \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} = 1 \times \det(\mathbf{I}_{n-1}) + 0 \times c_{12} + \cdots 0 \times c_{1n} = 1.$$

More generally, if **A** is lower triangular

$$\det \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = a_{11}a_{22}\cdots a_{nn}.$$

Exercise 3: $\mathbf{A}$ is a 4-by-4 matrix with only 4 nonzero entries (may assume them to be $1, 2, 3, 4$), what are the possible values for $\det \mathbf{A}$?

(Challenge part:) What is the probability that $\det \mathbf{A} = 24$?

Determinant as a volume function

We know already that if u and v are vectors in $\mathbb{R}^2$, defining a parallelogram **P**, $\det[u, v] = \mathrm{area}(\mathbf{P})$.
If we have 3 vectors v_1, v_2, v_3 in $\mathbb{R}^3$, they [usually] define a parallelepiped **P** [usually: means what here?]. One can show that

$$\mathrm{vol}(\mathbf{P}) = |\det[v_1, v_2, v_3]|.$$

Vector Calculus produces the same formula for the higher dimensional analogs.
Question: Do you like Calculus? Define a ball of radius R in $\mathbb{R}^n$ and find its volume and surface areas. [Or ask your other teacher!]

How clever have we been?

We are considering **alternating** n-linear functions on $\mathbf{F}^n$, i.e. functions $\mathbf{T}$ that take as inputs $n - tuples$ $(v_, \ldots, v_n)$ of vectors of $\mathbf{F}^n$. Obviously this is the same as

$$[v_1 \mid \cdots \mid v_n] = [a_{ij}],$$

an $n \times n$ matrix.

We have also proved that the set of all these functions is a vector space of dimension at most 1. It is forced on us to find one of them [nonzero] to have them all. The function $\mathbf{T}$ such that $\mathbf{T}(\mathbf{I}_n) = 1$ will be called **DETERMINANT**.

- Since we defined (a CANDIDATE) determinant recursively,

$$\det \mathbf{A} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j},$$

  we can easily use induction on the size of the matrices to check that this function is n-linear and skew-symmetric.
- There is an apparent drawback in this definition, we are using the cofactors of the first row of the matrix, so legitimate concern is what if we used a different row in this expansion, say row i

$$\det \mathbf{A} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij},$$

- If we call the first function det and the second DET, we proved that the space of all such functions has dimension 1, so one is a scalar multiple of the other

$$\det(\mathbf{A}) = c \cdot \mathrm{DET}(\mathbf{A}).$$

But if we evaluate them at $\mathbf{I}_n$, $\det(\mathbf{I}_n) = 1 = \mathrm{DET}(\mathbf{I}_n)$, so $c = 1$.
- We could also define in terms of the cofactors along a column

$$\det \mathbf{A} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}.$$

- Applying to matrices that are upper triangular [such as the rref of matrices] would be easy

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} = a_{11} a_{22} \cdots a_{nn}.$$

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Determinant of elementary matrices

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = -1 \quad \text{1 transposition}$$

$$\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix} = 1 \quad \text{triangular}$$

$$\det \begin{bmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = b \quad \text{expand along row 1}$$

Proposition

If $\mathbf{E}$ *is an elementary* $n \times n$ *matrix and* $\mathbf{A}$ *is also* $n \times n$*,*

$$\begin{aligned} \det(\mathbf{E} \cdot \mathbf{A}) &= \det(\mathbf{E}) \det(\mathbf{A}) \\ \det(\mathbf{A} \cdot \mathbf{E}) &= \det(\mathbf{E}) \det(\mathbf{A}). \end{aligned}$$

This looks innocuous, surely. But look at the consequence:
We know that given a matrix $\mathbf{A}$ there exists a sequence $\mathbf{E}_1, \ldots, \mathbf{E}_r$ such that $\mathbf{E}_r \cdots \mathbf{E}_1 \mathbf{A} = \mathbf{R} = \mathrm{rref}(\mathbf{A})$. So apply the rule repeatedly, [like in $\det(\mathbf{E}_2 \mathbf{E}_1 \mathbf{A}) = \det(\mathbf{E}_2) \det(\mathbf{E}_1 \mathbf{A}) = \det(\mathbf{E}_2) \det(\mathbf{E}_1) \det(\mathbf{A})$ we get

$$\det(\mathbf{E}_r) \cdots \det(\mathbf{E}_1) \det(\mathbf{A}) = \det(\mathbf{R})$$

Since $\mathbf{R}$ is triangular, its determinant is easy to find, we can get $\det(\mathbf{A})$.

The argument gives the following:

Corollary

If $\mathbf{A}$ *is a n-by-n matrix,* $\det(\mathbf{A}) = 0$ *if and only if* $\operatorname{rank}(\mathbf{A}) < n$*. In other words,* $\mathbf{A}$ *is invertible if and only* $\det(\mathbf{A}) \neq 0$*. Moreover, if* $\mathbf{A}$ *is invertible,* $\det(\mathbf{A}^{-1}) = (\det(\mathbf{A}))^{-1}$*.*

To prove it, we examine the effect of each of the 3 types **E** of elementary matrices: For convenience of [my] writing, we consider column operations: .

- Let $\mathbf{A} = [v_1|v_2|v_3]$ and $\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \det(\mathbf{E}_1) = -1$. Then $\mathbf{AE}_1 = [v_1|v_3|v_2]$, so

$$\det(\mathbf{AE}_1) = -\det(\mathbf{A}) = \det(\mathbf{A})\det(\mathbf{E}_1)$$

- $\mathbf{E}_2 = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det(\mathbf{E}_2) = a$. Then $\mathbf{AE}_1 = [av_1|v_2|v_3]$, so

$$\det(\mathbf{AE}_2) = a\det(\mathbf{A}) = \det(\mathbf{A})\det(\mathbf{E}_2)$$

- $\mathbf{E}_3 = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det(\mathbf{E}_3) = 1$. Then $\mathbf{AE}_3 = [v_1 | v_2 | v_3 + bv_1]$, so

$$\det(\mathbf{AE}_3) = \det[v_1|v_2|v_3] + b\underbrace{\det[v_1|v_2|v_1]}_{=0} = \det(\mathbf{E}_3)\det(\mathbf{A}).$$

Example: Given that the 4×4 matrix $A = [c_1|c_2|c_3|c_4]$ has determinant 3, find the determinant of the matrix

$$B = [c_2 + c_3|c_3 + c_4|c_4 + c_1|c_1 + c_2].$$

Answer: (a) $\det(B) = 0$ Explanation: If you subtract the first from the second column of B, and the third column from the fourth we get [without changing determinants)

$$B = [c_2+c_3|c_3+c_4|c_4+c_1|c_1+c_2] \to [c_2+c_3|-c_2+c_4|c_4+c_1|c_2-c_4].$$

But the last matrix has two linearly independent columns (one is the negative of the other), so its determinant is 0

Product rule

Theorem

If $\mathbf{A}$ *and* $\mathbf{B}$ *are n-by-n matrices,* $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

Proof.

1. We already know that this rule is valid if $\mathbf{A}$ is an elementary matrix $\mathbf{E}$. We also know that there exists a sequence $\mathbf{E}_1, \ldots, \mathbf{E}_r$ of elementary matrices such that

$$\mathbf{E}_r \cdots \mathbf{E}_1 \mathbf{A} = \mathbf{R} = \mathrm{rref}(\mathbf{A})$$

2. If $\mathrm{rank}\,(\mathbf{A}) < n$, we have seen that $\mathrm{rank}\,(\mathbf{AB}) < n$ also, so both $\det(\mathbf{AB})$ and $\det(\mathbf{A})$ are 0 and the formula is fine.
3. Thus we may assume $\mathrm{rank}\,(\mathbf{A}) = n$. But then $\mathbf{R} = \mathbf{I}_n$ and $\mathbf{A}$ is a product of elementary

$$\mathbf{A} = \mathbf{E}^{-1} \cdots \mathbf{E}^{-1}$$

Exercise 1:

Evaluate the determinant of the following matrix:

$$A = \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ e & f & 0 & 0 \\ g & h & 0 & 0 \end{bmatrix}$$

Exercise 2: If the 4×4 matrix $C = [c_1|c_2|c_3|c_4]$ has determinant 1, find the determinant of the matrix

$$B = [c_2 + c_3|c_3 + c_4|c_4 + 2c_1|c_1 + 2c_2].$$

Hint: The columns of B are combinations of the columns of C so we look for a matrix D such that $B = CD$. [There were several approaches.] Then use that $\det B = \det C \det D$.

Exercise 3:

Let **A** be the 4-by-4 matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix}$$

Show [Vandermonde] that
$\det(\mathbf{A}) = (d-a)(d-b)(d-c)(c-a)(c-b)(b-a)$.

Exercise 4: Let **A** be a 3-by-3 matrix with entries $0, 1$ or -1. How big can $\det \mathbf{A}$ be? What if **A** is 4-by-4?

Outline

1. Multilinear Algebra
2. Determinants
3. Determinant of a Product
4. Applications
5. Homework
6. HomeQuiz #5

Cramer's rule

Consider the system of equations $\mathbf{Ax} = \mathbf{b}$:

$$\begin{array}{ccccc} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array}$$

If the system is consistent, the column vector $\mathbf{b}$ of RHS entries can be written as a linear combination of the columns $\mathbf{a}_i$ of the system matrix

$$\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2.$$

Let us replace, for example, the first column of $\mathbf{A}$ by the vector $\mathbf{b}$ and calculate the determinant

$$\det[\mathbf{b}|\mathbf{a}_2] = \det[x_1\mathbf{a}_1 + x_2\mathbf{a}_2|\mathbf{a}_2] = x_1 \det[\mathbf{a}_1|\mathbf{a}_2] + x_2 \underbrace{\det[\mathbf{a}_2|\mathbf{a}_2]}_{=0}$$

and therefore

$$x_1 = \frac{\det[\mathbf{b}|\mathbf{a}_2]}{\det[\mathbf{a}_1|\mathbf{a}_2]}$$

Consider the system of equations $\mathbf{Ax} = \mathbf{b}$:

$$\begin{array}{ccccccc} a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ \vdots & + & \ddots & + & \vdots & & \vdots \\ a_{n1}x_1 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$$

If the system is consistent, the column vector $\mathbf{b}$ of RHS entries can be written as a linear combination of the columns $\mathbf{a}_i$ of the system matrix

$$\mathbf{b} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n.$$

Let us replace, for example, the first column of $\mathbf{A}$ by the vector $\mathbf{b}$ and calculate the determinant

$$\det[\mathbf{b}|\mathbf{a}_2|\cdots|\mathbf{a}_n] = \sum_{i=1}^{n} x_i \underbrace{\det[\mathbf{a}_i|\mathbf{a}_2|\cdots|\mathbf{a}_n]}$$

Observe that

$$\det[\mathbf{a}_i|\mathbf{a}_2|\cdots|\mathbf{a}_n] = 0$$

if $i = 2, 3, \ldots, n$, since the corresponding matrix would have two equal columns. We are left with the term

$$x_1 \det[\mathbf{a}_1|\mathbf{a}_2|\cdots|\mathbf{a}_n] = x_1 \det(\mathbf{A}).$$

Theorem (Cramer's Rule)

Let $\mathbf{Ax} = \mathbf{b}$ *be a n-by-n system of equations. If* $\det \mathbf{A} \neq 0$,

$$x_i = \frac{\det \mathbf{A}_i}{\det \mathbf{A}},$$

where $\mathbf{A}_i$ *is the matrix obtained by replacing the* i*th column of* $\mathbf{A}$ *with the* $\mathbf{b}$ *column.*

Example:

Solve the system of equations

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

for the variable x ONLY.

Answer: We use Cramer's rule: The determinant of the matrix $\mathbf{A}$ of the system is 1. By Cramer's

$$x = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})},$$

where $\mathbf{A}_1$ is the matrix obtained by replacing column 1 of $\mathbf{A}$ by the data vector. Note that $\mathbf{A}_1$ has two identical columns, so $\det(\mathbf{A}_1) = 0$. Thus $x = 0$.

Adjoint and Inverses

Let **A** be a matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

We defined the cofactors of **A** as $c_{ij} = (-1)^{i+j} \det \mathbf{A}_{ij}$, where $\mathbf{A}_{ij}$ is the matrix gotten by removing the row i and the column j of **A**. We can form the cofactors matrix

$$\text{cofactor}(\mathbf{A}) = [c_{ij}].$$

We introduce one additional terminology: The **adjoint** matrix of **A** is

$$\text{adj}\,(\mathbf{A}) = [c_{ij}]^t = \text{transpose of cofactor mat}$$

Theorem

Let **A** *be a n-by-n matrix. Then*

$$\mathbf{A} \cdot \mathrm{adj}(\mathbf{A}) = \mathrm{adj}(\mathbf{A}) \cdot \mathbf{A} = \det(\mathbf{A})\mathbf{I}_n.$$

In particular, if $\det(\mathbf{A}) \neq 0$,

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathrm{adj}(\mathbf{A}).$$

Proof. Let us inspect the entries p_{ij} of the product $\mathbf{A}\mathrm{adj}(\mathbf{A})$. For instance [keeping in mind that we flipped the matrix of cofactors]

$$p_{11} = a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n},$$

which is just the formula for $\det(\mathbf{A})$.

Let us try another entry:

$$p_{12} = a_{11}c_{21} + a_{12}c_{22} + \cdots + a_{1n}c_{2n},$$

This time we are multiplying the elements of row 1 of **A** by the cofactors of the row 2 of **A**. What is this? We argue that is 0: Suppose **B** is the matrix formed as follows: rows $1, 3, 4, \ldots, n$ are the same as in **A**, but row 2 is row 1 repeated

$$\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Of course, $\det(\mathbf{B}) = 0$. If we apply the determinant formula for **B** the along the second row we would get p_{12}. Thus $p_{12} = 0$. In a similar way, we get $p_{ij} = 0$ if $i \neq j$, and $p_{ii} = \det(\mathbf{A})$.

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Homework

Exercise 6: If $\det \mathbf{A} = 1$, show that $\mathrm{adj}\,(\mathrm{adj}\,(\mathbf{A})) = \mathbf{A}$.

Exercise 7: 4.2: 22, 26

Exercise 8: 4.3: $9 -> 15$, 21, 27, 28

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HomeQuiz #5

1. Evaluate the determinant of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ e & f & 0 & 0 \\ g & h & 0 & 0 \end{bmatrix}$$

   If $\det \mathbf{A} \neq 0$, what is $\mathbf{A}^{-1}$?

2. If the 4×4 matrix $C = [c_1|c_2|c_3|c_4]$ has determinant 1, find the determinant of the matrix

$$B = [c_2 + c_3|c_3 + c_4|c_4 + 2c_1|c_1 + 2c_2].$$

3. Let $\mathbf{A}$ be a 3-by-3 matrix with entries $0, 1$ or -1. How big can $\det \mathbf{A}$ be? Do same if $\mathbf{A}$ is 4-by-4. (*Google* Hadamard's Inequality.)