# Math 350: Linear Algebra

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Set 4

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# Today ...

- Matrix representation
- General Matrix Algebra
- Elementary matrices
- Algorithm for inverses
  - ○ Rank of a matrix
  - ○ Basics of systems of equations

# Outline

Let us consider 3 apparently simple functions from the vector space $\mathbf{F}^3$ into itself:

$$\begin{aligned} \mathbf{E}_1(x_1, x_2, x_3) &= (x_3, x_2, x_1) \\ \mathbf{E}_2(x_1, x_2, x_3) &= (x_1, x_2, x_3 + ax_1), \quad a \in \mathbf{F} \\ \mathbf{E}_3(x_1, x_2, x_3) &= (bx_1, x_2, x_3), \quad b \neq 0 \end{aligned}$$

It is very clear that they are linear transformations:

$$\mathbf{E}_i(c_1 v_1 + c_2 v_2) = c_1 \mathbf{E}_i(v_1) + c_2 \mathbf{E}_i(v_2).$$

Although we focused on the first and last coordinates of the vectors, we could pick different slots. They differ slightly from one another: For example, $\mathbf{E}_1$ is its own inverse: $\mathbf{E}_1 \circ \mathbf{E}_1 = \mathbf{I}$:

$$\mathbf{E}_1(\mathbf{E}_1)((x_1, x_2, x_3)) = \mathbf{E}_1(x_3, x_2, x_1) = (x_1, x_2, x_3)$$

The others have inverses still of the same kind.

Recall how we set up the matrix representation of a linear transformation. If $\mathbf{T} = \mathbf{V} \to \mathbf{V}$ is a L.T. and $\mathcal{A} = \{v_1, v_2, v_3\}$ is a basis of $\mathbf{V}$:

$$\begin{aligned} \mathbf{T}(v_1) &= a_{11}v_1 + a_{21}v_2 + a_{31}v_3 \\ \mathbf{T}(v_2) &= a_{12}v_1 + a_{22}v_2 + a_{32}v_3 \\ \mathbf{T}(v_3) &= a_{13}v_1 + a_{23}v_2 + a_{33}v_3 \end{aligned}$$

gives

$$[\mathbf{T}]_{\mathcal{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{31} \end{bmatrix}$$

If we choose the standard basis of $\mathbf{F}^3$, writing the vectors in columnar format we have the matrix representations [using same letters]

$$\mathbf{E}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{E}_1^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix} \qquad \mathbf{E}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & 0 & 1 \end{bmatrix}$$

$$\mathbf{E}_3 = \begin{bmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{E}_3^{-1} = \begin{bmatrix} 1/b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Elementary matrices

Look at how multiplication by these matrices works

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix} \quad \text{rows } 1 \leftrightarrow 3$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} c & b & a \\ f & e & d \\ i & h & g \end{bmatrix} \quad \text{cols } 1 \leftrightarrow 3$$

These are the so-called elementary matrices

# Gaussian moves

These are operations on matrices used in the algorithms to solve systems of linear equations, but have other applications as well. To a $m \times n$ matrix $\mathbf{A}$ these moves are:

- Interchange two rows [columns]
- Multiply one row [column] by a nonzero scalar
- Add to one row [column] a scalar multiple of another

**Proposition**

*Each one of these moves can be effected by pre-multiplying [post-multiplying in the column case]* $\mathbf{A}$ *by an elementary matrix* $\mathbf{E}$: $\mathbf{A} \to \mathbf{EA}$ [$\mathbf{A} \to \mathbf{AE}$].

Question: To what end? To give the answer we first describe a type of matrix **R** [open matrix?] that broadcasts all info it may code: row reduced echelon format:
It is put together from the following two rules:

- If the matrix $\mathbf{R} \neq O$, its leftmost nonzero entry is a 1 on row 1; all other entries in the corresponding column are zero.
- Apply the rule above on the matrix obtained from **R** by deleting row #1 until get the zero matrix/no matrix. The other entries of the columns of **R** with the blue 1 [called pivots] are zero.

$$\begin{bmatrix} 0 & \cdots & 1 & * & \cdots & * \\ 0 & \cdots & 0 & * & \cdots & * \\ \vdots & \ddots & 0 & * & \ddots & * \\ 0 & \cdots & 0 & * & \cdots & * \end{bmatrix}$$

### Theorem

*Given a rectangular $m \times n$ matrix $\mathbf{A}$ with entries in a field $\mathbf{F}$, there exists a finite sequence of elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_r$, such that*

$$\mathbf{E}_r \mathbf{E}_{r-1} \cdots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \mathbf{R},$$

*where $\mathbf{R}$ is in row reduced echelon format. Furthermore, $\mathbf{R}$ is unique, and is denoted $\mathbf{R} = \mathit{rref}(\mathbf{A})$. The rows of $\mathbf{R}$ form a basis of the row space of $\mathbf{A}$.*

Question: What other info is available? A lot! Let us take a quick look.

- $\mathbf{R}$ is unique: explain reader!
- The number of pivots is the rank of $\mathbf{A}$: recall that the range of $\mathbf{A}$ is its column space. That $\mathbf{A}$ and $\mathbf{E}_i\mathbf{A}$ may have different ranges but have the same rank. The positions of the pivots in $\mathbf{R}$ point to the columns of $\mathbf{A}$ that give a basis or its column space.
- What else do we get if column operations are used?

$$\mathbf{A} \to \mathbf{E}_r \cdots \mathbf{E}_1 \mathbf{A} = \mathbf{R} \to \mathbf{R}\mathbf{F}_1\mathbf{F}_2 \cdots \mathbf{F}_s = \left[\begin{array}{c|c} \mathbf{I}_p & O \\ \hline O & O \end{array}\right]$$

$$\mathbf{A} = \mathbf{E}_1^{-1} \cdots \mathbf{E}_r^{-1} \left[\begin{array}{c|c} \mathbf{I}_p & O \\ \hline O & O \end{array}\right] \mathbf{F}_s^{-1} \cdots \mathbf{F}_1^{-1}, \quad p = \mathrm{rank}(\mathbf{A}).$$

# Invertible matrices: Algorithm

It is now clear how decide whether a given $n \times n$ matrix $\mathbf{A}$ is invertible, and in the affirmative to give a procedure to determine $\mathbf{A}^{-1}$:

- $\mathbf{A}$ is invertible if and only if the corresponding linear transformation has rank $n$.
- According to our discussion, this means $\text{rref}(\mathbf{A}) = \mathbf{I}_n$, that is there is a sequence $\mathbf{E}_1, \ldots, \mathbf{E}_r$ of elementary matrices such that
$$\mathbf{E}_r \cdots \mathbf{E}_1 \mathbf{A} = \mathbf{R} = \mathbf{I}_n.$$
- This shows $\mathbf{A}^{-1} = \mathbf{E}_r \cdots \mathbf{E}_1$

The only issue is: How do we get hold of $\mathbf{A}^{-1} = \mathbf{E}_r \cdots \mathbf{E}_1$?

The point is that in the row reduction algorithm, we never get hold of the $\mathbf{E}_i$, we only record what they do. It takes just a tiny bit of cleverness: Write an identity matrix $\mathbf{I}_n$ alongside $\mathbf{A}$, and carry out row reduction on the $n \times 2n$ matrix [that is, we carry out the same row operations on $\mathbf{I}_n$ that we do on $\mathbf{A}$]:

$$[\mathbf{A} \mid \mathbf{I}_n] \to [\mathbf{I}_n \mid \mathbf{B}]$$

We are done when

$$\begin{aligned} \mathbf{E}_r \cdots \mathbf{E}_1 \mathbf{A} &= \mathbf{I}_n \\ \mathbf{E}_r \cdots \mathbf{E}_1 \mathbf{I}_n &= \mathbf{B} \end{aligned}$$

That is $\mathbf{B} = \mathbf{A}^{-1}$.

**Exercise:** Determine whether the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

is invertible. If so, find its inverse. We must do Gaussian elimination

$$[\mathbf{A}|\mathbf{I}_4] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \mathbf{R} = [\underbrace{\mathbf{I}_4}_{??}|\mathbf{B}]$$

If we get the identity in the first blook was because $\mathbf{E}_r \cdots \mathbf{E}_1 \mathbf{A} = \mathbf{I}_4$. Note that $\mathbf{E}_r \cdots \mathbf{E}_1 \mathbf{I}_4 = \mathbf{B}$, so $\mathbf{B} = \mathbf{A}^{-1}$. We actually got

$$\mathbf{A}^{-1} = \frac{1}{3}\left[\begin{array}{cccc} 1 & 1 & 1 & -2 \\ 1 & 1 & -2 & 1 \\ 1 & -2 & 1 & 1 \\ -2 & 1 & 1 & 1 \end{array}\right].$$

# Little puzzle

$$\mathbf{A} = \begin{bmatrix} 46 & 55 & 208 & -502 \\ 37 & 22 & 48 & 316 \\ 708 & 98 & 76 & 99 \\ -64 & 808 & 23 & 106 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathrm{rank}(\mathbf{B}) = 4$$

**B** is invertible—it implies **A** is also invertible!!! Explain or else...

# Outline

# Rank of a matrix

We already know the basics of rank: If $\mathbf{T}$ is a linear transformation $\mathbf{T} : \mathbf{V} \to \mathbf{W}$,

$$\text{rank}(\mathbf{T}) = \dim \mathbf{T}(\mathbf{V}) = \dim \text{image of } \mathbf{T}.$$

We are going to look at various properties of rank, when we consider products **AB** of matrices. It is more like a series of uncomplicated observations once we look at a matrix **A** as the linear transformation

$$\mathbf{L_A}(v) := \mathbf{A} \cdot v.$$

$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{L_A})$= dimension colspace of **A**= $\mathbf{A}(\mathbf{V})$: If $[c_1 \mid c_2 \mid \cdots \mid c_n]$ are the columns of **A**, range $\mathbf{A} = \text{span}\{c_1, \ldots, c_n\}$.

# Matrices of rank one

Let us gives a neat description of the matrices of rank 1. Let **A** be a $m \times n$ matrix given by its column vectors

$$\mathbf{A} = [v_1 | v_2 | \cdots | v_n].$$

If **A** has rank 1, the $v_i$ span a one-dimensional subspace of $\mathbf{F}^m$, in particular there is a vector $v$ and scalars $c_i$, $i = 1, \ldots, n$ such that $v_i = c_i v$ for all $i$

$$\mathbf{A} = [c_1 v | c_2 v | \cdots | c_n v].$$

If we write $v = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$, we have

## Proposition

*Any matrix* $\mathbf{A}$ *of rank one can be written*

$$\mathbf{A} = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix},$$

*that is, it the matrix product of a column matrix by a row matrix.*

# Ranks of products

Given are two matrices $\mathbf{A}, \mathbf{B}$ and the corresponding linear transformations

$$\mathbf{V} \xrightarrow{\mathbf{L_A}} \mathbf{W} \xrightarrow{\mathbf{L_B}} \mathbf{Z}$$

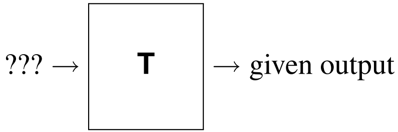
1. $\dim \mathbf{T}(\mathbf{V}) \leq \dim \mathbf{V}$ always: a L.T. never increases the dimension
2. The image of $\mathbf{B} \cdot \mathbf{A}$ is $\mathbf{B}(\mathbf{A}(\mathbf{V}))$, so $\mathrm{rank}(\mathbf{BA}) \leq \inf\{\mathrm{rank}(\mathbf{A}), \mathrm{rank}(\mathbf{B})\}$.
3. If $\mathbf{A}$ is invertible, $\mathrm{rank}(\mathbf{BA}) = \mathrm{rank}(\mathbf{B})$
4. If $\mathbf{B}$ is invertible, $\mathrm{rank}(\mathbf{BA}) = \mathrm{rank}(\mathbf{A})$
5. If $\mathbf{E}$ is an elementary matrix, $\mathrm{rank}(\mathbf{EB}) = \mathrm{rank}(\mathbf{B})$

# Outline

We have described a linear system of equations in the following manner:
Let **T** be a linear transformation of source **V** and target **W**,

$$\mathbf{T} : \mathbf{V} \to \mathbf{W}.$$

**Problem:** Given $w \in \mathbf{W}$ is there $v \in \mathbf{V}$ such $\mathbf{T}(v) = w$? Such $v$ is called **a** solution, or a special solution.



1. Do solutions exist? The answer, in the affirmative case [called CONSISTENT] carries consequences to the next questions.
2. If solutions exist, what is the nature of the set of solutions?
3. Among the solutions, which is the best?
4. How do we find these things anyway?

# Homogeneous systems

These are the systems of the form

$$\mathbf{T}(\mathbf{x}) = O.$$

They are always consistent: $\mathbf{T}(O) = O$.
The solution set **SolSet** is the **nullspace** of $\mathbf{T}$:

$$N(\mathbf{T}) = \{v \in \mathbf{V} \mid \mathbf{T}(v) = O.\}$$

$N(\mathbf{T})$ is a subspace of $\mathbf{V}$, and $\dim N(\mathbf{T}) = \dim \mathbf{V} - \mathrm{rank}(\mathbf{T})$.

# Non-Homogeneous systems

These are the systems of the form

$$\mathbf{T}(\mathbf{x}) = w, \quad w \neq O.$$

They may or may not be consistent. If $\mathbf{x}_0$ is a solution, $\mathbf{T}(\mathbf{x}_0) = w$, then any other solution $\mathbf{x}$, $\mathbf{T}(\mathbf{x}) = w$, has the property

$$\mathbf{T}(\mathbf{x} - \mathbf{x}_0) = \mathbf{T}(\mathbf{x}) - \mathbf{T}(\mathbf{x}_0) = w - w = O.$$

Thus $\mathbf{x} - \mathbf{x}_0 \in N(\mathbf{T})$, and consequently the solution set **SolSet**

$$\mathbf{SolSet} = \mathbf{x}_0 + N(\mathbf{T})$$

# Best solution

We will come to this topic next month. The point is the following: Suppose, for illustration, that **V** is some real space and the solution set

$$\mathbf{SolSet} = \mathbf{x}_0 + N(\mathbf{T})$$

has dimension 2, that is $N(\mathbf{T})$ is a plane. There are lots of vectors in **SolSet** and may want the **smallest** one.

# Finding the solution

- If **V** and **W** are spaces of **F**-tuples, $\mathbf{T}(\mathbf{x}) = w$, is the same as

$$\mathbf{A} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix},$$

  and we use the Gaussian algorithm or a variant.
- Some systems have features from other fields, requiring specialized techniques [e.g. certain diff eqs].
- Often, by picking coordinates, converts the problem into a standard one. Next we look at an example.

# Example

Let $\mathbb{R}_3[x]$ the set of real polynomials of degree at most 3. Find the **SolSet** of the equation

$$f'(x) + f(x+1) - f(x) = x^2 + x.$$

Is this a linear equation? Sure: the mapping $\mathbf{T}(f(x)) = f'(x) + f(x+1) - f(x)$ is a L.T.
Let us find a matrix representation for **T**. We are going to use the basis $\{1, x, x^2, x^3\}$:

$$\begin{aligned}
\mathbf{T}(1) &= 1' - 1 + 1 = 0 \\
\mathbf{T}(x) &= x' + (x+1) - x = 2 \\
\mathbf{T}(x^2) &= (x^2)' + (x+1)^2 - x^2 = 2x + (x^2 + 2x + 1) - x^2 = 4x + 1 \\
\mathbf{T}(x^3) &= (x^3)' + (x+1)^3 - x^3 = 3x^2 + (x^3 + 3x^2 + 3x + 1) - x^3 \\
&= 6x^2 + 3x + 1
\end{aligned}$$

The matrix formulation is:

$$\left[\begin{array}{cccc|c} 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 4 & 3 & 1 \\ 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

System is consistent, **SolSet**$\neq \emptyset$.

# Outline

# Homework

Section 3.1: 2
Section 3.2: 6a, 6f, 14, 18, 22

# Outline

# HomeQuiz #4

There are 4 Problems [in 3 frames]

1. (3 pts) Let $\mathbf{M}_2(\mathbf{F})$ be the vector space of all 2-by-2 matrices over the field **F**. Fix a matrix, say, $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and define the function **T** such that for any matrix $\mathbf{B} \in \mathbf{M}_2(\mathbf{F})$,

$$\mathbf{T}(\mathbf{B}) = \mathbf{AB} - \mathbf{BA}.$$

(a) Prove that **T** is a linear transformation.

(b) Show that the kernel of **T** is nonzero.

(c) Find a matrix representation of **T**.

2. (2 pts) Let **A** and **B** be $n \times n$ matrices. Prove that **AB** and **BA** have the same *trace*.

3. (3 pts) Find bases for the following subspaces of $\mathbf{F}^5$:

$$W_1 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbf{F}^5 : a_1 - a_3 - a_4 = 0\}$$

and

$$W_2 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbf{F}^5 : a_2 = a_3 = a_4 \quad \text{and} \quad a_1 + a_5 = 0\}.$$

What are the dimensions of $W_1$ and $W_2$? Argue that $W_1 \cap W_2 \neq (O)$.

4. (2 pts) Find all the real values for $t$ for which the resulting system of equations (a) has no solution, (b) a unique solution, and (c) infinitely many solutions.

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 2 \\ x & + & 2y & + & z & = & 3 \\ x & + & y & + & (t^2-5)z & = & t \end{array}$$

# Outline

# Oldies from Hourlies

Sample questions from old hourlies:

1. Given that a linear transformation $\mathbf{T} : \mathbb{R}^4 \to \mathbb{R}^3$ is represented by the matrix (in terms of the standard bases of $\mathbb{R}^4$ and of $\mathbb{R}^3$)

$$\begin{bmatrix} 2 & 1 & -5 & 4 \\ -1 & 1 & 3 & -5 \\ 1 & 2 & 0 & -1 \end{bmatrix}$$

(a) Find a basis of the nullspace of $T$.

(b) Find a basis of the range of $T$.

(c) Explain why any linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ must have a nullspace $\neq (O)$.

2. In $\mathbb{R}^3$, let $v_1 = (1, 1/2, 1/3)$, $v_2 = (1/2, 1/3, 1/4)$, $v_3 = (1/3, 1/4, 1/5)$.

(a) Prove that these vectors form a basis of the space.

(b) Find the coordinates of $v = (1, 1, 1)$ in terms of this basis.

3. Let $W$ be the subset of all real $2 \times 2$ matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + b + c + d = 0$.

(a) Show that $W$ is a subspace.

(b) Find a basis of $W$.

4. Let $\mathbf{V}$ be the set of all real polynomials of degree at most 3.

(a) Prove that the function $\mathbf{T} : \mathbf{V} \to \mathbf{V}$

$$\mathbf{T}(f) = f'' - f' + f,$$

is a linear transformation.

(b) Find a matrix representation of $\mathbf{T}$.

(c) Show that the nullspace of $\mathbf{T}$ is $(O)$.

(d) Prove that for any $g \in \mathbf{V}$, the differential equation

$$y'' - y' + y = g$$

has a solution in $\mathbf{V}$.

5. If $\mathbf{T} : \mathbf{V} \to \mathbf{W}$ is a linear transformation of vector spaces:

(a) Explain the meaning of **T** is: (i) **one-one**; (ii) **onto**; **isomorphism.**

(b) In each of the 3 cases, give an example **which is not an example of the other cases.**

(c) Prove that all vector spaces of the same dimension *n* are isomorphic.

6. (a) Define a linear transformation $\mathbf{T} : \mathbf{F}^9 \to \mathbf{F}^4$ of rank 3.

(b) Prove that for any two linear transformations $U, V : F^9 \to F^4$, we must have that

$$\text{nullspace of } U \cap \text{nullspace of } V \neq (O).$$

7. Given a linear transformation $T : V \to W$ of finite dimensional vector spaces, prove that

$$\text{nullity } T + \text{rank } T = \dim V.$$

Begin by explaining the notions of *nullity* and of *rank* of a linear transformation.

8. Let $V$ be the set of all $n \times n$ matrices over a field $F$.

(a) Fix a matrix $A$ and define the function $T : V \to V$ by $T(B) = AB - BA$. Prove that $T$ is a linear transformation.

(b) Show that the nullspace of $T$ is different from $(O)$.

9. Let $T$ be the linear operator on $\mathbb{R}^3$ defined by

$$T(x, y, z) = (-2x + y, 3x + z, -x + 2y + 4z).$$

(a) What is the matrix of $T$ in the standard basis of $\mathbb{R}^3$?

(b) What is the matrix of $T$ in the ordered basis

$$\{\alpha_1, \alpha_2, \alpha_3\}$$

where

$$\alpha_1 = (1, 0, 1), \qquad \alpha_2 = (-1, 2, 1), \qquad \text{and } \alpha_3 = (2, 1, 1)?$$

(c) Prove that $T$ is invertible and give a rule for $T^{-1}$ like the one which defines $T$.

10. Define succintly but clearly the following basic notions:
(a) A field **F**.
(b) A vector space **V** over the field **F**.
(c) A basis of the vector space **V**.
(d) Explain the notion of coordinates of a vector $v$ with respect to a basis $\mathcal{B}$.
(e) Prove that a vector space **V** cannot have a basis with 2 elements and another basis with 3 elements. Comment on why this [in general] allows one to define the *dimension* of a vector space.