Math 350: Linear Algebra

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Set 2

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Outline

A linear system of equations arises as follows: Let $v_1, \ldots, v_n$ be vectors of a vector space $\mathbf{V}$. Given another vector $v \in \mathbf{V}$, is v in the span of the v_i?
The question asks, is v a linear combination of the v_i? In other words, can we solve for scalars x_i the equation

$$v = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$$

Let us examine some examples.

Consider the vectors $v_1 = (1, 2, 3)$, $\quad v_2 = (2, 1, 4)$ and $v = (1, 5, 5)$ of $\mathbf{F}^3$. The condition $v = x_1 v_1 + x_2 v_2$ can be recast as the system of linear equations

$$\begin{array}{rcrcl} x_1 & + & 2x_2 & = & 1 \\ 2x_1 & + & x_2 & = & 5 \\ 3x_1 & + & 4x_2 & = & 5 \end{array}$$

Applying the Gaussian algorithm

$$\left.\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & 5 \\ 3 & 4 & 5 \end{array}\right. \quad \rightarrow \quad \left.\begin{array}{cc|r} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right.$$

So $x_1 = 3$, $x_2 = -1$.

Let us show the following: Every polynomial over $\mathbb{R}$ of degree at most two is a linear combination of the polynomials $p_1 = 1 + x + x^2$, $p_2 = 1 + 2x + 4x^2$ and $p_3 = 1 + 3x + 9x^2$. This means that we should be able to solve any relation of the form

$$a + bx + cx^2 = x_1p_1 + x_2p_2 + x_3p_3.$$

Matching coefficients of the powers of x, we must solve

$$\left.\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 2 & 3 & b \\ 1 & 4 & 9 & c \end{array}\right.$$

Gaussian elimination will show that the numerical matrix has rank 3, so the system can be solved for all choices of a, b, c.

Sometimes problems of this kind cannot be solved in this manner.

Exercise 1: Show that e^{3x} is not a linear combination of e^x and e^{2x}.

Solution: Suppose otherwise, that is we have real numbers a, b such that

$$e^{3x} = ae^x + be^{2x}.$$

Setting $x = 0$, we get the equation $1 = a + b$. Taking derivatives and setting $x = 0$, we get another equation $3 = a + 2b$. Taking second derivatives and setting $x = 0$ we get yet another equation $9 = a + 4b$.

The first two equations give $a = -1$, $b = 2$, which do not work for the third equation.

The following exercise yields to the same trick [but has a much better approach, using integrals instead]

Exercise 2: Prove that $\sin 2x$ is not a linear combination of $\sin x$, $\cos x$ and $\cos 2x$.

Gaussian algorithm

Consider the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n. \end{aligned}$$

over the field **F**.
The consistency [or existence of solutions] means that

$$\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} + \cdots + x_m \begin{bmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{bmatrix}$$

The general method to deal with this issue is Gaussian elimination. A first step is a representation of the system of equations by a matrix.

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & b_n \end{array}\right)$$

The system is simpler if it has a triangular shape [echelon] like

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} & b'_1 \\ 0 & a_{22} & a_{23} & \cdots & a_{2m} & b'_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{nm} & b'_n \end{array}\right)$$

Gaussian moves

That it is possible to pass to another system of linear with these properties but with the same solutions [an equivalent system] is a consequence of directed application of three reduction rules/elementary row operations:

- Interchange the order of two equations
- Multiply one equation by a nonzero scalar
- Add to one equation a scalar multiple of another

Obviously none of these reductions changes the solutions of the system [each is reversible].

Row reduced echelon matrix

$$\begin{bmatrix} 1 & 0 & 0 & a & b \\ 0 & 1 & 0 & c & d \\ 0 & 0 & 1 & e & f \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 4 & -1 & 4 \\ -2 & 1 & 5 \end{bmatrix} \xrightarrow{-2r_1+r_2} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ -2 & 1 & 5 \end{bmatrix} \xrightarrow{r_1+r_3} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2-2r_3} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1-r_3} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1+r_2}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem

Any matrix $n \times m$ matrix with entries in a field **F**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix}$$

can, after a finite sequence of Gaussian moves, be transformed into a [unique] matrix in row reduced echelon form

$$rref(A) = \begin{bmatrix} 1 & 0 & 0 & \cdots & a'_{1m} \\ 0 & 1 & 0 & \cdots & a'_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots \end{bmatrix}$$

There are several useful consequences [corollaries]:

- The elementary row operations are linear combinations in the row space of the matrix A. The nonzero rows of rref(A) span the row space of A.
- The columns of A where the pivots occur span the column space of A. [Note that the column space of A and of rref(A) are usually different.]

Outline

Linear dependence

Definition

A set of vectors $v_1, \ldots, v_m$ of a vector space **V** is linearly dependent if there is a relation

$$c_1 v_1 + c_2 v_2 + \cdots + c_m v_m = O,$$

where one of the scalars c_i is $\neq 0$.

This means simply: If, say, $c_1 \neq 0$,

$$v_1 = (-c_2/c_1)v_2 + \ldots + (-c_m/c_1)v_m,$$

that is, one of the vectors is a linear combination of the other vectors.

It is straightforward to set up a procedure to decide whether a set of vectors of $\mathbf{F}^n$ are linearly dependent. Say $v_1, \ldots, v_m \in \mathbf{F}^n$; we must see whether there is a nonzero solution [i.e. one of the x_i is nonzero] for

$$x_1 v_1 + x_2 v_2 + \cdots + x_m v_m = O.$$

We set it up in matrix form

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & 0 \end{array}\right]$$

and carry out Gaussian elimination.

Summary: If the rank of the matrix is $< m$, the vectors are linearly dependent. For instance, if $v_1, v_2, v_3, v_4 \in \mathbf{F}^3$, they are always linearly dependent since the matrix is 3×4 and so has rank at most 3 [at most 3 pivots].

Linear independence

Definition

A set of vectors $v_1, \ldots, v_m$ of a vector space $\mathbf{V}$ is linearly independent if whenever

$$c_1 v_1 + c_2 v_2 + \cdots + c_m v_m = O,$$

then all $c_i = 0$.

The method of Gaussian elimination permits us to decide whether any set of vectors $v_1, \ldots, v_m \in \mathbf{F}^n$ is linearly independent or not.

If we set up the vectors as column vectors, it will also tell us how to express some column vectors [if any] as a linear combination of the others.

The method will not work in all vector spaces. Let us examine one exercise

Exercise 3: Let $r_1, \ldots, r_n$ be distinct real numbers. Prove that the functions [vectors!] $e^{r_1 x}, \ldots, e^{r_n x}$ are linearly independent.

Solution: We solved already a special case. Suppose c_i are real numbers such that

$$c_1 e^{r_1 x} + \cdots + c_n e^{r_n x} = 0.$$

We will argue that all $c_i = 0$.

The trick is the following: We set $x = 0$ and get a scalar equation for the c_i

$$c_1 + \cdots + c_n = 0,$$

then take the derivative and set $x = 0$, all the way to the $n - 1$ derivative.

What we get is a system of equations

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The matrix of the system is invertible [Vandermonde] so $c_i = 0$. [This uses that the r_i are distinct.] Calculus gives variations to this approach which is only good to check whether the vectors are linearly independent.
Recall [n=3]

$$\det \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{bmatrix} = (r_2 - r_1)(r_3 - r_2)(r_3 - r_1)$$

Outline

We shall discuss distinguished sets of vectors of a vector space **V**. It has to do with bringing some efficacy to the calculus of vectors.

Definition

An ordered set of vectors $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis of **V** if it meets the two conditions:

- **V** is spanned by $\mathcal{B}$, that is any vector v of **V** is a linear combination

$$v = a_1 v_1 + \cdots + a_n v_n.$$

- The v_i are linearly independent.

Together these two conditions mean that if

$$v = b_1 v_1 + \cdots + b_n v_n,$$

is another linear combination,

$$v - v = 0 = (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n,$$

and therefore $a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$, and v is a UNIQUE linear combination of the v_i.

This implies that the vector v is completely determined by the basis that the scalars a_i: they are often called the coordinates of v and written

$$[v]_{\mathcal{B}} = (a_1, \ldots, a_n).$$

If u is another vector,

$$u = b_1 v_1 + \cdots + b_n v_n,$$

then

$$[v + u]_{\mathcal{B}} = (a_1 + b_1, \ldots, a_n + b_n),$$

$$[cv]_{\mathcal{B}} = c(a_1, \ldots, a_n).$$

Thus: a basis $\mathcal{B}$ provides the means for identifying a vector space $\mathbf{V}$ to the vector space $\mathbf{F}^n$.

A procedure to find a basis of a span

If $v_1, \dots, v_m$ are vectors of $\mathbf{F}^n$, a basis for their span can be obtained in two different ways. First, let

$$A = [v_1 | v_2 | \cdots | v_m]$$

be a matrix made up of the v_i as column vectors. Now find rref(A) to determine the columns where the pivots are. The v_i of the corresponding columns of A is the desired basis.

Second, set up the matrix B with the v_i as row vectors. The nonzero rows of rref(B) is a basis of the span.

Dimension of a vector space

We are going to derive several properties of this notion. We begin with

Theorem

Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ and $\mathcal{C} = \{u_1, \ldots, u_m\}$ be two bases of the vector space $\mathbf{V}$. Then $n = m$, that is all bases of $\mathbf{V}$ have the same cardinality. This number is called the dimension of $\mathbf{V}$.

The proof is an elegant argument; it will adapt to all vector spaces, even those with infinite bases. Let us give a special proof first.

Suppose we have two bases, $\{v_1, v_2, v_3\}$ and $\{u_1, u_2\}$.

1. Consider the set obtained by adding v_1 to the front of $u_1, u_2, \rightarrow \{v_1, u_1, u_2\}$:
2. This is not a basis because $\{u_1, u_2\}$ spans **V** and therefore v_1 can be written as $v_1 = c_1 u_1 + c_2 u_2$
3. Say $c_1 \neq 0$, so that $u_1 = (1/c_1)v_1 + (-c_2/c_1)u_2$
4. This shows that since **V** is spanned by u_1 and u_2, **V** is also spanned by v_1 and u_2
5. Now consider v_2. It can be written $v_2 = d_1 v_1 + d_2 u_2$, since $\{v_1, u_2\}$ spans **V**
6. d_2 cannot be zero as otherwise $v_2 = d_1 v_1$ and $\{v_1, v_2\}$ would be linearly dependent, but $\{v_1, v_2\}$ is part of a basis.
7. Thus, if $d_2 \neq 0$ we have $u_2 = (-d_1/d_2)v_1 + (1/d_2)v_2$.
8. Therefore since $\{v_1, u_2\}$ spans **V**, this equation shows that $\{v_1, v_2\}$ spans **V**, so would be a basis. This is a conbtradiction since $\{v_1, v_2\}$ is not a basis.

We will argue that $n \leq m$. Let us move the first vector of $\mathcal{B}$, v_1 to the front of $\mathcal{C}$,

$$v_1, u_1, u_2, \ldots, u_m.$$

Since $\mathcal{C}$ is a spanning set of $\mathbf{V}$, v_1 is a linear combination

$$v_1 = c_1 u_1 + \cdots + c_m u_m.$$

One of the $c_i \neq 0$, say $c_1 \neq 0$. We write

$$u_1 = (1/c_1)v_1 + (-c_2/c_1)u_2 + \cdots + (-c_m/c_1)u_m.$$

This relation implies that the set $v_1, u_2, \ldots, u_m$, in which we replaced u_1 by v_1 will also span $\mathbf{V}$.
Now we are going to insert v_2 in this list,

$$v_1, v_2, u_2, u_3, \ldots, u_m$$

and argue that we can delete another u_i and still get a spanning set with m elements.

Since $\{v_1, u_2, u_3, \ldots, u_m\}$ is a spanning set,

$$v_2 = c_1 v_1 + c_2 u_2 + \cdots + c_m u_m.$$

Since v_1 and v_2 are part of a same basis, they are linearly independent and so we must have one of $c_2, \ldots, c_m$ nonzero. Say $c_2 \neq 0$. Then as above we write u_2 as a linear combination of

$$v_1, v_2, u_3, \ldots, u_m,$$

a spanning set of m elements.
We go on like this until all v_i have been inserted and an equal number of u_i have been deleted. This shows $n \leq m$. Reversing their roles, woukld give $m \leq n$.

- $\dim \mathbf{M}_n(\mathbf{F}) = n^2$
- $\dim \mathbf{S} = \binom{n+1}{2}$: symmetric matrices

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

- $\dim \mathbf{U}_n = \binom{n+1}{2}$: upper triangular
- $\dim \mathbf{D}_n = n$: diagonal

- A vector space **V** of dimension 0 consists of $\{O\}$ alone.
- A vector space **V** of dimension 1 is called a line it consists of all multiples of any of its nonzero elements v,

$$\mathbf{V} = \{cv \mid c \in \mathbf{F}\}.$$

- A vector space **V** of dimension 2 is called a plane it consists of all linear combinations of any two elements v_1, v_2, with neither a multiple of the other,

$$\mathbf{V} = \{c_1 v_1 + c_2 v_2 \mid c_1, c_2 \in \mathbf{F}\}.$$

Theorem

Let $\mathbf{V}$ *be a vector space of dimension n and let* $\mathcal{C} = \{u_1, \ldots, u_m\}$ *be a set of linearly independent vectors. Then* $m \leq n$*, with equality if and only if the* u_i *form a basis of* $\mathbf{V}$*.*

Proof.

Let $\{v_1, \ldots, v_n\}$ be a basis of $\mathbf{V}$. If we insert v_1 in the other set,

$$u_1, \ldots, u_m, v_1$$

it may increase the span of $\mathcal{C}$–and we obtain a set of $m+1$ linearly independent vectors–or it does not change the span of $\mathcal{C}$, that is, v_1 is a combination of the u_i and we delete it. In either case we proceed with v_2 in the same manner. In this way we end up with a set of linearly independent vectors [all u_i plus some of the v_j] spanning $\mathbf{V}$, that is, with a basis. □

Corollary

Let $\mathbf{S}$ *be a subspace of a vector space* $\mathbf{V}$*. Any basis* $\mathcal{B} = \{u_1, \ldots, u_m\}$ *of* $\mathbf{S}$ *can be extended to a basis of* $\mathbf{V}$*. In particular,* $\dim \mathbf{S} \leq \dim \mathbf{V}$ *with equality if and only if* $\mathbf{S} = \mathbf{V}$*.*

Exercise 4: Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be subspaces of the vector space $\mathbf{V}$. Prove that

$$\dim \mathbf{S}_1 + \dim \mathbf{S}_2 = \dim(\mathbf{S}_1 \cap \mathbf{S}_2) + \dim(\mathbf{S}_1 + \mathbf{S}_2).$$

Hint: Begin by picking a basis for $\mathbf{S}_1 \cap \mathbf{S}_2$, and use it [applying the Corollary] to build bases for $\mathbf{S}_1$ and $\mathbf{S}_2$. Then ...

Exercise 5: Prove that every vector space has a basis. [Book gives a discussion.] Note that we only proved this for vector spaces spanned by a finite number of vectors.

Spaces of polynomials

Let $\mathbf{F}$ be a field and $\mathbf{F}_n[x]$ be the set of all polynomials over $\mathbf{F}$ of degree at most n,

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

This is a vector space spanned by the polynomials

$$1, x, x^2, \ldots, x^n.$$

Since they are linearly independent, $\dim \mathbf{F}_n[x] = n + 1$. This is a very nice basis but for some applications it is not great.

Let us consider another famous basis.

Lagrange interpolation polynomials

One of the best known polynomials arises as follows: [assume $\mathbf{F} = \mathbb{R}$] Let $P_1 = (c_1, b_1)$ and $P_2 = (c_2, b_2)$ be two points not on a vertical line. The first degree polynomial that passes through them is

$$f(x) = b_1 + \frac{b_2 - b_1}{c_2 - c_1}(x - c_1).$$

Suppose we ask the question: what is the polynomial, of degree at most n, whose graph passes through the $n + 1$ points $P_i = (c_i, b_i)$, where the c_i are distinct? That is, we look for a polynomial such that

$$f(c_i) = b_i, \quad i = 1 \dots n + 1.$$

Let us define another basis for $\mathbf{F}_n[x]$ which very appropriate here, the so-called Lagrange polynomials: Set

$$f_i(x) = \frac{\prod_{j \neq i}(x - c_j)}{\prod_{j \neq i}(c_i - c_j)}.$$

If $n = 1$,

$$f_1(x) = \frac{x - c_2}{c_1 - c_2}, \quad f_2(x) = \frac{x - c_1}{c_2 - c_1}$$

Note each $f_i(x)$ is a polynomial of degree n. Furthermore,

$$\begin{aligned} f_i(c_i) &= 1 \\ f_i(c_j) &= 0, \quad j \neq i. \end{aligned}$$

Proposition

$f_1(x), f_2(x), \ldots, f_{n+1}(x)$ are linearly independent. Therefore they form a basis of $\mathbf{F}_n[x]$.

To prove it, suppose there is some relation

$$\sum_{1 \le i \le n+1} a_i f_i(x) = 0, \quad a_i \in \mathbb{R}.$$

We claim all $a_i = 0$. To see see, it suffices to evaluate the summation at each c_j:

$$\sum_{1 \le i \le n+1} a_i f_i(c_j) = a_j = 0.$$

This completes the proof.

We can now write the explicit polynomial that passes through the points (c_j, b_j):

$$f(x) = \sum_{1 \le i \le n+1} b_i f_i(x).$$

Check: $f(c_j) = b_j$!

For those who like to check: Here is the equation of the line passing through two points

$$b_1 + \frac{b_2 - b_1}{c_2 - c_1}(x - c_1).$$

and here is the Lagrange polynomial

$$b_1 \frac{x - c_2}{c_1 - c_2} + b_2 \frac{x - c_1}{c_2 - c_1}$$

they are the same

Outline

Fancy Vector Spaces

Let us show one powerful method to create vector spaces. We will consider a very simple setting that contains the main ingridients of the method.

Let $\mathbb{R}^2$ be the usual real plane and let **L** be a line passing through the origin. [Carry an example in your mind.] **L** is a subspace of $\mathbb{R}^2$.

For any vector $v \in \mathbb{R}^2$, $v + \mathbf{L}$ is the set obtained by translating **L** by v. It is a line parallel to **L**. We are going to denote it $\mathbf{L}_v$ and the set of all such such lines we denote by **V**= all lines parallel to **L**.

A feature of the notation $\mathbf{L}_v$ is the following. Suppose $u \in \mathbf{L}$. Then $\mathbf{L}_u = \mathbf{L}_O = \mathbf{L}$. More generally,

$$\mathbf{L}_v = \mathbf{L}_{v+u}$$

v is said to be a representative of $\mathbf{L}_v$, but the observation says that $v + u$ is also a representative of $\mathbf{L}_v$. Essentially any vector in $\mathbf{L}_v$ serves as its representative.

This will be cause for confusion!

Let us define an ‘addition’ for this set of lines. We are going to use the ordinary ‘$+$’ when ‘$\oplus$’ or ‘$\otimes$’ might be more cautious. For any two lines $\mathbf{L}_v$ and $\mathbf{L}_w$

$$\mathbf{L}_v + \mathbf{L}_w := \mathbf{L}_{v+w}.$$

For instance $\mathbf{L}_v + \mathbf{L}_O = \mathbf{L}_v$.

Proposition

This composition does not depend of the representatives used, that is if $\mathbf{L}_v = \mathbf{L}_{v'}$ *and* $\mathbf{L}_w = \mathbf{L}_{w'}$ *then*

$$\mathbf{L}_v + \mathbf{L}_w = \mathbf{L}_{v'} + \mathbf{L}_{w'}.$$

It is obvious now that this composition is commutative and associative. The line $\mathbf{L}$ plays the role of O: $\mathbf{L}_v + \mathbf{L} = \mathbf{L}_v$, and $\mathbf{L}_{-v}$ is the negative of $\mathbf{L}_v$: $\mathbf{L}_v + \mathbf{L}_{-v} = \mathbf{L}$.

If we define scalar multiplication by

$$c\mathbf{L}_v = \mathbf{L}_{cv}$$

it will check easily that $\mathbf{V}$ is a vector space. It is called the quotient of $\mathbb{R}^2$ by $\mathbf{L}$. The notation $\mathbf{V} = \mathbb{R}^2/\mathbf{L}$ is used.

This method–and we see a tiny window into it–has many other uses. Let us create a field with 3 elements, the so-called $\mathbf{F}_3$.

Start with the set of integers $\mathbb{Z}$ and let $\mathbf{L}$ denote the set of multiples of 3. $\mathbf{L}$ is closed under addition and multiplication by integers. For each integer a, denote by $\mathbf{L}_a$ the subset of integers of the form $3n + a$, of multiples of 3 plus a. There are just three such sets, $\mathbf{L} = \mathbf{L}_0$, $\mathbf{L}_1$ and $\mathbf{L}_2$. [Note $\mathbf{L} = \mathbf{L}_3$, $\mathbf{L}_1 = \mathbf{L}_4$, for instance.]

We define an 'addition' and a 'multiplication' on this set of 3 elements by

$$\begin{aligned} \mathbf{L}_a + \mathbf{L}_b &:= \mathbf{L}_{a+b} \\ \mathbf{L}_a \cdot \mathbf{L}_b &:= \mathbf{L}_{ab} \end{aligned}$$

We again can check that these compositions do not depend on the chosen 'representatives.'

If we denote the set of these three subsets simply by $0, 1, 2$, and table the sums and products we get

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$2 + 2 = 1$

and

$\times$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$2 \times 2 = 1$

The same construction would work for any prime number p, and the field $\mathbf{F}_p$ [integers mod p] would arise.

Bases of Vector Spaces

Our aim is to prove the existence of bases in arbitrary vector spaces, wihout the restriction we used in our early discussion.

Theorem

Every vector space $\mathbf{V}$ *over a field* $\mathbf{F}$ *has a basis.*

The proof uses some elements of set theory [Zorn's Lemma] that many of you are already familiar with.

We are going to examine the set **A** of subsets $\mathcal{B}$ of **V** of linearly independent elements.

- If $\mathcal{B}$ is maximal [that is, not contained in any other such subset] then it is a basis. Indeed, let $v \in \mathbf{V}$. If v is not in the span of $\mathcal{B}$, we claim that the [larger] set $\mathcal{C} = \mathcal{B} \cup \{v\}$ is linearly independent: If not, we would have a relation

$$cv + c_1 u_1 + \cdots + c_n u_n = 0,$$

with some scalar nonzero. In case $c \neq 0$,

$$v = (-c_1/c)u_1 + \cdots + (-c_n/c)u_n,$$

and v wolujld be in the span of $\mathcal{B}$, which is again our hypothesis. So $c = 0$ and we would have a relation involving the u_i only; but these vectors are linearly independent so all the $c_i = 0$.

- A collection **L** of subsets $\mathcal{B}$ in **A** is linearly ordered if any two of them are comparable: If $\mathcal{B}_\alpha, \mathcal{B}_\beta \in \mathbf{L}$ then one contains the other. We have the following property: The subset
$$\bigcup \mathcal{B}_\alpha$$
is linearly independent. This is clear because if $u_1, \ldots, u_n$ are vectors in the union, $u_i \in \mathcal{B}_{\alpha_i}$, they will belong to larger of the $\mathcal{B}_{\alpha_i}$.
- The existence of a maximal subset in **A** is now asserted by Zorn's Lemma.

Outline

HomeQuiz #2

1. Let $\mathbf{F}$ be a field and let $\mathbf{V}$ be the vector space $\mathbf{M}_n(\mathbf{F})$. Let $\mathbf{W}$ be the set of all $v = [a_{ij}] \in \mathbf{V}$ such that $a_{11} + a_{22} + \cdots + a_{nn} = 0$, that is, the matrices of **trace** zero. Show that $\mathbf{W}$ is a subspace and describe precisely one of its bases.
2. Prove that the functions $\sin x, \cos x, \sin 2x, \cos 2x$ are linearly independent.
3. Let $f(x)$ be a real polynomial of degree n. Prove that $f(x)$ and its higher derivatives form a basis for the space $\mathbb{R}_n[x]$ (real polynomials of degree $\leq n$).
4. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be subspaces of the vector space $\mathbf{V}$. Prove that

$$\dim \mathbf{S}_1 + \dim \mathbf{S}_2 = \dim(\mathbf{S}_1 \cap \mathbf{S}_2) + \dim(\mathbf{S}_1 + \mathbf{S}_2).$$

Solutions

1: If $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ are two matrices of trace 0, the trace of $\mathbf{A} + \mathbf{B}$ is
$(a_{11} + b_{11}) + \cdots (a_{nn} + b_{nn}) = \sum_i a_{ii} + \sum_i b_{ii} = 0 + 0 = 0.$
There is a similar calculation that the trace of $c\mathbf{A}$ is 0. This is the subspace test for **W**.

To find the basis [will do here the case $n = 2$ only]. Just note if $\mathbf{A} = [a_{ij}]$ has trace 0, $a_{11} + a_{22} = 0$, so $a_{11} = a_{22}$ is the only equation to care. Thus

$$\mathbf{A} = \begin{bmatrix} -a_{22} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{22} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{21} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

These 3 matrices are linearly independent and span **W**. Think how you would do $n = 3$.

2: Must show that if $a \sin x + b \cos x + c \sin 2x + d \cos 2x = 0$, $a, b, c, d \in \mathbb{R}$, then $a = b = c = d = 0$. We convert this functional equation into numerical equations by picking convenient values for x:

- Setting $x = 0$, get $b + d = 0$
- Setting $x = \pi/2$, get $a - d = 0$
- Take derivative and set $x = 0$, get $a + 2c = 0$
- Take derivative and set $x = \pi/2$, get $-b - 2c = 0$

Solving these 4 linear equations gives the desired assertion [do it!]

3: If f is a polynomial of degree n, it suffices to show that

$$cf(x) + c_1 f'(x) + c_2 f''(x) + \cdots + c_n f^{(n)}(x) = 0$$

then $c = c_1 = c_2 = \cdots = c_n = 0$. This will show that $f(x)$ and its n derivatives are linearly in a space of dimension $n + 1$, so they will form a basis.

This is clear: Note that if $c \neq 0$, $cf(x)$ would be a combination of lower degree polynomials–not possible. So $c = 0$. Now argue the same way with $f'(x)$, a polynomial of degree $n - 1$. And so on [i.e. use descending induction].

4:

Some volunteer will do in class.