Math 311: Advanced Calculus

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Set 6

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Outline

Main Goal

Understand

Study of Sequences and Series of Functions

Motivation

Consider the function of last hourly

$$\mathbf{G}(x) = \int_0^x e^{t^2}\, dt.$$

Question: How to evaluate $\mathbf{G}(1)$?

We are going to make use of something we know already

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

and do lots of reckless arithmetic:

$$
\begin{aligned}
\mathbf{G}(1) &\stackrel{?}{=} \int_0^1 \Big(\sum_{n=0}^{\infty} \frac{(t^2)^n}{n!}\Big) dt \\
&\stackrel{?}{=} \sum_{n=0}^{\infty} \int_0^1 \frac{(t^2)^n}{n!} dt \\
&= \sum_{n=0}^{\infty} \int_0^1 \frac{t^{2n}}{n!} dt \\
&= \sum_{n=0}^{\infty} \frac{1}{n!(2n+1)} \\
&= \;?
\end{aligned}
$$

Outline

Convergence of Series

Given the series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots \quad ?$$

there are two sequences associated to it

- The sequence of **terms**, (a_n) and
- The sequence of **partial sums**, (s_n),

$$s_n = a_0 + a_1 + \cdots + a_n$$

- We say the **series converges** to $A \in \mathbb{R}$ if $\lim s_n = A$.We write this as

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots = A$$

A cautionary tale

We pick the alternating harmonic series–which we know to be convergent–and carry out arithmetic operations: See what happens

$$
\begin{aligned}
S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \\
\frac{1}{2}S &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots \\
S + \frac{1}{2}S &= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \cdots
\end{aligned}
$$

Thus $S + \frac{1}{2}S = \frac{3}{2}S$ is just a rearrangement of S! The arithmetic is saying instead that

$$\frac{3}{2}S = S!$$

Algebraic Limit Theorem for Series

Theorem

If $\sum_{k=1}^{\infty} a_k = A$ *and* $\sum_{k=1}^{\infty} b_k = B$, *then:*

1. $\sum_{k=1}^{\infty} ca_k = cA$ *for all* $c \in \mathbb{R}$ *and*
2. $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$.

Proof. (i) To show $\sum_{k=1}^{\infty} ca_k = cA$, we consider the sequence of partial sums

$$t_n = ca_1 + ca_2 + \cdots + ca_n.$$

Since $\sum_{k=1}^{\infty} a_k = A$, its sequence of partial sums

$$s_n = a_1 + a_2 + \cdots + a_n$$

converges to A. By the Algebraic Limit Theorem for Sequences, $\lim t_n = c \lim s_n = cA$.

(ii) To show that $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$, let $r_n = a_1 + \cdots + a_n$, $s_n = b_1 + \cdots + b_n$ be the partial sum terms of the series. The partial sum term of the addition of the two series is

$$t_n = (a_1+b_1)+\cdots+(a_n+b_n) = (a_1+\cdots+a_n)+(b_1+\cdots+b_n) = r_n+s_n.$$

By the Algebraic Limit Theorem for Sequences,

$$\lim t_n = \lim r_n + \lim s_n = A + B.$$

Product of Series

Other operations are harder:
Question: Given two series, $a_0 + a_1 + a_2 + \cdots + a_n + \cdots$ and $b_0 + b_1 + b_2 + \cdots + b_n + \cdots$, what is

$$(a_0 + a_1 + a_2 + \cdots + a_n + \cdots)(b_0 + b_1 + b_2 + \cdots + b_n + \cdots) = ?$$

Part of the issue arises from the **distributive rule.** We will offer a partial fix later.

Cauchy Criterion for Series

Definition

A sequence (a_n) is called a **Cauchy sequence** if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \epsilon$.

Recall:

Theorem

A sequence converges if and only if it is a Cauchy sequence.

We apply this criterion to the sequence (s_n) of partial sums of a series $\sum_{k=1}^{\infty} a_k$. Note that

$$|s_m - s_n| = |a_{m+1} + \cdots + a_n|$$

Cauchy Test for Series

Theorem

The series $\sum_{k=1}^{\infty} a_k$ converges if and only if given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n > m \geq N$ it follows that

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Proof. Just observe

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon,$$

and apply the Cauchy's Criterion for sequences. □

Corollary

If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.

Example

Consider the geometric series ($1 > q \geq 0$)

$$1 + q + q^2 + \cdots + q^n + \cdots$$

The difference of partial sums $s_n - s_m$ is

$$\begin{aligned} s_n - s_m &= q^{m+1} + \cdots + q^n \\ &= q^{m+1}(1 + q + \cdots + q^{n-m}) \\ &= q^{m+1}\frac{1 - q^{n-m+1}}{1-q} \\ &\leq q^{m+1}\frac{1}{1-q} \leq q^N \frac{1}{1-q}, \quad n, m \geq N \end{aligned}$$

Converse?

Question: Is a series whose sequence of terms a_n converges to 0 convergent? This one is easy:

Answer: No. The (harmonic) series

$$1 + 1/2 + 1/3 + \cdots + 1/n + \cdots$$

has $1/n \to 0$ but it is divergent.

Comparisons

Given two series $\sum_{k\geq 1} a_k$ and $\sum_{k\geq 1} b_k$ that loosely connected we seek to link their convergence/divergence:

Theorem (Comparison Test)

Assume $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are series satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$.

1. *If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.*
2. *If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.*

Proof. Both follow from Cauchy's Criterion applied to the partial sums

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |b_{m+1} + a_{m+2} + \cdots + b_n|$$

If, for instance, given $\epsilon > 0$ we can find N so that for $n, m > N$ $|b_{m+1} + a_{m+2} + \cdots + b_n| < \epsilon$, then the same condition will apply to the a_n.

Example

1. We know that the **harmonic series,** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. It is clear that the same happens if we form the series $\sum_{n=N}^{\infty} \frac{1}{n}$ where N is some fixed number $N \geq 1$.
2. If a and b are positive numbers, consider the series [called generalized harmonic series] whose terms are given by the rule:
$$\frac{1}{a}, \frac{1}{a+b}, \frac{1}{a+2b}, \ldots, \frac{1}{a+nb}, \cdots$$
3. We claim that this series is also divergent: We compare the terms to a multiple of the harmonic series
$$\frac{1}{a+bn} \geq \frac{1}{n+bn} = \frac{1}{b+1}\frac{1}{n}, \quad n \geq a$$

Absolute Convergence Test

If $\sum_{n=1}^{\infty} a_n$ is a series of non-negative terms, its partial sums

$$s_n = a_1 + a_2 + \cdots + a_n, \quad s_{n+1} = s_n + a_n$$

is a monotone sequence. Therefore, by the criterion, the series converges exactly when the sequence (s_n) is bounded.

We make use of this:

Theorem (Absolute Convergence Test)

If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges as well.

Proof of the Absolute Convergence Test

1. We make use of Cauchy criterion for series: Let $\epsilon > 0$. Since the series $\sum_{k=1}^{\infty} |a_k|$ converges, there exists N so that

$$|a_{n+1}| + |a_{n+1}| + \cdots + |a_m| < \epsilon \quad m \geq n > N$$

2. By the **triangle inequality** (one that say $|a + b| \leq |a| + |b|$), we get

$$|a_{n+1} + a_{n+1} + \cdots + a_m| < \epsilon \quad m \geq n > N$$

3. Therefore the series $\sum_{k=1}^{\infty} a_k$ satisfies the Cauchy condition and therefore converges.

Converse?

The series

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots (-1)^{n-1}\frac{1}{n} + \cdots$$

is convergent (alternating harmonic series) (the one that won a Grammy's Award), but the series of the absolute values is

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots ,$$

is divergent.

Alternating Series

An alternating series is one with consecutive terms have opposite signs. One group of them is easy to study:

Theorem (Alternating Series Test)

Let (a_n) be a sequence satisfying

1. $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$, *and*
2. $(a_n) \to 0$.

Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

In other words: If (a_n) is a decreasing sequence of positive terms then

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n \quad \text{converges if and only if} \quad \lim a_n = 0$$

Proof. Observe the odd and even sequences of partial sums

$$s_1 = a_1 \geq s_3 = a_1 - (a_2 - a_3) \geq s_5 = s_3 - (a_4 - a_5), \ldots$$

$$s_2 = a_1 - a_2 \leq s_4 = s_2 + (a_3 - a_4) \leq s_5 = s_3 + (a_5 - a_6), \ldots$$

They are monotone and bounded: Since $(a_n) \to 0$, there exists $a_n \leq K$, $s_{2n} = s_{2n-1} + a_{2n} \leq s_{2n-1} + K \leq a_1 + K$, therefore the even sequence is increasing and bounded. Thus it has a limit ℓ_1. Similarly, the other sequence is decreasing and with a lower bound, so it has a limit ℓ_2. Since $\pm a_n = s_n - s_{n-1}$ converges to 0, $\ell_1 = \ell_2$.

Rearrangements

Definition

Let $\sum_{k\geq 1} a_k$ be a series. A series $\sum_{k\geq 1} b_k$ is said to be a **rearrangement** of $\sum_{k\geq 1} a_k$ if there exists a 1–1, onto function $\mathbf{f} : \mathbb{N} \to \mathbb{N}$ such that $b_{\mathbf{f}(k)} = a_k$ for all $k \in \mathbb{N}$.

Consider the geometric series of ratio q

$$1 + q + q^2 + q^3 + \cdots + q^n + \cdots$$

Now we shuffle the terms

$$q + 1 + q^3 + q^2 + q^5 + q^4 + \cdots$$

This is not a geometric series, but we should expect its fate linked to the first series. The next result says this.

Series of Positive Terms

Theorem (Dirichlet)

The sum of a series of positive terms [convergence/divergence] is the same in whatever order [rearrangement] the terms are taken.

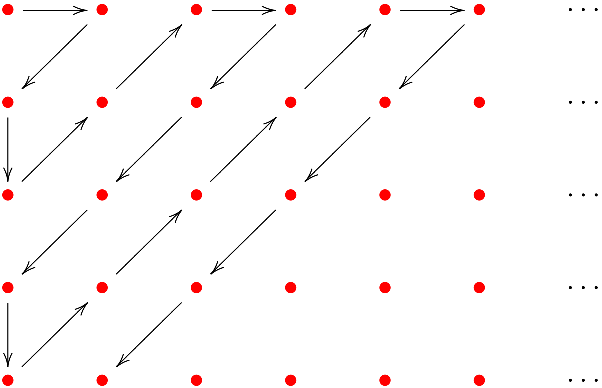
Proof. Let $a_0 + a_1 + a_2 + \cdots + a_n + \cdots$ be a series of positive terms of sum s. Then any partial sum of rearrangement $b_0 + b_1 + b_2 + \cdots + b_n + \cdots$ is bounded by s. Thus the second is convergent and its sum t is bound by s.
We reverse the roles to obtain $s \leq t$. □

Product of Series

Question: Given two series, $a_0 + a_1 + a_2 + \cdots + a_n + \cdots$ and $b_0 + b_1 + b_2 + \cdots + b_n + \cdots$, what is

$$(a_0 + a_1 + a_2 + \cdots + a_n + \cdots)(b_0 + b_1 + b_2 + \cdots + b_n + \cdots) = ?$$

The issue is: we have all the poducts $a_m b_n$ that can be organized into many different series, and then grouped. For instance, if we list the $a_m b_n$ as the double array, we



We could try the following: **Define** the product as the series

$$a_0b_0 + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + \cdots$$

Makes sense? [Discuss] Will see another rearrangement soon.

$$\begin{array}{ccccc} a_0b_0 & a_1b_0 & a_2b_0 & a_3b_0 & \ldots \\ a_0b_1 & a_1b_1 & a_2b_1 & a_3b_1 & \ldots \\ a_0b_2 & a_1b_2 & a_2b_2 & a_3b_2 & \ldots \\ a_0b_3 & a_1b_3 & a_2b_3 & a_3b_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

The partial sums remind us how polynomials are multiplied

$$(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)(b_0 + b_1x + b_2x^2 + \cdots + b_mx^m)$$

$$= \sum_{k=0}^{m+n} \Big(\sum_{0 \leq i \leq k} a_i b_{k-i} \Big) x^k$$

$a_0b_0, a_0b_1 + a_1b_0$, $a_0b_2 + a_1b_1 + a_2b_2, \ldots$
Another aspect of this definition is:

Theorem

If $\sum_{n\geq 0} a_n$ and $\sum_{n\geq 0} b_n$ are two convergent series of positive terms, and s and t are their respective sums, then the third series is convergent and has the sum st.

Out of all products $a_m b_n$, the 'product' above is given in terms of the diagonals

$$\begin{array}{ccccc} a_0 b_0 & a_1 b_0 & a_2 b_0 & a_3 b_0 & \ldots \\ a_0 b_1 & a_1 b_1 & a_2 b_1 & a_3 b_1 & \ldots \\ a_0 b_2 & a_1 b_2 & a_2 b_2 & a_3 b_2 & \ldots \\ a_0 b_3 & a_1 b_3 & a_2 b_3 & a_3 b_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

$a_0 b_0, a_0 b_1 + a_1 b_0$, $a_0 b_2 + a_1 b_1 + a_2 b_2, \ldots$ whose partial sums don't write conveniently:

$$p_n = (a_0 b_0) + (a_1 b_0 + a_1 b_0) + (a_2 b_0 + a_1 b_1 + a_0 b_2) + \cdots$$

We want to re-write the terms of the product series differently:

$$\begin{array}{ccccc} a_0 b_0 & a_1 b_0 & a_2 b_0 & a_3 b_0 & \dots \\ a_0 b_1 & a_1 b_1 & a_2 b_1 & a_3 b_1 & \dots \\ a_0 b_2 & a_1 b_2 & a_2 b_2 & a_3 b_2 & \dots \\ a_0 b_3 & a_1 b_3 & a_2 b_3 & a_3 b_3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{array}$$

$a_0 b_0, (a_0 + a_1)(a_0 + a_1) - a_0 b_0$,
$(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) - (a_0 + a_1)(b_0 + b_1), \dots$ whose nth partial sum is

$$(a_0 + a_1 + \cdots + a_n)(b_0 + b_1 + \cdots + b_n),$$

a sequence that converges to st by the Algebraic Limit Theorem.

Observe that

$$p_n = (a_0 b_0) + (a_1 b_0 + a_0 b_1) + \cdots + (a_0 b_n + \cdots + a_n b_0) \leq$$

$$(a_0 + a_1 + \cdots + a_n)(b_0 + b_1 + \cdots + b_n)$$

on one hand and

$$p_n \geq (a_0 + a_1 + \cdots + a_{n/2})(b_0 + b_1 + \cdots + b_{n/2})$$

Since the terms at the ends converge to st, $(p_n) \to st$ as well.

Theorem

If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then any rearrangement of this series converges to the same limit.

Proof. Assume $\sum_{k\geq 1} a_k$ converges absolutely to A, and let $\sum_{k\geq 1} b_k$ be an rearrangement of $\sum_{k\geq 1} a_k$. Let

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

and

$$t_n = \sum_{k=1}^{n} b_k = b_1 + b_2 + \cdots + b_n$$

be the corresponding partial sums.
Let $\epsilon > 0$. Since $(s_n) \to A$, choose N_1 such that

$$|s_n - A| < \epsilon/2$$

Because the convergence is absolute, we can choose N_2 so that

$$\sum_{k=m+1}^{n} |b_k| < \epsilon/2$$

for all $n > m \geq N_2$. Take $N = \max\{N_1, N_2\}$. We know that the terms $\{a_1, a_2, \ldots, a_N\}$ must all appear in the rearranged series, and we move far out enough in the series $\sum_{k\geq 1} b_k$ that these terms are all included. Thus, choose
$M = \max\{f(k) \mid 1 \leq k \leq N\}$.
It is clear that if $m \geq M$, then $(t_m - s_N)$ consists of a finite number of terms, the absolute values of which appear in the tail of $\sum_{k=N+1}^{\infty} |a_k|$. The earlier choice of N_2 guarantees $|t_m - s_N| < \epsilon/2$, and so

$$\begin{aligned} |t_m - A| &= |t_m - s_N + s_N - A| \\ &\leq |t_m - s_N| + |s_N - A| \leq \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

Integral Test

Theorem (Integral Test)

Let $\sum_{n\geq 0} a_n$ be a series of positive terms. If there is a decreasing function $\mathbf{f}(x)$ such that $a_n \leq \mathbf{f}(n)$ for large n and

$$\int_{x=1}^{\infty} \mathbf{f}(x)dx < \infty,$$

then $\sum_{n\geq 0} a_n$ converges.

Proof. If $a_n \leq \mathbf{f}(n)$ for $n \geq n_0$, since $\mathbf{f}(x)$ is decreasing, $a_n \leq \int_{n-1}^{n} \mathbf{f}(x)dx, \quad n > n_0$. From this, and the assumption that $\int_1^{\infty} \mathbf{f}(x)dx < \infty$, we get that the partial sums of the series $\sum_{n\geq 0} a_n$ are bounded, and therefore converge by the theorem on bounded monotone sequences. □

Zeta Function

The series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots,$$

for $p > 1$ will always converge. Its sum is denoted by $\zeta(p)$.

For example, $\zeta(2) = \frac{\pi^2}{6}$.
This function is actually defined for all complex numbers p whose real part is > 1. It is known as **Riemann zeta function.**It is probably the most famous function of Mathematics.

Convergence

Let us show that

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots ,$$

for $p > 1$ will always converge.

We are going to bound each term $1/n^p$ by the terms of another series, and then argue the new series converges.

Consider the function $\mathbf{f}(x) = 1/x^p$, $x \geq 2$. This is a decreasing function (draw the graph).
Observe

$$1/n^p \leq \int_{x=n-1}^{n} 1/x^p dx$$

Therefore its partial sums are bounded by

$$s_n \leq 1 + \int_{x=1}^{n} \frac{dx}{x^p} = 1 + \frac{1}{p-1}\left[1 - \frac{1}{n^{p-1}}\right] < 1 + \frac{1}{p-1}$$

Examples

The series in earlier Workshop satisfies

$$\sum_{n\geq 1} \frac{1}{n(n+1)} \leq \sum_{n\geq 1} \frac{1}{n^2}$$

which is convergent.
In the same manner, if

$$\sum_{n\geq 1} \frac{p(n)}{q(n)},$$

where $p(n)$ and $q(n)$ are positive polynomial expressions with $\deg q \geq 2 + \deg p$, then the series converges by the same reason. Do it!

Exam Type Exercises

1. Show that
$$\sum_{n\geq 0}(-1)^n \frac{2n+3}{(n+1)(n+2)} = 1.$$
2. Determine the values of q for which the series
$$q + 2q^2 + 3q^3 + \cdots + nq^n + \cdots$$
is convergent.
3. Show that $\sum_{n\geq 2} \frac{1}{n(\ln n)^p}$ converges if $p > 1$, and diverges if $p \leq 1$.

Ratio Tests

There are very useful tests involving the ratio a_{n+1}/a_n of two successive terms of a series. Sometimes we compare the ratio a_{n+1}/a_n to another b_{n+1}/b_n. In these we suppose that a_n and b_n are strictly positive.

Suppose $a_n, b_n > 0$ and that $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for sufficiently large n, that is for $n \geq n_0$.

Then

$$\begin{aligned} a_n &= \frac{a_{n_0+1}}{a_{n_0}} \frac{a_{n_0+2}}{a_{n_0+1}} \cdots \frac{a_n}{a_{n-1}} a_{n_0} \\ &\leq \frac{b_{n_0+1}}{b_{n_0}} \frac{b_{n_0+2}}{b_{n_0+1}} \cdots \frac{b_n}{b_{n-1}} a_{n_0} = \frac{a_{n_0}}{b_{n_0}} b_n \\ &= Cb_n, \quad C = a_{n_0}/bn_0. \end{aligned}$$

Here are some applications:

Theorem

Let $\sum a_n$ and $\sum b_n$ be series of positive terms.

1. *If for $n \geq n_0$*
$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n},$$
and the series $\sum b_n$ converges, then $\sum a_n$ converges also.
2. *If for $n \geq n_0$*
$$\frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n},$$
and the series $\sum a_n$ diverges, then $\sum b_n$ diverges also.

Theorem (d'Alambert Test)

The series $\sum a_n$ is convergent if $a_{n+1}/a_n \leq r$, where $r < 1$, for all sufficiently large n.

Theorem

Given a series $\sum_{n\geq 1} a_n$ with $a_n \neq 0$, if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1,$$

then the series converges absolutely.

Proof.

1. Let r' satisfy $r < r' < 1$. For $\epsilon = r' - r$, there is N such that for $n \geq N$ $||a_{n+1}/a_n| - r| < \epsilon$, and therefore
$$|a_{n+1}/a_n| - r \leq ||a_{n+1}/a_n| - r| < \epsilon = r' - r,$$
giving $|a_{n+1}| \leq r'|a_n|$ for $n \geq N$.
2. The above shows that the series $\sum_{n=N}^{\infty} |a_n|$ satisfies $|a_n| \leq |a_N|(r')^{n-N}$, a geometric series of ratio $r' < 1$, which converges.

Exponential

A quick application of the ratio test:
We claim that the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

converges for all values of x.

For the ratio of consecutive terms

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}/(n+1)!}{x^n/n+!} = \frac{x}{n+1}$$

so that for any x, $\lim a_{n+1}/a_n = 0$.

This is a well used technique for power series.

Examples

1. For the series $\sum_{n\geq 1}\frac{n}{2^n}$ we invoke the ratio test:

$$\frac{a_{n+1}}{a_n}=\frac{n+1}{2^{n+1}}\Big/\frac{n}{2^n}=\frac{n+1}{n}\frac{1}{2}$$

   which has limit $1/2<1$. So the series converges.

2. Decide [with justification] whether the series

$$\sum_{n\geq 1}\frac{n!}{n^n},$$

   is convergent or divergent?

Exercises

1. Show that if $a_n > 0$ and $\lim na_n = L$, with $L \neq 0$, then the series $\sum a_n$ diverges.
2. Show that if $a_n > 0$ and $\lim n^2 a_n = L$, with $L \neq 0$, then the series $\sum a_n$ converges.
3. Find examples of two series $\sum a_n$ and $\sum b_n$ both of which diverge but for which $\sum \min\{a_n, b_n\}$ converges. To make it more difficult, choose examples where (a_n) and (b_n) are positive and decreasing.

Root Test

Let $\sum_{n\geq 1} a_n$ be a series of positive terms. We are going to examine how the limit

$$\lim_{n\to\infty} \sqrt[n]{a_n}$$

is used to decide convergence. We recall one special calculation of these limits: If $x > 0$

$$\lim_{n\to\infty} \sqrt[n]{x} = 1$$

Recall another limit: $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Root Test

Theorem

If $\sum_{n\geq 1} a_n$ is a series of positive terms and $\lim_{n\to\infty} \sqrt[n]{a_n} = r < 1$, then the series converges.

Proof. Let $r < r' < 1$ and pick $\epsilon = r' - r$. This is the same subtle point we used above.

1. There is N so that for $n > N$
$$|\sqrt[n]{a_n} - r| < \epsilon$$
2. This implies that $\sqrt[n]{a_n} < r + \epsilon = r' < 1$ for $n > N$. As a consequence
$$a_n < (r')^n$$

Example

Consider the series (for $q > 0$)

$$1 + q + 2q^2 + \cdots + nq^n + \cdots$$

We invoke the **root test**

$$\lim_{n \to \infty} \sqrt[n]{nq^n} = q \lim_{n \to \infty} \sqrt[n]{n} = q$$

Therefore it converges if $q < 1$
Let us calculate the sum of the series. For that we must have an inkling on how the series arose from the geometric series. At these times we replace q by x and recall:

Nice calculation

1. Differentiate the ‘equality’

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

2. To get almost our series

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots$$

3. Now multiply by x and add 1

$$1 + \frac{x}{(1-x)^2} = 1 + x + 2x^2 + \cdots + nx^n + \cdots$$

4. Thus for $0 < q < 1$ the series sums to $1 + \frac{q}{(1-q)^2}$.

Outline

Workshop #10: This page and next

1. If a is a positive integer, prove that the series

$$\sum_{n\geq 1} \frac{1}{n(a+n)}$$

converges. Find its sum.

2. If $b > a > 0$, do the same for the series

$$\sum_{n\geq 1} \frac{1}{n(a+n)(b+n)}.$$

3: Argue by induction that for any sequence of integers $0 < a_1 < a_2 < \ldots < a_r$, the series

$$\sum_{n\geq 1} \frac{1}{n(a_1+n)(a_2+n)\cdots(a_r+n)}$$

converges and its sum can be effectively computed.

4: Given the series

$$\sum_{n\geq 0} \frac{1}{n^2+1}$$

- Prove by comparison and by a direct application of the integral test that it converges.
- Try to find its sum somehow/somewhere.
- Google it.

Outline

Sequences of Functions

Let $\mathbf{f}_n : A \to \mathbb{R}$, $n \in \mathbb{N}$, be a set of functions. For each $x \in A$ they define a numerical sequence $(\mathbf{f}_n(x))$. If $\mathbf{f}_n(x) \to L$, we say that $(\mathbf{f}_n)$ converges at x. We are greatly interested in case it converges to all $x \in A$, as the limit

$$\mathbf{f}_n(x) \to \mathbf{f}(x)$$

will define a function $\mathbf{f} : A \to \mathbb{R}$.

1. If the $\mathbf{f}_n$ are continuous, when is $\mathbf{f}$ continuous?
2. If the $\mathbf{f}_n$ are differentiable, when is $\mathbf{f}$ differentiable?

Example

Let $\mathbf{f}_n(x) = x^n$, $n \in \mathbb{N}$, be the sequence of powers of x as functions on $[0, 1]$. For any x in this interval, we have

$$\begin{aligned} \lim_{n\to\infty} \mathbf{f}_n(x) &= 0, \quad 0 \leq x < 1 \\ \lim_{n\to\infty} \mathbf{f}_n(x) &= 1, \quad x = 1 \end{aligned}$$

Thus $\lim_{n\to\infty} \mathbf{f}_n$ exists for all $x \in [0, 1]$, but it is not a continuous function on the interval.
We need a rule that guarantees that $\lim_{n\to\infty} \mathbf{f}_n$ is continuous.

Pointwise and Uniform Convergence

Definition

The sequence of functions $(\mathbf{f}_n(x))$ converges **pointwise** to $\mathbf{f}(x)$ if for every x $\mathbf{f}_n(x)$ converges to $\mathbf{f}(x)$. For a given x, this means that given $\epsilon > 0$ there is $N = N(x) \in \mathbb{N}$ such that for $n \geq N$, $|\mathbf{f}_n(x) - \mathbf{f}(x)| < \epsilon$.

Another definition of convergence is much more restrictive:

Definition

The sequence of functions $(\mathbf{f}_n(x))$ converges uniformly to $\mathbf{f}(x)$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$|\mathbf{f}_n(x) - \mathbf{f}(x)| < \epsilon.$$

Example: Let $\mathbf{f}_n(x) = 1/n(1+x^2)$. Then $\mathbf{f}(x) = \lim_{n\to\infty} \mathbf{f}_n(x) = 0$. Given $\epsilon > 0$

$$|\mathbf{f}_n(x) - \mathbf{f}(x)| < 1/n$$

Thus if $N \geq 1/\epsilon$,

$$|\mathbf{f}_n(x) - \mathbf{f}(x)| < \epsilon$$

for $n \geq N$.

Cauchy Criterion for Uniform Convergence

Theorem

A sequence of functions $(\mathbf{f}_n(x))$ defined on a set $A \subset \mathbb{R}$ converges uniformly on A if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|\mathbf{f}_n(x) - \mathbf{f}_m(x)| < \epsilon$ for all $n, m \geq N$ and all $x \in A$.

Proof. $\Rightarrow$: For each $x \in A$, the numerical Cauchy sequence $(\mathbf{f})_n(x))$ converges: Call the limit $\mathbf{f}(x)$. Now we argue that $\mathbf{f}_n$ converges to $\mathbf{f}$ uniformly. Let $\epsilon > 0$ and let N be such that $|\mathbf{f}_n(x) - \mathbf{f}_m(x)| < \epsilon$ for $n, m \geq N$.
Now we use the argument used in the numerical case.

Let $\epsilon > 0$. Because the sequence $\mathbf{f}_n$ is Cauchy, there exists N such that for all $n, m \geq N$ and all $x \in A$,

$$|\mathbf{f}_n(x) - \mathbf{f}_m(x)| < \epsilon/2.$$

On the other hand, for each $x \in A$ the sequence $\mathbf{f}_n(x) \to \mathbf{f}(x)$, so there is N_K

$$|\mathbf{f}_{N_K}(x) - \mathbf{f}(x)| < \epsilon/2.$$

Thus for all $x \in A$ and all $n \geq N_K$

$$\begin{aligned} |\mathbf{f}_n(x) - \mathbf{f}(x)| &= |\mathbf{f}_n(x) - \mathbf{f}_{N_k}(x) + \mathbf{f}_{N_k}(x) - \mathbf{f}(x)| \\ &\leq |\mathbf{f}_n(x) - \mathbf{f}_{N_k}(x)| + |\mathbf{f}_{N_k}(x) - \mathbf{f}(x)| < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

Uniform Convergence and Continuity

Theorem

If the sequence of continuous functions $(\mathbf{f}_n(x))$ converges uniformly to $\mathbf{f}(x)$, then $\mathbf{f}(x)$ is continuous (on the same domain).

Proof. Let $x = c$ be a point in the domain. Given $\epsilon > 0$, we must show that there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|\mathbf{f}(x) - \mathbf{f}(c)| < \epsilon$. The idea is to write

$$\mathbf{f}(x) - \mathbf{f}(c) = (\mathbf{f}(x) - \mathbf{f}_n(x)) + (\mathbf{f}_n(x) - \mathbf{f}_n(c)) + (\mathbf{f}_n(c) - \mathbf{f}(c))$$

and use uniform convergence on the first and third terms and continuity on the second.

$$\begin{aligned}|\mathbf{f}(x) - \mathbf{f}(c)| &\leq |\mathbf{f}(x) - \mathbf{f}_n(x)| + |\mathbf{f}_n(x) - \mathbf{f}_n(c)| \\ &+ |\mathbf{f}_n(c) - \mathbf{f}(c)|\end{aligned}$$

$$\begin{aligned}|\mathbf{f}(x) - \mathbf{f}_n(x)| &< \epsilon/3, \quad n \geq N \\ |\mathbf{f}_n(x) - \mathbf{f}_n(c)| &< \epsilon/3, \quad 0 < |x - c| < \delta \\ |\mathbf{f}(c) - \mathbf{f}_n(c)| &< \epsilon/3, \quad n \geq N\end{aligned}$$

Thus, for $0 < |x - c| < \delta$,

$$|\mathbf{f}(x) - \mathbf{f}(c)| < \epsilon.$$

Uniform Convergence and Differentiability

Theorem

Let $\mathbf{f}_n \to \mathbf{f}$ pointwise on interval $[a, b]$ and assume each $\mathbf{f}_n$ is differentiable. If $(\mathbf{f}'_n)$ converges uniformly on $[a, b]$ to a function $\mathbf{g}$, then $\mathbf{f}$ is differentiable and $\mathbf{f}' = \mathbf{g}$.

Proof. Let $\epsilon > 0$ and fix $c \in [a, b]$. We will argue that $\mathbf{f}'(c)$ exists and it is equal to $\mathbf{g}(c)$. We begin with

$$\mathbf{f}'(c) = \lim_{x \to c} \frac{\mathbf{f}(x) - \mathbf{f}(c)}{x - c}$$

and claim we can find $\delta > 0$ so that for $0 < |x - c| < \delta$

$$\left| \frac{\mathbf{f}(x) - \mathbf{f}(c)}{x - c} - \mathbf{g}(c) \right| < \epsilon.$$

$$\left|\frac{\mathbf{f}(x)-\mathbf{f}(c)}{x-c}-\mathbf{g}(c)\right| \leq \left|\frac{\mathbf{f}(x)-\mathbf{f}(c)}{x-c}-\frac{\mathbf{f}_n(x)-\mathbf{f}_n(c)}{x-c}\right| + \left|\frac{\mathbf{f}_n(x)-\mathbf{f}_n(c)}{x-c}-\mathbf{f}_n'(c)\right| + |\mathbf{f}_n'(c)-\mathbf{g}(c)|$$

We will argue that we can find δ so that each of the three terms $< \epsilon/3$.

Apply the MVT to $\mathbf{f}_n - \mathbf{f}_m$ on $[c, x]$: there exists $\alpha \in (c, x)$ such that

$$\mathbf{f}'_n(\alpha) - \mathbf{f}'_m(\alpha) = \frac{(\mathbf{f}_n(x) - \mathbf{f}_m(x)) - (\mathbf{f}_n(c) - \mathbf{f}_m(c))}{x - c}.$$

By Cauchy Criterion for Unif Conv, there exists $N \in \mathbb{N}$ such that for $n, m \geq N_1$,

$$|\mathbf{f}'_n(\alpha) - \mathbf{f}'_m(\alpha)| < \epsilon/3$$

Together we have

$$\left| \frac{\mathbf{f}_n(x) - \mathbf{f}_m(x)}{x - c} - \frac{\mathbf{f}_n(c) - \mathbf{f}_m(c)}{x - c} \right| < \epsilon/3$$

for all $m, n \geq N_1$, and all $x \in [a, b]$. If we take the limit $\mathbf{f}_m \to \mathbf{f}$ (making use of the Order Limit Theorem)

$$\left|\frac{\mathbf{f}(x)-\mathbf{f}(c)}{x-c}-\frac{\mathbf{f}_n(x)-\mathbf{f}_n(c)}{x-c}\right| \leq \epsilon/3$$

Finally, choose N_2 large enough so that

$$|\mathbf{f}'_m(c)-\mathbf{g}(c)| < \epsilon/3$$

for all $m \geq N_2$, and let $N = \max\{N_1, N_2\}$ Use that $\mathbf{f}_N$ is differentiable to produce $\delta > 0$ for which

$$\left|\frac{\mathbf{f}_N(x)-\mathbf{f}_N(c)}{x-c}-\mathbf{f}'_N(c)\right| < \epsilon/3$$

whenever $0 < |x-c| < \delta$. Substituting in the original expression,

$$\left|\frac{\mathbf{f}(x)-\mathbf{f}(c)}{x-c}-\mathbf{g}(c)\right| < \epsilon$$

Theorem

Let $(\mathbf{f}_n)$ be a sequence of differentiable functions defined on the interval $[a, b]$ and assume that $(\mathbf{f}'_n)$ converges uniformly on $[a, b]$ to a function $\mathbf{g}$. If there exists a point $x_0 \in [a, b]$ where $(\mathbf{f}_n(x_0))$ is convergent, then $(\mathbf{f}_n)$ converges uniformly on $[a, b]$.

Proof. For any $x \in [a, b]$, we have

$$|\mathbf{f}_n(x)-\mathbf{f}_m(x)| \leq |(\mathbf{f}_n(x)-\mathbf{f}_m(x))-(\mathbf{f}_n(x_0)-\mathbf{f}_m(x_0))|+|\mathbf{f}_n(x_0)-\mathbf{f}_m(x_0)|$$

One reduces to the previous proof by applying the MVT to $\mathbf{f}_n - \mathbf{f}_m$ on $[x_0, x]$: there exists $\alpha \in (x_0, x)$ such that

$$\mathbf{f}'_n(\alpha) - \mathbf{f}'_m(\alpha) = \frac{(\mathbf{f}_n(x) - \mathbf{f}_m(x)) - (\mathbf{f}_n(x_0) - \mathbf{f}_m(x_0))}{x - x_0}.$$

Combining the two theorems we get

Theorem

Let $(\mathbf{f}_n)$ *be a sequence of differentiable functions defined on the interval* $[a, b]$ *and assume that* $(\mathbf{f}_n')$ *converges uniformly on* $[a, b]$ *to a function* $\mathbf{g}$. *If there exists a point* $x_0 \in [a, b]$ *where* $(\mathbf{f}_n(x_0))$ *is convergent, then* $(\mathbf{f}_n)$ *converges uniformly on* $[a, b]$. *Moreover, the limit function* $\mathbf{f} = \lim \mathbf{f}_n$ *is differentiable and satisfies* $\mathbf{f}' = \mathbf{g}$.

Basics on Limits and Derivatives

1. Let $\mathbf{f}_n \to \mathbf{f}$ pointwise on interval $[a, b]$ and assume each $\mathbf{f}_n$ is differentiable. If $(\mathbf{f}_n')$ converges uniformly on $[a, b]$ to a function $\mathbf{g}$, then $\mathbf{f}$ is differentiable and $\mathbf{f}' = \mathbf{g}$.
2. Let $(\mathbf{f}_n)$ be a sequence of differentiable functions defined on the interval $[a, b]$ and assume that $(\mathbf{f}_n')$ converges uniformly on $[a, b]$ to a function $\mathbf{g}$. If there exists a point $x_0 \in [a, b]$ where $(\mathbf{f}_n(x_0))$ is convergent, then $(\mathbf{f}_n)$ converges uniformly on $[a, b]$.
3. Let $(\mathbf{f}_n)$ be a sequence of differentiable functions defined on the interval $[a, b]$ and assume that $(\mathbf{f}_n')$ converges uniformly on $[a, b]$ to a function $\mathbf{g}$. If there exists a point $x_0 \in [a, b]$ where $(\mathbf{f}_n(x_0))$ is convergent, then $(\mathbf{f}_n)$ converges uniformly on $[a, b]$. Moreover, the limit function $\mathbf{f} = \lim \mathbf{f}_n$ is differentiable and satisfies $\mathbf{f}' = \mathbf{g}$.

Outline

Series of Functions

Question: What do we see in the Infinite Series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots = ?$$

Answer: At least two things

- The sequence of terms, (a_n) and
- The sequence of partial sums, (s_n),

$$s_n = a_0 + a_1 + \cdots + a_n$$

- We say the **series converges** to $S \in \mathbb{R}$ if $\lim s_n = S$. By abuse of notation, we then replace the ? by S.

Question: What do we see in the Infinite Series of Functions $\mathbf{f}_n : A \to \mathbb{R}$

$$\sum_{n=0}^{\infty} \mathbf{f}_n = \mathbf{f}_0 + \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 + \cdots = ?$$

Answer: At least three things

- The sequence of terms, $(\mathbf{f}_n)$
- The sequence of partial sums, (s_n),

$$s_n = \mathbf{f}_0 + \mathbf{f}_1 + \cdots + \mathbf{f}_n$$

- We say the **series converges** to $\mathbf{f}(x) \in \mathbb{R}$ if $\lim \mathbf{f}_n(x) = \mathbf{f}(x)$.
- Main question: Properties of **f** ? continuous ? differentiable

Reasons Why

Two quick reasons why series of functions are widely (and wildly) used:

1. There are equations for which we do not have explicit (short) formulas of their solutions, e.g.

$$x^5 + 5x + 6 = 0,$$

   yet we are still able to write the solutions as the limits of numerical series

$$x = \sum_{n \geq 0} a_n.$$

2. Series gives the means to break down some functions into basic blocks:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

Noteworthy Examples

1. Geometric series

$$1 + x + x^2 + \cdots + x^n + \cdots$$

2. Exponential series

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

3. Arctangent series

$$x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

It is legitimate to evaluate the last series for $x = 1$ in order to get

$$\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} + \cdots$$

Continuity of Series of Functions

The guiding theorems:

Theorem

Let $\mathbf{f}_n$ be continuous functions on a set $A \subset \mathbb{R}$, and assume $\sum_{n=1}^{\infty} \mathbf{f}_n$ converges uniformly to a function $\mathbf{f}$. Then, $\mathbf{f}$ is continuous on A.

We need the means to test when the sequence of partial sums

$$s_n(x) = \mathbf{f}_0(x) + \mathbf{f}_1(x) + \cdots + \mathbf{f}_n(x)$$

converges uniformly.

Cauchy Criterion

Theorem

A series $\sum_{n=1}^{\infty} \mathbf{f}_n$ converges uniformly on $A \subset \mathbb{R}$ if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n > m \geq M$,

$$|\mathbf{f}_{m+1}(x) + \mathbf{f}_{m+2}(x) + \cdots + \mathbf{f}_n(x)| < \epsilon$$

for all $x \in A$.

Weierstrass M-Test

Theorem

For each $n \in \mathbb{N}$, let $\mathbf{f}_n$ be a function defined on a set $A \subset \mathbb{R}$, and let M_n be a real number satisfying

$$|\mathbf{f}_n(x)| \leq M_n$$

for all $x \in A$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} \mathbf{f}_n(x)$ converges uniformly on A.

This reduces to Cauchy's Criterion since

$$|\mathbf{f}_{m+1}(x) + \mathbf{f}_{m+2}(x) + \cdots + \mathbf{f}_n(x)| \leq M_{m+1} + \cdots + M_n,$$

for all $x \in A$. Now we use the Cauchy Criterion for numerical series.

Example

Let

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2},$$

whose terms are bounded by the terms of the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. It converges uniformly to a continuous function $\mathbf{f}(x)$.

The series of derivatives

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n},$$

diverges at $x = 0$ (becomes the harmonic series).

Derivative of a Series

The following gives us a criterion of when we can differentiate a series:

Theorem

Let $\mathbf{f}_n$ *be differentiable functions defined on the interval* $[a, b]$*, and assume that* $\sum_{n=1}^{\infty} \mathbf{f}'_n$ *converges uniformly on* $[a, b]$ *to a function* $\mathbf{g}$ *on* $[a, b]$*. If there exists a point* $x_0 \in [a, b]$ *where* $\sum_{n=1}^{\infty} \mathbf{f}_n(x_0)$ *is convergent, then the series* $\sum_{n=1}^{\infty} \mathbf{f}_n$ *converges uniformly to a differentiable function* $\mathbf{f}(x)$ *satisfying* $\mathbf{f}' = \mathbf{g}$ *on* $[a, b]$*. In other words,*

$$\mathbf{f}(x) = \sum_{n=1}^{\infty} \mathbf{f}_n(x), \quad \mathbf{f}'(x) = \sum_{n=1}^{\infty} \mathbf{f}'_n(x)$$

Outline

Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Sometimes instead of x^n on has $(x-a)^n$.

These series have, unlike more general series, amenable properties: It will be much simpler to study their convergence, continuity and differentiability.

Basic Theorem

Part of the simplicity is grounded on the following:

Theorem

If a power series $\sum_{n=0} a_n x^n$ converges at some point $x_0 \in \mathbb{R}$, then it converges absolutely for any x satisfying $|x| < |x_0|$.

Proof. If $\sum_{n=0} a_n {x_0}^n$ converges, then the sequence $a_n x_0^n$ is bounded (in fact, by Cauchy's, converges to 0). Let $M > 0$ satisfy $|a_n x_0^n| \leq M$ for all $n \in \mathbb{N}$. If $|x| < |x_0|$,

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leq M \left|\frac{x}{x_0}\right|^n$$

The geometric series

$$\sum_{n=0}^{\infty} M \left| \frac{x}{x_0} \right|^n$$

converges since its ratio is < 1, so by the Comparison Test, the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely. □

Radius of Convergence

Here is a surprising property of power series: If we have a power series

$$\sum_{n=0}^{\infty} a_n x^n,$$

what is like the set of all x (besides $x = 0$) where it converges? Here is part of the answer:

Corollary

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. The possible sets of points where it converges are: 0 only; all of $\mathbb{R}$; or an interval $(-R, R)$, possibly with one or both of its boundary points.

R: **radius of convergence** : the largest nonnegative number such that $\sum_{n=0}^{\infty} a_n x^n$ converges for all $|x| < R$.

Theorem

The radius of convergence of the series $\sum a_n x^n$ is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided the limit exists or is $+\infty$.

Proof. We make use of the Ratio Test: The series converges if the limit

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = L|x| < 1$$

and diverges if $L|x| > 1$.

From this we conclude: $R = 1/L$ if $L \neq 0$. Also, $R = \infty$ if $L = 0$, and $R = 0$ if $L = \infty$.

1. For the exponential series $\sum \frac{x^n}{n!}$, $R = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty$
2. For the geometric series $\sum x^n$, $R = 1$
3. For $\sum n!x^n$, $R = 0$

Radius of Convergence and Differentiation/Integration

Let $\mathbf{f}(x) = \sum a_n x^n$, $\sum_{n\geq 1} n a_n x^{n-1}$, and $\sum_{n\geq 1} \frac{1}{n+1} x^{n+1}$

Theorem

The three series have the same radii of convergence.

Proof. Suppose R and R' are the radii of convergence of the first two series. Suppose $|x| < R$, and choose $|x| < |x_0| < R$. Then the first series is convergent with $x = x_0$, and consequently $|a_n x_0^n| \leq A$ for all n.

Then

$$na_nx^{n-1} = \frac{n}{x_0}a_nx_0^n\left(\frac{x}{x_0}\right)^{n-1},$$

$$|na_nx^{n-1}| \leq \frac{A}{|x_0|}nr^{n-1},$$

where

$$r = \frac{|x|}{|x_0|} < 1.$$

The series

$$\frac{A}{|x_0|}nr^{n-1}$$

is convergent, for the limit of the ratio of the terms is

$$\lim_{n\to\infty}\frac{n+1}{n}r < 1.$$

This proves that the series na_nx^{n-1} converges and therefore $R \leq R'$

Now we show that $R' > R$ is impossible. Otherwise, pick x so that $R < |x| < R'$. Then the series $\sum na_n x^{n-1}$ is absolutely convergent for this x and the first series is divergent. Now

$$|a_n x^n| = |na_n x^{n-1}| \left| \frac{x}{n} \right| < |na_n x^{n-1}|$$

as soon as $n > |x|$. This comparison shows that the series $\sum |a_n x^n|$ is convergent, a contradiction.

Root Formula

Exercise: Prove that the radius of convergence of the series

$$\sum_{n=0}^{\infty} a_n x^n$$

is given by

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{a_n}.$$

Note: In some early Workshops we had several examples of $\lim_{n\infty} \sqrt[n]{\text{something}}$: $\sqrt[n]{n}$, $\sqrt[n]{a^n + b^n + c^n}$

Note also the consequence: the series of indefinite integrals will have the same radius of convergence

$$\sum_{n=1}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

Uniform Convergence

Theorem

If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at a point $|x_0|$, then it converges uniformly on the closed interval $[-c, c]$, where $c = |x_0|$.

Proof. We use Cauchy Criterion for Uniform Convergence of Series.

By assumption, $\sum_{n=0}^{\infty} |a_n x^n| < \infty$ so that in particular, for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n > m \geq N$

$$|a_{m+1} c^{m+1}| + \cdots |a_n c^n| < \epsilon$$

which implies that for all $x \in [-c, c]$

$$|a_{m+1} x^{m+1} + \cdots a_n x^n| \leq |a_{m+1} c^{m+1}| + \cdots |a_n c^n| < \epsilon$$

Abel's Lemma

Lemma

Let b_n satisfy $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$, and let $\sum_{n=1}^{\infty} a_n$ be a series for which the partial sums are bounded. In other words, assume there exists $A > 0$ such that

$$|a_1 + a_2 + \cdots + a_n| < A$$

for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$

$$|a_1 b_1 + a_2 b_2 + \cdots + a_n b_n| \leq 2A.$$

The proof uses a technique called **summation by parts**. Let (x_n) and (y_n) be sequences and let $s_n = x_1 + x_2 + \cdots + x_n$. Note that $x_j = s_j - s_{j-1}$. Now we verify that

$$\sum_{j=m+1}^{n} x_j y_j = s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j (y_j - y_{j+1}).$$

Note that the two sides as sums $\sum a_{i,j} x_i y_j$, where $a_{i,j}$ are integers. To verify this is an identity, it is enough to check that for each j in the range $m+1 \leq i, j \leq n+1$, taking the partial derivative relative to x_i followed by that of y_j we get the same values:

$$\frac{\partial^2}{\partial x_i \partial y_j} \sum a_{i,j} x_i y_j = a_{i,j}$$

Abel's Theorem

Theorem

Let $\mathbf{g}(x) = \sum_{n=1}^{\infty} a_n x^n$ be a power series that converges at the point $x = R > 0$. Then the series converges uniformly on the interval $[0, R]$. A similar result holds if the series converges at $x = -R$.

Proof. We use Cauchy Criterion for Uniform Convergence of Series: Set

$$\mathbf{g}(x) = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_n R^n \left(\frac{x}{R}\right)^n.$$

We must show that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n > m \geq N$

$$|a_{m+1} R^{m+1} \left(\frac{x}{R}\right)^{m+1} + \cdots + a_n R^n \left(\frac{x}{R}\right)^n | < \epsilon$$

Because we are assuming that $\sum_{n=1}^{\infty} a_n R^n$ converges, by Cauchy Criterion for convergent numerical series there exists $N \in \mathbb{N}$ such that

$$|a_{m+1} R^{m+1} + \cdots + a_n R^n| < \epsilon/2$$

for all $n > m \geq N$. By Abel's Lemma

$$|a_{m+1} R^{m+1} (x/R)^{m+1} + \cdots + a_n R^n (x/R)^n| < 2\epsilon/2 = \epsilon$$

Outline

Taylor Series

Let $\mathbf{f}(x)$ be a function defined on a neighborhood of $x = a$, let us assume its derivatives of all orders exist at $x = a$, $\mathbf{f}^{(n)}(a)$, $n \geq 0$. We can assemble these derivatives into several series, the most important being the **Taylor series** of **f** at $x = a$:

$$\sum_{n=0}^{\infty} \frac{\mathbf{f}^{(n)}(a)}{n!}(x-a)^n.$$

1. For what values of x, in addition to $x = a$, does the series converge?
2. When will it converge to $\mathbf{f}(x)$?

The partial sums of this series are the polynomials

$$s_n(x) = \sum_{i=0}^{n} \frac{\mathbf{f}^{(i)}(a)}{i!}(x-a)^i.$$

To see whether $s_n(x) \to \mathbf{f}(x)$, we must examine the difference

$$\mathbf{f}(x) - s_n(x)$$

This is called the **remainder** of the Taylor series.

Note that the series expresses a relationship between values of **f** at different points. We recall a basic result of this kind:

1. If $\mathbf{f} : [a, b] \to \mathbb{R}$ is continuous and $\mathbf{f}'(x)$ exists in (a, b), the MVT says that

$$\mathbf{f}(b) = \mathbf{f}(a) + (b - a)\mathbf{f}'(c),$$

for some $c \in (a, b)$.

2. If we assume more: Suppose $\mathbf{f}'(x)$ is continuous on $[a, b]$ and $\mathbf{f}''(x)$ exists in (a, b):

$$\mathbf{f}(b) = \mathbf{f}(a) + (b - a)\mathbf{f}'(a) + \frac{(b - a)^2}{2}\mathbf{f}''(c),$$

for some $c \in (a, b)$.

To prove this, consider the function

$$\mathbf{g}(x) = \mathbf{f}(b)-\mathbf{f}(x)-(b-x)\mathbf{f}'(x)-\frac{(b-x)^2}{(b-a)^2}(\mathbf{f}(b)-\mathbf{f}(a)-(b-a)\mathbf{f}'(a)).$$

Note that it vanishes for $x = a$ and $x = b$. Since it is differentiable, by Rolle's Theorem

$$\mathbf{g}'(c) = 0$$

for some $c \in (a, b)$. Since

$$\mathbf{g}'(x) = -(b-x)\mathbf{f}''(x) - \frac{2(b-x)}{(b-a)^2}(\mathbf{f}(b) - \mathbf{f}(a) - (b-a)\mathbf{f}'(a)),$$

and we get: $\mathbf{f}(b) - \mathbf{f}(a) - (b-a)\mathbf{f}'(a) = \frac{\mathbf{f}''(c)}{2!}(b-a)^2$.

Taylor's Theorem

This can be proved in all degrees:

Theorem

Suppose that $\mathbf{f} : [a, b] \to \mathbb{R}$ *is n-times differentiable on* $[a, b]$ *and* $\mathbf{f}^{(n)}$ *is continuous on* $[a, b]$ *and differentiable on* (a, b)*. Assume* $x_0 \in [a, b]$*. Then for each* $x \in [a, b]$ *with* $x \neq x_0$*, there is c between x and* x_0 *such that*

$$\mathbf{f}(x) = \mathbf{f}(x_0) + \sum_{k=1}^{n} \frac{\mathbf{f}^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{\mathbf{f}^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Proof of Taylor's

Define the function

$$\mathbf{F}(t) = \mathbf{f}(t) + \sum_{k=1}^{n} \frac{\mathbf{f}^{(k)}(t)}{k!}(x-t)^k + M(x-t)^{n+1},$$

where M is chosen so that $\mathbf{F}(x_0) = \mathbf{f}(x)$. This is possible because $x - x_0 \neq 0$.
$\mathbf{F}$ is continuous on $[a, b]$ and differentiable on (a, b), and

$$\mathbf{F}(x) = \mathbf{f}(x) = \mathbf{F}(x_0).$$

By Rolle's Theorem,

$$\mathbf{F}'(c) = 0, \quad \text{for } c \text{ between } x \text{ and } x_0$$

$$0 = \mathbf{F}'(c) = \frac{\mathbf{f}^{(n+1)}(c)}{n!}(x-c)^n - (n+1)M(x-c)^n.$$

This gives

$$M = \frac{\mathbf{f}^{(n+1)}(c)}{(n+1)!}$$

and

$$\mathbf{f}(x) = \mathbf{F}(x_0) = \mathbf{f}(x_0) + \sum_{k=1}^{n} \frac{\mathbf{f}^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{\mathbf{f}^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

$$\begin{aligned}\mathbf{f}(b) &= \mathbf{f}(a) + (b-a)\mathbf{f}'(a) + \frac{(b-a)^2}{2}\mathbf{f}''(a) \\ &+ \cdots + \frac{(b-a)^{n-1}}{(n-1)!}\mathbf{f}^{(n-1)}(a) + \frac{(b-a)^n}{n!}\mathbf{f}^{(n)}(c),\end{aligned}$$

for some $c \in (a, b)$. To prove this, consider the function

$$\mathbf{g}(x) = \mathbf{F}_n(x) - \left(\frac{b-x}{b-a}\right)^n \mathbf{F}_n(a)$$

where

$$\mathbf{F}_n(x) = \mathbf{f}(b) - \mathbf{f}(x) - (b-x)\mathbf{f}'(x) - \cdots - \frac{(b-x)^{n-1}}{(n-1)!}\mathbf{f}^{(n-1)}(x).$$

The function $\mathbf{g}(x)$ vanishes at $x = a$ and $x = b$.

Its derivative is

$$\frac{n(b-x)^{n-1}}{(b-a)^n}\left(\mathbf{F}_n(a)-\frac{(b-a)^n}{n!}\mathbf{f}^{(n)}(x)\right),$$

which must vanish by Rolle's Theorem for some $a < c < b$.

This gives the formula

$$\mathbf{f}(x)=\sum_{i=0}^{n-1}\frac{\mathbf{f}^{(i)}(a)}{i!}(x-a)^i+R_n(x)$$

We must control the term (**remainder**)

$$R_n(x)=\frac{(b-a)^n}{n!}\mathbf{f}^{(n)}(c), \quad a < c < x$$

to study Taylor's.

Example

Problem: Compute the first 5 decimals of e.
The Taylor series of e^x around $x_0 = 0$ is

$$1 + x + \cdots + \frac{x^n}{n!} + \cdots$$

The remainder term is

$$\frac{\mathbf{f}^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}, \quad c \in [0, x].$$

We want to find n so that the remainder (for $x = 1$) is $< 10^{-6}$. We know that the derivatives of e^x are e^x, so $e^c \leq e < 4$. As $(1-c) \leq 1$, the remainder is smaller than

$$\frac{4}{(n+1)!}$$

We pick n so that

$$\frac{4}{(n+1)!} < 10^{-6}$$

That is,

$$(n+1)! > 4 \times 10^{6}$$

$$\begin{aligned} 7! &= 5040 \\ 10! &= 720 \times 5040 \\ 11! &> 4 \times 10^{6} \end{aligned}$$

Example

Let $\mathbf{f}(x) = \log(1+x)$, $a = 0$: Then

$$
\begin{aligned}
\mathbf{f}'(x) &= \frac{1}{1+x} \\
\mathbf{f}''(x) &= \frac{-1}{(1+x)^2} \\
&\vdots \\
\mathbf{f}^{(n)}(x) &= (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}
\end{aligned}
$$

Thus

$$|R_n(x)| = \frac{1}{n}\left|\frac{1}{|1+x|^n}\right| \leq \frac{1}{n}, \quad 0 \leq x$$

Example

Let $\mathbf{f}(x) = \arctan x$, $a = 0$: Then

$$\begin{aligned} \mathbf{f}'(x) &= \frac{1}{1+x^2} \\ \mathbf{f}''(x) &= \frac{-2x}{(1+x^2)^2} \\ &\vdots \\ \mathbf{f}^{(n)}(x) &= ? \end{aligned}$$

We will be tricky: Consider the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

Exercises

1. Decide whether the series converges or diverges

$$\sum_{n\geq 1} \frac{\sqrt{n+1}-\sqrt{n}}{n}$$

2. Write the Taylor series of $\ln x$ using powers of $x-1$
3. Prove that $e^x \geq 1+x$ for all x.
4. Use induction to show that $1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} \geq \sqrt{n}$. Which other way?
5. Chapter 6: 9, 19, 22, 24(a,b), 37, 41b, 42

Outline

Workshop #11

1. Observe that the series

$$\mathbf{f}(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

converges for on $[0, 1)$ but not when $x = 1$. For fixed $x_0 \in (0, 1)$, use the M-test to prove that **f** is continuous at x_0.

2. Let

$$\mathbf{f}(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}$$

1: Show that **f** is a continuous function defined on all of $\mathbb{R}$.
2: Is **f** differentiable? If so, is $\mathbf{f}'$ continuous?

Outline

Old Finals

1. (10 pts) State carefully and prove the **Mean Value Theorem.**

2. (8 pts)

   1. What is a countable set? Show that the set of rational numbers is countable.
   2. Show that the set of irrational numbers is not countable.

3. (8 pts)

1. What is a monotone sequence of real numbers?
2. If (a_n) is a bounded monotone sequence, prove that it converges.

4. (8 pts) Let $x_1 = 1$ and $x_{n+1} := 1 + \frac{1}{x_n}$. Show that (x_n) is a convergent sequence and find its limit.

5. (8 pts) If $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ is a nonzero function satisfying $\mathbf{f}(x+y) = \mathbf{f}(x) + \mathbf{f}(y)$ and $\mathbf{f}(xy) = \mathbf{f}(x)\mathbf{f}(y)$ for any $x, y \in \mathbb{R}$, prove:

1. $\mathbf{f}(m/n) = m/n$ for every $m/n \in \mathbb{Q}$.
2. For $a \in \mathbb{R}$, if $a > 0$ then $\mathbf{f}(a) > 0$. (Note that every positive number is a square.)
3. Use (2) to prove that if $x > y$ then $\mathbf{f}(x) > \mathbf{f}(y)$.
4. Use (1), (3), the Density of $\mathbb{Q}$ and NIP, to prove that $\mathbf{f}(x) = x$ for every $x \in \mathbb{R}$.

6. (8 pts) Let $\mathbf{f} : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). If $\mathbf{f}(a) = \mathbf{f}(b) = 0$, show that for any $k \in \mathbb{R}$ there is $c \in (a, b)$ such that

$$\mathbf{f}'(c) = k\mathbf{f}(c).$$

Hint: Consider $\mathbf{f}(x)e^{-kx}$

7. (8 pts)

1. Describe the Cantor set C.
2. Show that C is uncountable.
3. Show that $1/4 \in C$.

8. (8 pts) [Topology]

1. What is an open set of $\mathbb{R}$?
2. If A and B are subsets of $\mathbb{R}$, $A + B = \{a + b \mid a \in A, \quad b \in B\}$. If $A = (1, 3)$ and $B = (2, 5)$, what is $A + B$?
3. If A and B are open, prove that $A + B$ is also open.
4. Prove (3) assuming only that B is open.

9. (8 pts) Find the Taylor series of $\arctan x$ and determine where it converges.

10. (8 pts) What is the **radius of convergence** of a power series $\sum_{n\geq 1} a_n x^n$?

If $\mathbf{f}(x) = x^2 + x + 1$, and $a_n = \mathbf{f}(n)$ for $n \in \mathbb{N}$, find the radius of convergence of the corresponding series.

11. (8 pts) Let

$$\mathbf{f}(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}.$$

1. Show that $\mathbf{f}(x)$ is differentiable and that its derivative $\mathbf{f}'(x)$ is continuous.
2. Can we determine if $\mathbf{f}$ is twice differentiable? [Explain]

12. (10 pts) Explain [as in prove] why the Riemann integral, $\int_a^b \mathbf{f}$, of a continuous function $\mathbf{f}$ on the closed interval $[a, b]$ exists.