Math 311–03: Advanced Calculus

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Set 5

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Outline

Partitions and Riemann Sums

Given a

$$\boxed{\mathbf{f} : [a, b] \to \mathbb{R}}$$

bounded, that is, with $|\mathbf{f}(x)| \leq M$ for $x \in [a, b]$, what is

$$\int_a^b \mathbf{f}(x)?$$

Partitions

Definition

A partition $\mathcal{P}$ of the closed interval $[a, b]$ is the choice of points $x_i \in [a, b]$,

$$a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_{n-1} \leq x_n = b.$$

It is denoted by

$$\mathcal{P} = \{x_0, x_1, \ldots, x_{n-1}, x_n\}.$$

It is just a partitioning of $[a, b]$ as a union of a special sequence of subintervals

$$[a, b] = \bigcup_{k=1}^{n} [x_{k-1}, x_k]$$

Regular partition

- A very simple partition of $[a, b]$ is $\mathcal{P} = \{a, b\}$
- We get a regular partition by choosing regularly spaced points:For a positive integer n

$$\mathcal{P} = \{x_0 = a, x_1 = a+(b-a)/n, \ldots, x_i = a+i(b-a)/n, \ldots, x_n = b\}$$

- We emphasize the ability to pick partitions where the distance between consecutive points can be as small as we wish.

Riemann Sum

Start with the following data

1. $\mathbf{f}$ is a bounded function defined on $[a, b]$, that is $|\mathbf{f}(x)| \leq M$
2. $\mathcal{P}$ is a partition of $[a, b]$,

$$\mathcal{P} = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$$

3. For each $[x_{k-1}, x_k]$ pick a point y_k, $x_{k-1} \leq y_k \leq x_k$

Definition

Riemann sum of $\mathbf{f}$ is an expression of the form

$$\sum_{k=1}^{n} \mathbf{f}(y_k)(x_k - x_{k-1})$$

Data for a Riemann sum

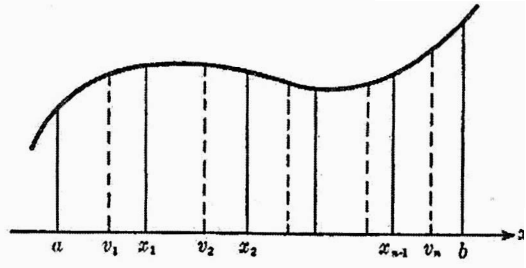


Fig. 262. Arbitrary subdivision in the general definition of integra

Observe the geometric interpretations:

- The rectangle of base the interval $[x_{k-1}, x_k]$, and height $\mathbf{f}(y_k)$ has for area
$$\mathbf{f}(y_k)(x_k - x_{k-1})$$
It gives an indication of the area under the graph of **f** over the interval. It is actually an algebraic area, that is a signed area.
- The sum
$$\sum_{k=1}^{n} \mathbf{f}(y_k)(x_k - x_{k-1})$$
can be viewed as an approximation of the area under the graph over $[a, b]$.
- These expressions as called **Riemann sums** of **f**. The textbook writes $S(\mathcal{P}, \mathbf{f})$ for them, even as they depend on the y_k.

Examples

Let $\mathbf{f} : [a, b] \to \mathbb{R}$ be the constant function $\mathbf{f}(x) = C$. For any partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$, we have, for any y_k,

$$\begin{aligned} S(\mathcal{P}, \mathbf{f}) &= \sum_{k=1}^{n} \mathbf{f}(y_k)(x_k - x_{k-1}) \\ &= C \sum_{k=1}^{n} (x_k - x_{k-1}) = C(b - a) \end{aligned}$$

Let $\mathbf{f} : [a, b] \to \mathbb{R}$ be the function $\mathbf{f}(x) = x$. For the partition $\mathcal{P} = \{x_k = a + k(b-a)/n,\ k = 0, \ldots, n\}$, and $y_k = x_k$,

$$\begin{aligned} S(\mathcal{P}, \mathbf{f}) &= \sum_{k=1}^{n} \mathbf{x}_k (x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} (a + k(b-a)/n)((b-a)/n) \\ &= \sum_{k=1}^{n} a(b-a)/n + ((b-a)/n)^2)) \sum_{k=1}^{n} k \\ &= a(b-a) + ((b-a)/n)^2 (n(n+1)/2) \end{aligned}$$

a quantity that approaches $(b^2 - a^2)/2$ as n grows.

Parabolas

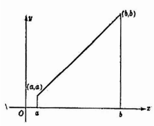


Fig. 263. Area of a trapezoid.

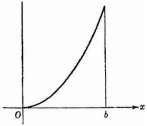


Fig. 264. Area under a parabola.

$\Delta x = b/n$. Since $x_j = j \cdot \Delta x$ and $f(x_j) = j^2(\Delta x)^2$, we obtain for S_n the expression

$$S_n = \sum_{j=1}^{n} f(j\Delta x)\Delta x = [1^2 \cdot (\Delta x)^2 + 2^2 \cdot (\Delta x)^2 + \cdots + h^2(\Delta x)^2] \cdot \Delta x$$
$$= (1^2 + 2^2 + \cdots + n^2)(\Delta x)^3.$$

Now we can actually calculate the limit. Using the formula

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

established on page 14, and making the substitution $\Delta x = b/n$, we obtain

$$S_n = \frac{n(n+1)(2n+1)}{6} \cdot \frac{b^3}{n^3} = \frac{b^3}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right).$$

This preliminary transformation makes the passage to the limit an easy matter, since $1/n$ tends to zero as n increases indefinitely. Thus we obtain as limit simply $\frac{b^3}{6} \cdot 1 \cdot 2 = \frac{b^3}{3}$, and thereby the result

$$\int^b x^2\, dx = b^3/3.$$

Difficulty

Question: How are we to handle the limit

$$\lim_{\mathcal{P}} S(\mathcal{P}, \mathbf{f})$$

We could consider only regular partition and take finer and finer meshes, an strategy that Archimedes used already. It is not good enough. What we are going to do is to define two new types of Riemann sums associated to $\mathcal{P}$, that have the property

$$L(\mathcal{P}, \mathbf{f}) \leq S(\mathcal{P}, \mathbf{f}) \leq U(\mathcal{P}, \mathbf{f}),$$

and then define two numbers $L(\mathbf{f})$ and $U(\mathbf{f})$ using the **Axiom of Completeness**, and study when they are equal. It will be the desired limit for the $S(\mathcal{P}, \mathbf{f})$.

Special Riemann Sums

To make this more natural, we are going to make special choices for y_k, or rather of the heights of the basic rectangles. They will also be called of Riemann sums. Consider a partition of $[a, b]$

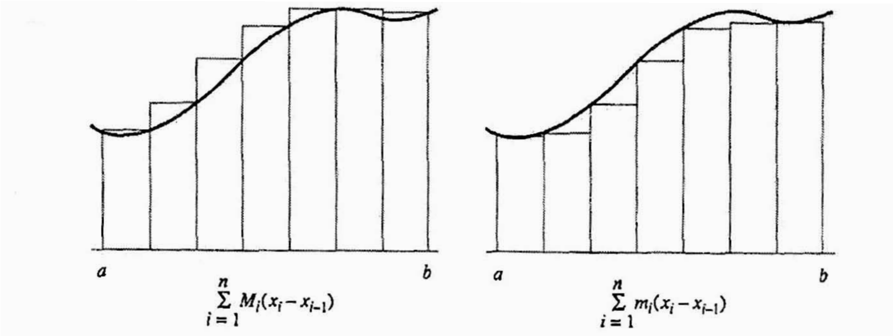
$$\mathcal{P} = \{x_0, x_1, \ldots, x_{n-1}, x_n\}.$$

For each subinterval $[x_{k-1}, x_k]$, in addition to y_k, let m_k be the **greatest lower bound** of **f** on $[x_{k-1}, x_k]$, and M_k be the **least upper bound** of **f** on $[x_{k-1}, x_k]$.

Recall that since $|\mathbf{f}(x)| \leq M$, both m_k and M_k exist by the **Axiom of Completeness.** These values determine two rectangles.

Upper and Lower Riemann sums

Chapter 5 The Riemann Integral



Note that from

$$m_k \leq \mathbf{f}(y_k) \leq M_k,$$

we have

$$m_k(x_k - x_{k-1}) \leq \mathbf{f}(y_k)(x_k - x_{k-1}) \leq M_k(x_k - x_{k-1})$$

$$\sum_{k=1}^{n} m_k(x_k - x_{k-1}) \leq \sum_{k=1}^{n} \mathbf{f}(x_k)(x_k - x_{k-1}) \leq \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

If we denote the first sum by $L(\mathcal{P}, \mathbf{f})$ and the last by $U(\mathcal{P}, \mathbf{f})$ we have a comparison of 'areas'

$$L(\mathcal{P}, \mathbf{f}) \leq \sum_{k=1}^{n} \mathbf{f}(x_k)(x_k - x_{k-1}) \leq U(\mathcal{P}, \mathbf{f})$$

Key idea

This will permit us to define two real numbers:

$$\begin{aligned} L(\mathbf{f}) &= \sup_{\mathcal{P}} L(\mathcal{P}, \mathbf{f}) \\ U(\mathbf{f}) &= \inf_{\mathcal{P}} U(\mathcal{P}, \mathbf{f}) \end{aligned}$$

We want to see when these two numbers are the same. This is so remarkable that it will have a name: The Riemann integral.

Refinement of Partitions

Now we examine what happens to $L(\mathcal{P}, \mathbf{f})$ and $U(\mathcal{P}, \mathbf{f})$ when we change $\mathcal{P}$. By a **refinement** of

$$\mathcal{P} = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$$

we mean a partition obtained by further partitioning of the intervals $[x_{k-1}, x_k]$:

$$\mathcal{P}' = \{z_0, z_1, \ldots, z_{m-1}, z_m\}$$

where the z_i includes the x_k.

Let us see what happens when the refinement consists of adding a single point z, $\mathcal{P}' = \mathcal{P} \cup \{z\}$

$$x_{k-1} < z < x_k$$

If m_k is the greatest lower bound for $\mathbf{f}(x)$ on $[x_{k-1}, x_k]$, we now also have numbers m'_k the greatest lower bound for $\mathbf{f}(x)$ on $[x_{k-1}, z]$ and m''_k the greatest lower bound for $\mathbf{f}(x)$ on $[z, x_k]$. We have

$$m_k(x_k - x_{k-1}) = m_k(x_k - z) + m_k(z - x_{k-1}) \leq m''_k(x_k - z) + m'_k(z - x_{k-1})$$

This shows that

$$L(\mathcal{P}, \mathbf{f}) \leq L(\mathcal{P}', \mathbf{f})$$

Obviously for a general refinement one can add a point at a time and obtain the same inequality. In exactly the same manner, we get the inequality for upper Riemann sums

$$U(\mathcal{P}, \mathbf{f}) \geq U(\mathcal{P}', \mathbf{f})$$

Note the reversal.

This gives the picture:

$$L(\mathbf{f},\mathcal{P}) \le L(\mathbf{f},\mathcal{P}') \to ? \quad ? \leftarrow U(\mathbf{f},\mathcal{P}') \le U(\mathbf{f},\mathcal{P})$$

Thus as we refine the partitions the lower sums don't decrease and the upper sums don't increase. Always $L(\mathbf{f},\mathcal{P}) \le U(\mathbf{f},\mathcal{P}')$. In particular, we

$$\sup L(\mathbf{f},\mathcal{P}) = L(\mathbf{f}) \le U(\mathbf{f}) = \inf U(\mathbf{f},\mathcal{P})$$

Criterion for the equality $L(\mathbf{f}) = U(\mathbf{f})$

The following is our main test to see whether $\int_a^b \mathbf{f}(x)dx$ exists:

Proposition

Let $\mathbf{f}$ *be a bounded function on the interval* $[a, b]$*. Then* $L(\mathbf{f}) = U(\mathbf{f})$ *if for any* $\epsilon > 0$ *there exists a partition* $\mathcal{P}_\epsilon$ *so that*

$$U(\mathcal{P}_\epsilon, \mathbf{f}) - L(\mathcal{P}_\epsilon, \mathbf{f}) < \epsilon.$$

Proof. Suppose $L(\mathbf{f}) = U(\mathbf{f})$. By definition there exist partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ such that

$$L(\mathbf{f}) - L(\mathcal{P}_1, \mathbf{f}) \leq \epsilon/2$$

$$U(\mathcal{P}_2, \mathbf{f}) - U(\mathbf{f}) \leq \epsilon/2$$

Consider the partition $\mathcal{P}_\epsilon = \mathcal{P}_1 \cup \mathcal{P}_2$, which refines both partitions.
The assertion that

$$U(\mathcal{P}_\epsilon, \mathbf{f}) - L(\mathcal{P}_\epsilon, \mathbf{f}) < \epsilon$$

follows from the inequalities

$$U(\mathcal{P}_2, \mathbf{f}) \geq U(\mathcal{P}_\epsilon, \mathbf{f}) \geq U(\mathbf{f}) = L(\mathbf{f}) \geq L(\mathcal{P}_\epsilon, \mathbf{f}) \geq L(\mathcal{P}_1, \mathbf{f}).$$

Conversely, suppose for each $\epsilon >$ there is a partition $\mathcal{P}_\epsilon$ so that

$$U(\mathcal{P}_\epsilon, \mathbf{f}) - L(\mathcal{P}_\epsilon, \mathbf{f}) < \epsilon.$$

We need to show that $L(\mathbf{f}) = U(\mathbf{f})$.

Let us argue by contradiction. If not, for $\epsilon = U(\mathbf{f}) - L(\mathbf{f})$, it suffices to look at the inequalities

$$U(\mathcal{P}_\epsilon, \mathbf{f}) \geq U(\mathbf{f}) \geq L(\mathbf{f}) \geq L(\mathcal{P}_\epsilon, \mathbf{f}), .$$

showing that $U(\mathbf{f}) - L(\mathbf{f}) < \epsilon$, a contradiction.

Outline

Riemann Integral

Let $\mathbf{f} : [a, b] \to \mathbb{R}$ be a bounded function. If $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$ consider the lower and upper Riemann sums $L(\mathcal{P}, \mathbf{f})$ and $U(\mathcal{P}, \mathbf{f})$.

We will say that the **diameter** Δ of $\mathcal{P}$ is the largest of the lengths $|x_k - x_{k-1}|$. We will want to make sense of the **limits** of such Riemann sums as $\Delta \to 0$.

Definition

If $L(\mathbf{f}) = U(\mathbf{f})$, we denote this value $\int_a^b \mathbf{f}$, or $\int_a^b \mathbf{f}(x)dx$, and call it the Riemann integral of $\mathbf{f}$ on $[a, b]$.

Monotone Functions are Integrable

Let us give an example of a class of functions, not necessarily continuous, that have Riemann integrals.

Theorem

If $\mathbf{f} : [a, b] \to \mathbb{R}$ *is monotone, then* $\int_a^b \mathbf{f}(x)dx$ *exists.*

Proof. Suppose $\mathbf{f}$ is monotone increasing (i.e. if $x_0 \le x_1$ then $\mathbf{f}(x_0) \le \mathbf{f}(x_1)$). Choose $\epsilon > 0$. There is a k such that $k(\mathbf{f}(b) - \mathbf{f}(a)) < \epsilon$. Choose a partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ so that

$$x_i - x_{i-1} < k, \quad i = 1, 2, \ldots, n.$$

Since $\mathbf{f}$ is increasing, $M_i = \mathbf{f}(x_i)$ and $m_i = \mathbf{f}(x_{i-1})$. Now

$$\begin{aligned} U(\mathcal{P},\mathbf{f}) - L(\mathcal{P},\mathbf{f}) &= \sum_{i=1}^{n} [\mathbf{f}(x_i) - \mathbf{f}(x_{i-1})][x_i - x_{i-1}] \\ &\leq \sum_{i=1}^{n} [\mathbf{f}(x_i) - \mathbf{f}(x_{i-1})]k = k[\mathbf{f}(b) - \mathbf{f}(a)] < \epsilon \end{aligned}$$

By the criterion, $\int_a^b \mathbf{f}$ exists. The functions for which the integral exists are called *Riemann integrable*. The textbook denotes this set by $\mathcal{R}[a, b]$. There are other types of **integral.**

Example

Consider the sequence $\{x_n = 1 - 1/n : n \geq 1\}$ and let us build a monotone function **f** on $[0, 1]$ as follows:

$$
\begin{aligned}
\mathbf{f}(x) &= 1/2, \quad 0 \leq x < 1/2 \\
\mathbf{f}(x) &= 1 - 1/3, \quad 1/2 \leq x < 1 - 1/3 \\
&\vdots \\
\mathbf{f}(x) &= 1 - 1/n, \quad 1 - 1/(n-1) \leq x < 1 - 1/n \\
\mathbf{f}(1) &= 1
\end{aligned}
$$

f is monotone but not continuous: its graph has an infinite number of breaks.

Exercise: Its Riemann integral is what ?.

Dirichlet Function

Consider the function

$$\mathbf{f}(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

and let us try to determine its integral on $[0, 1]$.
For any partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$, in the interval $[x_{i-1}, x_k]$ there are **rational** and **non rational** points. This implies that $m_k = 0$ and $M_k = 1$, and therefore

$$\begin{aligned} L(\mathcal{P}, \mathbf{f}) &= 0 \\ U(\mathcal{P}, \mathbf{f}) &= 1 \end{aligned}$$

Thus $L(\mathbf{f}) = 0 \neq U(\mathbf{f}) = 1$, and $\int_0^1 \mathbf{f}$ does not exist.

Continuous Functions are Integrable

Theorem

If $\mathbf{f}$ *is continuous on* $[a, b]$ *then* $\int_a^b \mathbf{f}(x)dx$ *exists.*

Proof. It will suffice to prove that for given $\epsilon > 0$ there is a partition $\mathcal{P}$ such that $U(\mathcal{P}, \mathbf{f}) - L(\mathcal{P}, \mathbf{f}) < \epsilon$.
We know that $\mathbf{f}$ is uniformly continuous on $[a, b]$. Thus given $\epsilon > 0$ there exists $\delta > 0$ such that for $x, y \in [a, b]$,

$$|x - y| < \delta \Rightarrow |\mathbf{f}(x) - \mathbf{f}(y)| < \frac{\epsilon}{b - a}.$$

Let $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ be a partition of diameter $< \delta$, that is $|x_k - x_{k-1}| < \delta$. If m_k and M_k are the minimum and the maximum of $\mathbf{f}$ on this interval, by the **Extreme Value Theorem** there are $y_k, z_k \in [x_{k-1}, x_k]$ so that

$$m_k = \mathbf{f}(y_k), \quad M_k = \mathbf{f}(z_k)$$

Proof cont'd

$$\begin{aligned} U(\mathcal{P}, \mathbf{f}) &= \sum_{k=1}^{n} \mathbf{f}(z_k)(x_k - x_{k-1}) \\ L(\mathcal{P}, \mathbf{f}) &= \sum_{k=1}^{n} \mathbf{f}(y_k)(x_k - x_{k-1}) \\ U(\mathcal{P}, \mathbf{f}) - L(\mathcal{P}, \mathbf{f}) &= \sum_{k=1}^{n} (\mathbf{f}(z_k) - \mathbf{f}(y_k))(x_k - x_{k-1}) \end{aligned}$$

Since $\mathbf{f}(z_k) - \mathbf{f}(y_k) < \frac{\epsilon}{b-a}$, we have

$$\begin{aligned} U(\mathcal{P}, \mathbf{f}) - L(\mathcal{P}, \mathbf{f}) &< \sum_{k=1}^{n} \frac{\epsilon}{b-a}(x_k - x_{k-1}) = \frac{\epsilon}{b-a} \sum_{k=1}^{n} (x_k - x_{k-1}) \\ &= \frac{\epsilon}{b-a}(b-a) = \epsilon \end{aligned}$$

Outline

Old Hourlies #2

- 1. State and prove the **Intermediate Value Theorem** theorem.
  2. Let $\mathbf{f} : [0,1] \to \mathbb{R}$ be a continuous function. If $\mathbf{f}(0) \neq \mathbf{f}(1)$, prove that the image of **f**, $\mathbf{f}([0,1])$, is uncountable.
- Assume $\mathbf{f}_1(x) \geq \mathbf{f}_2(x)$ for all x in some set A on which $\mathbf{f}_1$ and $\mathbf{f}_2$ are defined. Show that for any limit point c of A (what are these, anyway?) we must have

$$\lim_{x \to c} \mathbf{f}_1(x) \geq \lim_{x \to c} \mathbf{f}_2(x).$$

  Moreover, if $\mathbf{f}_1(x) < \mathbf{f}_2(x)$, can equality in the limits occur?
- 1. If $\mathbf{f} : A \to \mathbb{R}$, what does it mean to say that **f** is **uniformly continuous** on A?
  2. Which result guarantees that $\sqrt{x}$ is uniformly continuous on $[0,1]$?
  3. Prove that $\sqrt{x}$ is uniformly continuous on $[0,\infty)$. (Why can't use (2) here?).

- Evaluate the following limits. You must state the reasons.
  1. $\lim_{x\to\infty} \frac{e^x}{x^{10}}$
  2. $\lim_{x\to\infty} x \ln \frac{x+1}{x-1}$
- 1. State (clearly and completely) the **Mean Value Theorem**.
  2. Show that if $\mathbf{f}$ is a function that is differentiable on an interval with $\mathbf{f}'(x) \neq 1$, then there exists at most one point where $\mathbf{f}(c) = c$.
- Study algebraic limits–read proofs
- (Substitute Questions) Chapter 5, Miscellaneous (p. 168): 39, 40, 41

Study proofs and illustrations of the following:

1. IVT, Darboux
2. EVT
3. Rolle, MVT
4. Uniform continuity of continuous functions
5. Riemann sums, integrals of monotone and continuous functions
6. L'Hospital rules and the calculation of limits
7. Review your [relevant] workshops

Outline

The Fundamental Theorem of Calculus

It is highly desirable to have the means to actually find the value of the integral $\int_a^b \mathbf{f}(x)dx$, not merely to assert that it exists. The **FTC** does this for a huge collection of functions in a very effective manner. It converts the consideration of finding a very difficult limit process into the task of using a lookup table.

Here it is:

FTC

Theorem (FTC)

Let $\mathbf{f} : [a, b] \to \mathbb{R}$ *be a function such that* $\int_a^b \mathbf{f}$ *exists. If* $\mathbf{F}$ *is a function such that* $\mathbf{F}'(c) = \mathbf{f}(c)$ *for all* $c \in [a, b]$*, then*

$$\int_a^b \mathbf{f}(x)dx = \mathbf{F}(b) - \mathbf{F}(a).$$

Proof. Let $\mathcal{P}$ be a partition of $[a, b]$. By the **MVT**,

$$\mathbf{F}(x_k) - \mathbf{F}(x_{k-1}) = \mathbf{F}'(y_k)(x_k - x_{k-1}) = \mathbf{f}(y_k)(x_k - x_{k-1}), \quad y_k \in [x_{k-1}, x_k]$$

Adding both sides over the terms of the partition, and taking into account the telescoping terms on the left, we get

$$\sum_{k=1}^{n} \mathbf{F}(x_k) - \mathbf{F}(x_{k-1}) = \mathbf{F}(b) - \mathbf{F}(a) = \sum_{k=1}^{n} \mathbf{f}(y_k)(x_k - x_{k-1}),$$

This means that $\mathbf{F}(b) - \mathbf{F}(a)$ equals a Riemann sum associated to $\mathcal{P}$. Since

$$L(\mathcal{P}, \mathbf{f}) \leq \mathbf{F}(b) - \mathbf{F}(a) = \sum_{k=1}^{n} \mathbf{f}(y_k)(x_k - x_{k-1}) \leq U(\mathcal{P}, \mathbf{f}),$$

we have

$$L(\mathcal{P}, \mathbf{f}) \leq \mathbf{F}(b) - \mathbf{F}(a) \leq U(\mathcal{P}, \mathbf{f}).$$

This implies that $\mathbf{F}(b) - \mathbf{F}(a)$ is arbitrarily close to $\int_a^b \mathbf{f}(x)dx$ since $U(\mathcal{P}, \mathbf{f}) - L(\mathcal{P}, \mathbf{f})$ can be made arbitrarily small. This shows

$$\mathbf{F}(b) - \mathbf{F}(a) = \int_a^b \mathbf{f}(x)dx.$$

Another Version of FTC

Theorem

If $\mathbf{f} : [a, b] \to \mathbb{R}$ *satisfies* $|\mathbf{f}(x)| \leq M$ *and is integrable, then*

$$\mathbf{G}(x) = \int_a^x \mathbf{f}(t)dt$$

is a continuous function on $[a, b]$*. Moreover, if* $\mathbf{f}(x)$ *is continuous at* $x = c$*, then*

$$\mathbf{G}'(c) = \mathbf{f}(c).$$

This **FTC** is about the derivative of the integral. The earlier version was about the integral of the derivative.

This says that $\mathbf{G}(x)$ is a ‘nicer’ function than $\mathbf{f}(x)$. Consider the following example.

Example

$$f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 2 & 1 < x \end{cases}$$

$$G(x) = \int_0^x f(t)\,dt$$

$$= \begin{cases} x & 0 \le x \le 1 \\ 1 + 2(x-1) & x > 1 \end{cases}$$

We need the following result as preparation:

Proposition

Assume that $\mathbf{f} : [a, b] \to \mathbb{R}$ *is bounded. If* $a < c < b$ *then* $\mathbf{f} \in \mathcal{R}[a, b]$ *if and only if* $\mathbf{f} \in \mathcal{R}[a, c]$ *and* $\mathbf{f} \in \mathcal{R}[c, b]$. *In this case,*

$$\int_a^b \mathbf{f} = \int_a^c \mathbf{f} + \int_c^b \mathbf{f}.$$

The basic idea is the following: If $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions of $[a, c]$ and $[c, b]$, respectfully, then $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ is a partition of $[a, b]$. Conversely, if $\mathcal{P}$ is a partition of $[a, b]$, by adding the point $x = c$, if needed, and thus obtaining a refinement of $\mathcal{P}$, we can express it as a union partitions as above.

The argument is now standard. For example, if $\int_a^c \mathbf{f}$ and $\int_c^b \mathbf{f}$ exist, given $\epsilon > 0$, pick partitions partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ and select points in the subintervals so that the Riemann sums $S(\mathcal{P}_1, \mathbf{f})$ and $S(\mathcal{P}_2, \mathbf{f})$ satisfy

$$|S(\mathcal{P}_1, \mathbf{f}) - \int_a^c \mathbf{f}| < \epsilon/2, \quad |S(\mathcal{P}_2, \mathbf{f}) - \int_a^c \mathbf{f}| < \epsilon/2$$

Now $S(\mathcal{P}_1, \mathbf{f}) + S(\mathcal{P}_2, \mathbf{f})$ is a Riemannn sum of $\mathbf{f}$ over $[a, b]$ that is closer to $\int_a^c \mathbf{f} + \int_c^b \mathbf{f}$ by less that ϵ,

$$|S(\mathcal{P}_1, \mathbf{f}) + S(\mathcal{P}_2, \mathbf{f}) - \int_a^c \mathbf{f} - \int_c^b \mathbf{f}| < \epsilon$$

Together with the earlier note, this implies the equality

$$\int_a^b \mathbf{f} = \int_a^c \mathbf{f} + \int_c^b \mathbf{f}.$$

The converse is similar.

Another version of FTC

Theorem

If $\mathbf{f} : [a, b] \to \mathbb{R}$ *satisfies* $|\mathbf{f}(x)| \leq M$ *and is integrable, then*

$$\mathbf{G}(x) = \int_a^x \mathbf{f}(t)dt$$

is a continuous function on $[a, b]$*. Moreover, if* $\mathbf{f}(x)$ *is continuous at* $x = c$*, then*

$$\mathbf{G}'(c) = \mathbf{f}(c).$$

Proof. $\mathbf{G}$ is well-defined since $\mathbf{f}$ is integrable on $[a, c]$ for $a < c < b$, by a previous proposition. Observe that for $x, y \in [a, b]$

$$\mathbf{G}(x) - \mathbf{G}(y) = \int_y^x \mathbf{f}(t)dt$$

In particular, $\mathbf{G}$ is uniformly continuous. Suppose $\mathbf{f}$ is continuous at $x = c$. Consider

$$\frac{\mathbf{G}(x) - \mathbf{G}(c)}{x - c} = \frac{\int_c^x \mathbf{f}(t)dt}{x - c}.$$

$$\frac{\mathbf{G}(x) - \mathbf{G}(c)}{x - c} - \mathbf{f}(c) = \frac{\int_c^x (\mathbf{f}(t) - \mathbf{f}(c))dt}{x - c}.$$

Since $\mathbf{f}$ is continuous at $x = c$, for any $\epsilon > 0$ there exists a neighborhood $V_\delta(c)$ of c so that $|\mathbf{f}(t) - \mathbf{f}(c)| < \epsilon$ for $t \in V_\delta(c)$. This gives for $x \in V_\delta(c)$

$$|\frac{\mathbf{G}(x) - \mathbf{G}(c)}{x - c} - \mathbf{f}(c)| \leq \frac{\int_c^x |\mathbf{f}(t) - \mathbf{f}(c)|dt}{x - c} < \epsilon.$$

Example

The function

$$\mathbf{G}(x) = \int_0^x e^{-t^2} dt$$

satisfies

$$\mathbf{G}'(x) = e^{-x^2}$$

Note that we did not know much about $\mathbf{G}(x)$!

Example

For a function $\mathbf{F}(x)$, we can define

$$\mathbf{G}(x) = \int_a^{\mathbf{F}(x)} \mathbf{f}(t)dt,$$

which can be interpreted as a composition

$$\mathbf{g}(u) = \int_a^{u} \mathbf{f}(t)dt, \quad u = \mathbf{F}(x).$$

If $\mathbf{F}(x)$ is differentiable, the **chain rule** gives

$$\mathbf{G}'(x) = (\mathbf{g} \circ \mathbf{F})'(x) = \mathbf{f}(\mathbf{F}(x))\mathbf{F}'(x).$$

As an exercise, if $\mathbf{F}(x)$ and $\mathbf{H}(x)$ are differentiable, find the derivative of

$$\int_{\mathbf{H}(x)}^{\mathbf{F}(x)} \mathbf{f}(t)dt$$

Outline

Algebra of Integrable Functions

The set $\mathcal{R}[a,b]$ contains all the continuous functions on $[a,b]$ but many others. Some operations on these produce other integrable functions: Clearly

$$\mathbf{f}, \mathbf{g} \in \mathcal{R}[a,b] \Rightarrow \mathbf{f} + \mathbf{g} \in \mathcal{R}[a,b]$$

but it is not so clear that

$$\mathbf{f}, \mathbf{g} \in \mathcal{R}[a,b] \Rightarrow \mathbf{f} \cdot \mathbf{g} \in \mathcal{R}[a,b],$$

or that $|\mathbf{f}(x)| \in \mathcal{R}[a,b]$.
Let us analyze a piece of algebra to see what is needed to derive those facts. Of course if $\mathbf{f}(x)$ is continuous we do not bother since composites of continuous is continuous.But it is a fact that the composite of two integrable functions may not be integrable (see p. 156).

If $\mathbf{f}(x)$ and $\mathbf{g}(x)$ are integrable, then both $\mathbf{f}(x) + \mathbf{g}(x)$ and $\mathbf{f}(x) - \mathbf{g}(x)$ are integrable. Suppose we knew that the square of an integrable function is integrable. This would imply that

$$(\mathbf{f}(x) + \mathbf{g}(x))^2 - (\mathbf{f}(x) - \mathbf{g}(x))^2 = 4\mathbf{f}(x)\mathbf{g}(x)$$

is integrable, giving our goal. What we need is:

Theorem

Suppose $\mathbf{f} : [a, b] \to S$ is integrable on $[a, b]$ and $\psi : S \to \mathbb{R}$ is continuous with S compact. Then the composite $\phi \circ \mathbf{f} \in \mathcal{R}[a, b]$.

In the argument above, we would use $\phi(x) = x^2$, while for $\int_a^b |\mathbf{f}|$ we would invoke $\phi(x) = |x|$: Both are continuous and since $\mathbf{f}$ is bounded we may assume $\mathbf{f}[a, b]$ is contained in a compact set S.

Proof

Choose $\epsilon > 0$, let $K = \sup\{|\phi(t)| : t \in S\}$, and let $\epsilon'[b - a + 2K] \leq \epsilon$.

By Uniform Continuity of ϕ on S, there is $0 < \delta < \epsilon'$ such that $s, t \in S$ and $|s - t| < \delta$ implies $|\phi(s) - \phi(t)| < \epsilon'$.

Since $\mathbf{f}(x)$ is integrable on $[a, b]$, there is a partition $\mathcal{P}$ of $[a, b]$ such that

$$U(\mathcal{P}, \mathbf{f}) - L(\mathcal{P}, \mathbf{f}) < \delta^2.$$

We want to argue that

$$U(\mathcal{P}, \phi \circ \mathbf{f}) - L(\mathcal{P}, \phi \circ \mathbf{f}) < \epsilon.$$

If $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$, split the indices *i*'s into two sets:

$$\begin{aligned} A &= \{i : M_i(\mathbf{f}) - m_i(\mathbf{f}) < \delta\} \\ B &= \{i : M_i(\mathbf{f}) - m_i(\mathbf{f}) \geq \delta\} \end{aligned}$$

For $i \in A$ and $s, t \in [x_{i-1}, x_i]$, $|\phi(s) - \phi(t)| < \delta$, so

$$|\phi(\mathbf{f}(s)) - \phi(\mathbf{f}(t))| < \epsilon';$$

that is

$$M_i(\phi \circ \mathbf{f}) - m_i(\phi \circ \mathbf{f}) \leq \epsilon'.$$

For $i \in B$, $M_i(\mathbf{f}) - m_i(\mathbf{f}) \geq \delta$, hence

$$\begin{aligned} \delta \sum_{i \in B} (x_i - x_{i-1}) &\leq \sum_{i \in B} [M_i(\mathbf{f}) - m_i(\mathbf{f})](x_i - x_{i-1}) \\ &\leq U(\mathcal{P}, \mathbf{f}) - L(\mathcal{P}, \mathbf{f}) \leq \delta^2 \end{aligned}$$

Thus $\sum_{i \in B} (x_i - x_{i-1}) \leq \delta$.

$$
\begin{aligned}
U(\mathcal{P}, \phi \circ \mathbf{f}) - L(\mathcal{P}, \phi \circ \mathbf{f}) &= \sum_{i=1}^{n} [M_i(\phi \circ \mathbf{f}) - m_i(\phi \circ \mathbf{f})](x_i - x_{i-1}) \\
&= \sum_{i \in A} [M_i(\phi \circ \mathbf{f}) - m_i(\phi \circ \mathbf{f})](x_i - x_{i-1}) \\
&+ \sum_{i \in B} [M_i(\phi \circ \mathbf{f}) - m_i(\phi \circ \mathbf{f})](x_i - x_{i-1}) \\
&\leq \epsilon'(b-a) + 2K\delta' \leq \epsilon'(b-a+2K) \leq \epsilon
\end{aligned}
$$

Thus $\phi \circ \mathbf{f}$ is integrable on $[a, b]$.

Integration by Parts

Theorem

Suppose $\mathbf{f}$ *and* $\mathbf{g}$ *are differentiable on* $[a, b]$ *and* $\mathbf{f}'$ *and* $\mathbf{g}'$ *are integrable on* $[a, b]$*. Then* $\mathbf{f}\mathbf{g}'$ *and* $\mathbf{f}'\mathbf{g}$ *are integrable on* $[a, b]$ *and*

$$\int_a^b \mathbf{f}'\mathbf{g}\,dx = \mathbf{f}(b)\mathbf{g}(b) - \mathbf{f}(a)\mathbf{g}(a) - \int_a^b \mathbf{f}\mathbf{g}'\,dx.$$

Proof. $\mathbf{f}$ and $\mathbf{g}$ being differentiable are also continuous. Since the product of a continuous function and an integrable function is integrable, $\mathbf{f}'\mathbf{g}$ and $\mathbf{f}\mathbf{g}'$ are integrable.

Applying the **FTC** to the function **fg**, and using Leibnitz rule we have

$$\begin{aligned}\mathbf{f}(b)\mathbf{g}(b)-\mathbf{f}(a)\mathbf{g}(a) &\overset{\textbf{FTC}}{=} \int_a^b (\mathbf{fg})' dx \\ &= \int_a^b (\mathbf{f}'\mathbf{g}+\mathbf{fg}')dx \\ &= \int_a^b \mathbf{f}'\mathbf{g}dx + \int_a^b \mathbf{fg}'dx.\end{aligned}$$

Outline

Exercises to be Handed in

1. Sec. 5.2: 9. Assume $\mathbf{f}:[a,b]\to\mathbb{R}$ is continuous and $\mathbf{f}(x)\geq 0$ for all $x\in[a,b]$. Prove that if $\int_a^b \mathbf{f}(x)dx = 0$, then $\mathbf{f}(x)=0$ for all $x\in[a,b]$.
2. Sec. 5.3: 12. Suppose $\mathbf{f}$ in integrable on $[0,1]$. Define

$$a_n = \frac{1}{n}\sum_{k=1}^{n}\mathbf{f}(k/n)$$

for all n. Prove that (a_n) converges to $\int_0^1 \mathbf{f}dx$.
3. Sec. 5.5: 20
4. Sec. 5.5: 25
5. Sec. 5.6: 32

Outline

Workshop #9

The goal here is to create $\ln x$ and rediscover the number e. Let

$$L(x) = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$

1. Prove that L is continuous, differentiable and $1-1$.
2. Prove that L is increasing.
3. Prove that $L(ab) = L(a) + L(b)$, for $a, b > 0$.
4. Prove that $L(x^n) = nL(x)$, for $n \in \mathbb{Z}$.
5. Prove that $L(a/b) = L(a) - L(b)$, for $a, b > 0$.
6. Prove that the image of L is $\mathbb{R}$.

7. We defined *e* as the limit of the sequence $(1+1/n)^n, \quad n \geq 1$. Show that $L(e) = 1$.
8. Let E be the inverse of L (explain why E exists):$E : \mathbb{R} \to (0, \infty)$. Prove that E is continuous, differentiable and increasing.
9. Prove that $E(x+y) = E(x)E(y)$.
10. If $x > 0$ and $y \in \mathbb{R}$, define $x^y = E(yL(x))$, or as usual $x^y = e^{y \ln x}$. Show that x^x is differentiable and find $\mathbf{f}'(x)$ $(x > 0)$.

Outline

MVT for Integrals

Theorem

If $\mathbf{f} : [a, b] \to \mathbb{R}$ *is continuous and* $\mathbf{g}$ *is integrable on* $[a, b]$ *and* $\mathbf{g}(x) \geq 0$, *then there is a* $c \in [a, b]$ *such that*

$$\int_a^b \mathbf{f}(x)\mathbf{g}(x)dx = \mathbf{f}(c) \int_a^b \mathbf{g}(x)dx.$$

Proof. Let $m = \inf\{\mathbf{f}(x) : x \in [a, b]\}$ and $M = \sup\{\mathbf{f}(x) : x \in [a, b]\}$. Since $\mathbf{g}(x) \geq 0$,

$$m \int_a^b \mathbf{g}(x)dx \leq \int_a^b \mathbf{f}(x)\mathbf{g}(x)dx \leq M \int_a^b \mathbf{g}(x)$$

The only case of interest is when $\int_a^b \mathbf{g}(x)dx \neq 0$. Write the inequality as

$$m \leq \frac{\int_a^b \mathbf{f}(x)\mathbf{g}(x)dx}{\int_a^b \mathbf{g}(x)dx} \leq M.$$

Now apply the **IVT** to the function $\mathbf{f}(x)$ on $[a, b]$ to get $c \in [a, d]$ such that

$$\mathbf{f}(c) = \frac{\int_a^b \mathbf{f}(x)\mathbf{g}(x)dx}{\int_a^b \mathbf{g}(x)dx}.$$

Corollary

If $\mathbf{f} : [a, b] \to \mathbb{R}$ *is continuous, then there is* $c \in [a, b]$ *such that*

$$\int_a^b \mathbf{f}(x)dx = \mathbf{f}(c)(b - a).$$

Second MVT for Integrals

Theorem

Suppose $\mathbf{f} : [a, b] \to \mathbb{R}$ *is monotone. Then there* $c \in [a, b]$ *such that*

$$\int_a^b \mathbf{f}(x)dx = \mathbf{f}(a)(c-a) + \mathbf{f}(b)(b-c).$$

Proof. Since $\mathbf{f}$ is monotone, $\int_a^b \mathbf{f}(x)dx$ exists. Define the linear function $\mathbf{h} : [a, b] \to \mathbb{R}$, $\mathbf{h}(x) = \mathbf{f}(a)(x-a) + \mathbf{f}(b)(b-x)$. Note that $\int_a^b \mathbf{f}(x)dx$ lies between $\mathbf{h}(a) = \mathbf{f}(b)(b-a)$ and $\mathbf{h}(b) = \mathbf{f}(a)(b-a)$, since $\mathbf{f}$ is monotone. Applying the **IVT** to $\mathbf{h}(x)$, we have

$$\int_a^b \mathbf{f}(x)dx = \mathbf{f}(a)(c-a) + \mathbf{f}(b)(b-c), \quad c \in [a, b].$$

Change of Variables

Theorem

Suppose $\phi : [a, b] \to \mathbb{R}$ is differentiable and ϕ' is continuous. Further assume that $\phi([a, b]) = [c, d]$ with $\phi(a) = c$ and $\phi(b) = d$. If $\mathbf{f} : [c, d] \to \mathbb{R}$ is continuous, then

$$\int_a^b \mathbf{f}(\phi(t))\phi'(t)dt = \int_c^d \mathbf{f}(x)dx.$$

Proof. Define $\mathbf{F}(u) = \int_c^u \mathbf{f}(x)dx$ and $\mathbf{G}(s) = \int_a^s \mathbf{f}(\phi(t))\phi'(t)dt$. Since $\mathbf{f}$, ϕ and ϕ' are continuous, both $\mathbf{F}$ and $\mathbf{G}$ are differentiable and

$$\begin{aligned} \mathbf{F}'(u) &= \mathbf{f}(u), \quad u \in [c, d] \\ \mathbf{G}'(s) &= \mathbf{f}(\phi(s))\phi'(s), \quad s \in [a, b] \end{aligned}$$

Since $\mathbf{F} \circ \phi$ is differentiable, by the Chain Rule,

$$(\mathbf{F} \circ \phi)'(s) = \mathbf{F}'(\phi(s))\phi'(s) = \mathbf{f}(\phi(s))\phi'(s) = \mathbf{G}'(s).$$

This means that there is a constant K such that for $s \in [a, b]$

$$(\mathbf{F} \circ \phi)(s) = \mathbf{G}(s) + K.$$

Setting $s = a$,

$$0 = \mathbf{F}(c) = \mathbf{F}(\phi(a)) = \mathbf{G}(a) + K = 0 + K, \quad \Rightarrow K = 0,$$

as desired.

Area of the Circle

A simple application gives the area of the (quarter circle):

- Equation of the top semicircle: $y = \sqrt{r^2 - x^2}$, $0 \leq x \leq r$
- Change the variable: $x = r\cos t, \quad 0 \leq t \leq \pi/2$
- Evaluate

$$\begin{aligned}
\int_0^r y\,dx &= \int_{\pi/2}^0 \sqrt{r^2 - r^2\cos^2 t}(-r\sin t)dt \\
&= -r^2 \int_{\pi/2}^0 \sin^2 t dt = -r^2 \int_{\pi/2}^0 1/2(1 - \cos 2t)dt \\
&= \frac{\pi r^2}{4}
\end{aligned}$$

Approximations of Integrals

There are several formulas that seek approximations for

$$\int_a^b \mathbf{f}(x)dx$$

The simplest replaces the area under the graph by the area of the quadrilateral formed by the points $(a, 0)$, $(b, 0)$, $(a, \mathbf{f}(a))$, $(b, \mathbf{f}(b))$:

$$\frac{(b-a)(\mathbf{f}(b)+\mathbf{f}(a))}{2}$$

If $\mathbf{f}$ has two derivatives, one can show that the Error is

$$\text{Error} = -\frac{(b-a)^3\mathbf{f}''(c)}{12}, \quad a \leq c \leq b.$$

Simpson's Rule

A much better formula is:

Theorem (Simpson's Rule)

If $\mathbf{f}(x)$ *has four derivatives, then*

$$\int_a^b \mathbf{f}(x)dx = \frac{1}{6}(b-a)(\mathbf{f}(a) + 4\mathbf{f}(\frac{a+b}{2}) + \mathbf{f}(b)) + \mathrm{Error},$$

where

$$\mathrm{Error} = -\frac{1}{2880}(b-a)^5\mathbf{f}^{iv}(c), \quad a \leq c \leq b.$$

Proof. Write $a = m - h$, $b = m + h$ (i.e. $m = (a+b)/2$ and $h = (b-a)/2$),and consider the function

$$\phi(t) = \psi(t) - (t/m)^5\psi(m),$$
$$\psi(t) = \int_{m-t}^{m+t} \mathbf{f}(x)dt - \frac{1}{3}t(\mathbf{f}(m+t) + 4\mathbf{f}(m) + \mathbf{f}(m-t)).$$

Differentiating three times we get:

$$\begin{aligned}
\phi'(t) &= \frac{2}{3}(\mathbf{f}(m+t) - 2\mathbf{f}(c) + \mathbf{f}(m-t)) - \frac{1}{3}t(\mathbf{f}'(m+t) - \mathbf{f}'(m-t)) \\
&- \frac{5t^4}{m^5}\psi(m) \\
\phi''(t) &= \frac{1}{3}(\mathbf{f}'(m+t) - \mathbf{f}'(m-t)) - \frac{1}{3}t(\mathbf{f}''(m+t) + \mathbf{f}''(m-t)) \\
&- \frac{20t^3}{m^5}\psi(m) \\
\phi'''(t) &= -\frac{1}{3}t(\mathbf{f}'''(m+t) - \mathbf{f}'''(m-t)) - \frac{60t^2}{m^5}\psi(m)
\end{aligned}$$

One uses the **MVT** applied to the function $\mathbf{f}'''$,

$$\begin{aligned}\mathbf{f}'''(m+t)-\mathbf{f}'''(m-t) &= \mathbf{f}^{iv}(\theta)((m+t)-(m-t)) \\ &= 2t\mathbf{f}^{iv}(\theta), \quad m-t<\theta<m+\theta\end{aligned}$$

that gives

$$\phi'''(t)=-\frac{2}{3}t^2(\mathbf{f}^{iv}(\theta)+\frac{90}{m^5}\psi(m)), \quad m-t<\theta<m+t.$$

Next, one observes that $\phi(0)=\phi(m)=0$, which by Rolle's theorem, $\phi'(t_1)=0$, $t_1\in(0,m)$. Also $\phi'(0)=0$, and therefore $\phi''(t_2)=0$, $t_2\in(0,t_1)\subset(0,m)$. Finally, $\phi''(0)=0$, and therefore $\phi'''(t_3)=0$ for $t_3\in(0,m)$. This gives

$$\mathbf{f}^{iv}(c)=-\frac{90}{m^5}\psi(m), \quad c\in(m-t_3,m+t_3)$$

Some more manipulation...

Differentiation under the integral

Let $\mathbf{f}(x, y)$ be a function of two variables defined on the rectangle $a \leq x \leq b$, $c \leq y \leq d$. Denote the rectangle by R. Consider

$$\mathbf{F}(y) = \int_a^b \mathbf{f}(x, y)dx.$$

One of its properties is:

Theorem

If $\mathbf{f}(x, y)$ is continuous at each point of the rectangle R, then $\mathbf{F}(y)$ is continuous at each point of the interval $[c, d]$

This follows because $\mathbf{f}(x, y)$ is uniformly continuous.

Theorem

Suppose $\mathbf{f}(x, y)$ *is an integrable function of* x *for each value of* y*, and the partial derivative* $\frac{\partial \mathbf{f}(x,y)}{\partial y}$ *exists and is a continuous function of* x *and* y *in the rectangle* R*. Then*

$$\mathbf{F}(y) = \int_a^b \mathbf{f}(x, y)dx$$

has a derivative given by

$$\mathbf{F}'(y) = \int_a^b \frac{\partial \mathbf{f}(x, y)}{\partial y} dx.$$

Proof. We use the notation $\mathbf{f}_2(x, y)$ for the partial derivative of $\mathbf{f}(x, y)$ with respect to y.

We must show that

$$\lim_{h\to 0}\left(\frac{\mathbf{F}(y+h)-\mathbf{F}(y)}{h}-\int_a^b \mathbf{f}_2(x,y)dx\right)=0.$$

Since

$$\mathbf{F}(y+h)-\mathbf{F}(y)=\int_a^b(\mathbf{f}(x,y+h)-\mathbf{f}(x,y))dx,$$

we apply the **MVT**

$$\mathbf{f}(x,y+h)-\mathbf{f}(x,y)=h\mathbf{f}_2(x,y+\theta h)$$

where θ depends on x, y and h and is such that $0<\theta<1$.This says that

$$\frac{\mathbf{F}(y+h)-\mathbf{F}(y)}{h}-\int_a^b \mathbf{f}_2(x,y)dx=\int_a^b(\mathbf{f}_2(x,y+\theta h)-\mathbf{f}_2(x,y))dx.$$

Now we make use of the fact that $\mathbf{f}_2$ is uniformly continuous **Why Justin?** Suppose $\epsilon > 0$. Pick $\delta > 0$ so that the values of $\mathbf{f}_2$ at different points of R differ by less than ϵ if the distance between the points is less than δ. Then

$$|\mathbf{f}_2(x, y + \theta h) - \mathbf{f}_2(x, y)| < \epsilon, \quad |h| < \delta.$$

This gives

$$\left| \frac{\mathbf{F}(y + h) - \mathbf{F}(y)}{h} - \int_a^b \mathbf{f}_2(x, y)dx \right| < \epsilon(b - a)$$

if $0 < |h| < \delta$. Since ϵ is as small as we please, the proof is complete.

Example

Find $\mathbf{F}'(y)$ if

$$\mathbf{F}(y) = \int_0^1 \ln(x^2 + y^2)dx$$

Note that this function is well defined if $0 \leq x \leq 1$ and on any closed interval not containing $y = 0$. The theorem asserts

$$\mathbf{F}'(y) = \int_0^1 \frac{2y}{x^2 + y^2}dx$$

Therefore

$$\mathbf{F}'(y) = 2\tan^{-1}(\frac{1}{y}).$$

Exercise: Find $\mathbf{F}'(y)$ if

$$\mathbf{F}(y) = \int_{\sin y}^{e^y} \sqrt{1+x^3}dx.$$

Write

$$\mathbf{G}(u,v) = \int_u^v \sqrt{1+x^3}dx, \quad u = \sin y, v = e^y$$

Then make use of

$$\frac{d\mathbf{F}}{dy} = \frac{\partial \mathbf{G}}{\partial u}\frac{du}{dy} + \frac{\partial \mathbf{G}}{\partial v}\frac{dv}{dy}$$

$$\mathbf{F}'(y) = -\sqrt{1+\sin^3 y}\cos y + \sqrt{1+e^{3y}}e^y$$