Math 311: Advanced Calculus

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Set 4

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Outline

Some Goals

Understand useful functions

$$\boxed{\mathbf{f} : A \subset \mathbb{R} \longrightarrow \mathbb{R}}$$

Building Blocks

After building the real number set $\mathbb{R}$, we treated various notions that will be used intensively:

1. Sequences (a_n) and their limits (or lack of) $(a_n) \to a$
2. Distinguished subsets of $\mathbb{R}$: neighborhoods, open sets etc
3. If $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ and (a_n) is a sequence then $(\mathbf{f}(a_n))$ is also a sequence:

$$a_n \to \boxed{\mathbf{f}} \to \mathbf{f}(a_n)$$

If (a_n) is an interesting sequence (what does this mean?) , for what types of functions $\mathbf{f}$ will $(\mathbf{f}(a_n))$ be interesting?

Dirichlet Function

It might be a good idea to have wonderful functions at hand:

1. (Dirichlet Function)

$$\mathbf{f}(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

2. $$\mathbf{f}(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

3. Let $\mathbf{f}(x)$ be your favorite function: polynomials, rational functions, trig functions, $\zeta(x)$?

Outline

Functional Limit

Now we look at a notion at the root of Calc:

Definition

Let $\mathbf{f} : A \to \mathbb{R}$, and let c be a limit point of the domain A. We say $\lim_{x \to c} \mathbf{f}(x) = L$ provided that, for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|\mathbf{f}(x) - L| < \epsilon$.

Let us walk through the functional limit template:

1. Let $\mathbf{f} : A \to \mathbb{R}$, c limit point of A
2. Given $\epsilon > 0$ that is, ϵ arbitrary
3. There is $\delta > 0$ that is, δ is a function of ϵ and c
4. Such that

$$0 < |x - c| < \delta \Rightarrow |\mathbf{f}(x) - L| < \epsilon$$

Example

$$\mathbf{f}(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Let us examine its continuity at $c = 0$: Let $\epsilon > 0$

$$|\mathbf{f}(x) - \mathbf{f}(0)| = |x\sin(1/x) - 0| \leq |x|$$

Thus if we choose $\delta = \epsilon$, whenever $|x - 0| = |x| < \delta$,

$$|\mathbf{f}(x) - \mathbf{f}(0)| < \epsilon.$$

Thus $\mathbf{f}(x)$ is continuous at $x = 0$.

Example

Let $\mathbf{f}(x) = 2x + 1$.

$$\lim_{x \to 3} \mathbf{f}(x) = 7$$

Let $\epsilon > 0$. We must produce $\delta > 0$ so that for $0 < |x - 3| < \delta$ we have $|\mathbf{f}(x) - 7| < \epsilon$.

$$|\mathbf{f}(x) - 7| = |2x + 1 - 7| = |2x - 6| = 2|x - 3|$$

Thus if we choose $\delta = \min\{\epsilon/2, 1\}$, then $|x - 3| < \delta$ implies $|\mathbf{f}(x) - 7| < \epsilon$.

Example

Let $\mathbf{f}(x) = x^2$.

$$\lim_{x \to 3} \mathbf{f}(x) = 9$$

Let $\epsilon > 0$. We must produce $\delta > 0$ so that for $0 < |x - 3| < \delta$ we have $|\mathbf{f}(x) - 9| < \epsilon$.

$$|\mathbf{f}(x) - 9| = |x^2 - 9| = |x - 3||x + 3|$$

To choose $|x - 3|$ small, we need to bound $|x + 3|$. Since we are interested in neighborhoods of $x = 3$, we may restrict ourselves to $2 < x < 4$, so that $|x + 3| \leq 7$.

$$|\mathbf{f}(x) - 9| = |x^2 - 9| = |x - 3||x + 3| \leq 7|x - 3|, \quad |x| \leq 4$$

Thus if we choose $\delta = \epsilon/7$, then $|x - 3| < \delta$ implies $|\mathbf{f}(x) - 9| < \epsilon$.

Functional Limit Template

These examples illustrate the functional limit template:

1. Let $\mathbf{f} : A \to \mathbb{R}$, c limit point of A
2. Given $\epsilon > 0$ that is, ϵ arbitrary
3. There is $\delta > 0$ that is, δ is a function of ϵ and c
4. Such that

$$0 < |x - c| < \delta \Rightarrow |\mathbf{f}(x) - L| < \epsilon$$

5. Note who comes first: the value for δ comes in response to the requirement on ϵ. Thus δ is a function of ϵ (and of c and $\mathbf{f}$ as well)

Topological Version

Definition

Let $\mathbf{f} : A \to \mathbb{R}$, and let c be a limit point of the domain A. We say $\lim_{x \to c} \mathbf{f}(x) = L$ provided that for every ϵ-neighborhood $V_\epsilon(L)$ there exists a δ-neighborhood $V_\delta(c)$ such that for all $x \in V_\delta(c)$ different from c (and $x \in A$) it follows that $\mathbf{f}(x) \in V_\epsilon(L)$.

Definition

Let $\mathbf{f} : A \to \mathbb{R}$, and let c be a limit point of the domain A. We say $\lim_{x \to c} \mathbf{f}(x) = L$ provided that, for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|\mathbf{f}(x) - L| < \epsilon$.

Sequential Criteria

Theorem

Given $\mathbf{f} : A \to \mathbb{R}$ and a limit point c of A, the following statements are equivalent:

(i) $\lim_{x\to c} \mathbf{f}(x) = L$;

(ii) *For **all** sequences $(x_n) \subset A$ satisfying $x_n \neq c$ and $x_n \to c$, it follows that $\mathbf{f}(x_n) \to L$.*

Proof. (i) $\Rightarrow$ (ii): Consider one arbitrary sequence (x_n), $x_n \neq c$, converging to c.

Let $\epsilon > 0$. By (i), there exists $V_\delta(c)$ such that $\mathbf{f}(x) \in V_\epsilon(L)$ for all $x \in V_\delta(c)$, $x \neq c$. Since $x_n \to c$, there is a point x_N such that $x_n \in V_\delta(c)$ for $n \geq N$. It follows that $\mathbf{f}(x_n) \in V_\epsilon(L)$ for $n \geq N$.

(ii) $\Rightarrow$ (i): We are going to argue by contradiction that if $\lim_{x\to c} \mathbf{f}(x) \neq L$, then there is a sequence $x_n \to c$ such that $(\mathbf{f}(x_n))$ does not converge to L.

This assumption is that there is an $\epsilon > 0$ for which no $\delta > 0$ will work in (i).

Let $\delta_n = 1/n$. Then there is $x_n \in V_{\delta_n}(c)$, $x_n \neq c$, such that $\mathbf{f}(x_n) \notin V_\epsilon(L)$.
This creates a sequence $(x_n) \to c$, $x_n \neq c$, such that $\mathbf{f}(x_n)$ does not converge to L. $\square$

Example: Dirichlet Function

$$\mathbf{f}(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

Composing (a_n), a sequence of elements of A, with $\mathbf{f} : A \to \mathbb{R}$, gives another sequence $(\mathbf{f}(a_n))$.

1. Consider the two convergent sequences $a_n = \frac{1}{n}$ and $b_n = \frac{\sqrt{2}}{n}$, both converging to 0
2. If **f** is Dirichlet function, $\mathbf{f}(a_n)$ and $\mathbf{f}(b_n)$ are also convergent but to different limits.
3. **f** is not continuous.

Algebraic Limit Theorem for Functional Limits

Theorem

Let $\mathbf{f}$ *and* $\mathbf{g}$ *be functions on a domain* $A \subset \mathbb{R}$, *and assume* $\lim_{x\to c} \mathbf{f}(x) = L$ *and* $\lim_{x\to c} \mathbf{g}(x) = M$ *for some limit point of A. Then*

1. $\lim_{x\to c} k\mathbf{f}(x) = kL$ *for all* $k \in \mathbb{R}$;
2. $\lim_{x\to c}[\mathbf{f}(x) + \mathbf{g}(x)] = L + M$;
3. $\lim_{x\to c}[\mathbf{f}(x)\mathbf{g}(x)] = LM$;
4. $\lim_{x\to c}[\mathbf{f}(x)/\mathbf{g}(x)] = L/M$, *provided* $M \neq 0$.

Let us prove one of these statements,

$$\lim_{x\to c}[\mathbf{f}(x)\mathbf{g}(x)] = LM$$

Algebraic Limit Theorem for Sequences

Recall the following result about sequences:

Theorem

Let $\lim a_n = a$ *and* $\lim b_n = b$. *Then*

(i) $\lim ca_n = ca$, *for all* $c \in \mathbb{R}$;

(ii) $\lim(a_n + b_n) = a + b$;

(iii) $\lim(a_n b_n) = ab$;

(iv) $\lim(a_n / b_n) = a/b$ *provided* $b_n \neq 0$ *and* $b \neq 0$.

Proof that $\lim \mathbf{f}(x)\mathbf{g}(x) = \lim \mathbf{f}(x) \lim \mathbf{g}(x)$

Since $\lim_{x\to c} \mathbf{f}(x) = L$ and $\lim_{x\to c} \mathbf{g}(x) = M$, by the **sequential criterion** for functional limits, for any sequence $(x_n) \to c$, $x_n \in A$, $\lim_{n\to\infty} \mathbf{f}(x_n) = L$ and $\lim_{n\to\infty} \mathbf{g}(x_n) = M$. Now we use the Algebraic Limit Theorem for Sequences,

$$\begin{aligned} \lim_{n\to\infty} \mathbf{f}(x_n)\mathbf{g}(x_n) &= \lim_{n\to\infty} \mathbf{f}(x_n) \lim_{n\to\infty} \mathbf{g}(x_n) \\ &= LM \end{aligned}$$

Because we used an arbitrary sequence, by the Sequential Criterion for Functional Limits,

$$\lim_{x\to c} (\mathbf{fg})(x) = LM.$$

Example

Let $\mathbf{f} : (0,1) \to \mathbb{R}$ defined by $\mathbf{f}(x) = \frac{\sqrt{4+x}-2}{x}$. Note that $\mathbf{f}$ is not defined at $x = 0$, but this is a limit point of $(0,1)$. Let us determine $\lim_{x\to 0} \mathbf{f}(x)$:

$$\begin{aligned}\mathbf{f}(x) &= \frac{\sqrt{4+x}-2}{x} = \frac{(\sqrt{4+x}-2)(\sqrt{4+x}+2)}{x(\sqrt{4+x}+2)} \\ &= \frac{4+x-4}{x(\sqrt{4+x}+2)} = \frac{1}{\sqrt{4+x}+2}\end{aligned}$$

Both numerator and denominator of this expression of $\mathbf{f}$ have a limit when $x \to 0$. By the Algebraic Limit Theorem for Functional Limits, $\lim_{x\to 0} \mathbf{f}(x) = \frac{1}{4}$.

Exercises to be handed in

1. Section 2.1: 5, 7
2. Section 2.3: 17
3. Miscellaneous: 26

Outline

Continuous Functions

Finally, here it is the jewel of the crown:

Definition

A function $\mathbf{f} : A \to \mathbb{R}$ is **continuous** at a point $c \in A$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) it follows that $|\mathbf{f}(x) - \mathbf{f}(c)| < \epsilon$. If **f** is continuous at every point of A, then **f** is said to be continuous on A.

- Note that the **limit point** c is now required to belong to the domain of **f**
- Shorthand for the definition is

$$\lim_{x \to c} \mathbf{f}(x) = \mathbf{f}(c)$$

Examples

If $\mathbf{f}$ is the constant function $\mathbf{f}(x) = K$, then for $\epsilon > 0$ ANY $\delta > 0$ works:

$$|x - c| < \delta \Rightarrow |\mathbf{f}(x) - \mathbf{f}(c)| = 0 < \epsilon$$

If $\mathbf{f}$ is the identity function $\mathbf{f}(x) = x$, then for any $\epsilon > 0$, $\delta = \epsilon$ works:

$$|x - c| < \delta = \epsilon \Rightarrow |\mathbf{f}(x) - \mathbf{f}(c)| = |x - c| < \epsilon$$

Noteworthy Example

$\mathbf{f}(x) = \sin x$ is continuous at any $c \in \mathbb{R}$: Write $x = c + h$, so give $\epsilon > 0$ we must find $\delta > 0$ (may depend on c) so that

$$|h| < \delta \Rightarrow |\sin(c+h) - \sin c| < \epsilon$$

$$\begin{aligned} |\sin(c+h) - \sin c| &= |\sin c \cos h + \cos c \sin h - \sin c| \\ &\leq |\sin c||\cos h - 1| + |\cos c||\sin h| \end{aligned}$$

We use the formulas (h in radians): $|\sin h| \leq |h|$ and $1 - \cos h = 2\sin^2(h/2)$ to get (we assume $|h| \leq 1$)

$$|\sin(c+h) - \sin c| \leq |h| + 2(h/2)^2 \leq 2|h|$$

so if we take $\delta = \min\{1, \epsilon/2\}$

$$|x - c| < \delta \Rightarrow |\mathbf{f}(x) - \mathbf{f}(c)| < \epsilon$$

The other Trig functions could also be approached directly, but that can be avoided: For example, $\cos x$ is the composite

$$x \to \pi/2 - x \to \sin(\pi/2 - x)$$

where the first arrow is continuous (obviously) and the observation we will be making soon that the composite of continuous functions is continuous.
The other functions arise from basic algebraic operations on $\sin x$ and $\cos x$, which are also covered by general observations later.

Characterizations of Continuity

There are many different ways to express the notion:

Theorem

Let $\mathbf{f} : A \to \mathbb{R}$, and let $c \in A$ be a limit point of A. The function $\mathbf{f}$ is continuous at c if and only if any of the following conditions is met:

1. *[Basic] For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies $|\mathbf{f}(x) - \mathbf{f}(c)| < \epsilon$;*
2. *[Shorthand] $\lim_{x \to c} \mathbf{f}(x) = \mathbf{f}(c)$;*
3. *[Topological] For all $V_\epsilon(\mathbf{f}(c))$, there exists $V_\delta(c)$ with the property that $x \in V_\delta(c)$ (and $x \in A$) implies $\mathbf{f}(x) \in V_\epsilon(\mathbf{f}(c))$;*
4. *[Sequential] If $(x_n) \to c$ (with $x_n \in A$), then $\mathbf{f}(x_n) \to \mathbf{f}(c)$.*

Algebra of Continuous Functions

The following assembles some tools to build continuous functions:

Theorem

Assume $\mathbf{f} : A \to \mathbb{R}$ *and* $\mathbf{g} : A \to \mathbb{R}$ *are continuous functions at a point* $c \in A$. *Then,*

1. $k\mathbf{f}(x)$ *is continuous at* c *for all* $k \in \mathbb{R}$;
2. $\mathbf{f}(x) + \mathbf{g}(x)$ *is continuous at* c;
3. $\mathbf{f}(x)\mathbf{g}(x)$ *is continuous at* c;
4. $\mathbf{f}(x)/\mathbf{g}(x)$ *is continuous at* c, *provided the quotient is defined.*

A refinement of Part (iv)

The assertion that $\mathbf{f}(x)/\mathbf{g}(x)$ is continuous at c does not require that $\mathbf{f}(x)/\mathbf{g}(x)$ be always defined, but only that $\mathbf{g}(c) \neq 0$. This follows from the following:

Lemma

If $\mathbf{g}(x)$ is a continuous function at c and $\mathbf{g}(c) \neq 0$ then there are $\delta > 0$ and $\alpha > 0$ such that $|\mathbf{g}(x)| \geq \alpha$ for $x \in V_\delta(c)$.

Proof. Choose $\epsilon = \alpha = \frac{|\mathbf{g}(c)|}{2} > 0$. By hypothesis, there is $\delta > 0$ such that for $|x - c| < \delta$, $x \in A$, we have $|\mathbf{g}(x) - \mathbf{g}(c)| < \alpha$. Thus

$$|\mathbf{g}(x)| \geq |\mathbf{g}(c)| - |\mathbf{g}(x) - \mathbf{g}(c)| \geq |\mathbf{g}(c)| - \alpha = \alpha.$$

Polynomials

The fact that constant functions and $\mathbf{f}(x) = x$ are continuous this leads immediately to: polynomials

$$\mathbf{f}(x) = a_n x^n + \cdots + a_0,$$

are continuous functions (everywhere). For example,

$$4x^3 + 5x = [4][x][x][x] + [5][x]$$

Similar argument applies to rational functions

$$\mathbf{f}(x) = \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0},$$

are continuous at all c which are not roots of the denominator.

$x \sin(1/x)$

$$\mathbf{g}(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Let us examine the continuity at $x = 0$.

$$|\mathbf{g}(x) - \mathbf{g}(0)| = |x \sin(1/x) - 0| \leq |x|$$

Given $\epsilon > 0$, set $\delta = \epsilon$ so that $|x - 0| = |x| < \delta$ implies $|\mathbf{g}(x) - \mathbf{g}(0)| < \epsilon$

$\sqrt{x}$

Let $\mathbf{f}(x) = \sqrt{x}$ defined on $A = \{x \in \mathbb{R} \mid x \geq 0\}$.

1. If $c = 0$, $|\sqrt{x} - \sqrt{0}| = |\sqrt{x}| < \epsilon$ if $x < \epsilon^2$, so choose $\delta = \epsilon^2$.
2. For $c > 0$,

$$|\sqrt{x}-\sqrt{c}| = |\sqrt{x}-\sqrt{c}| \left(\frac{|\sqrt{x} + \sqrt{c}|}{|\sqrt{x} + \sqrt{c}|} \right) = \frac{|x - c|}{|\sqrt{x} + \sqrt{c}|} \leq \frac{|x - c|}{\sqrt{c}}$$

Let $\delta = \epsilon\sqrt{c}$. Then $|x - c| < \delta$ implies

$$|\sqrt{x} - \sqrt{c}| < \frac{\epsilon\sqrt{c}}{\sqrt{c}} = \epsilon$$

Composition of Continuous Functions

Theorem

Given $\mathbf{f} : A \to \mathbb{R}$ *and* $\mathbf{g} : B \to \mathbb{R}$, *assume that* $\mathbf{f}(A) \subset B$, *so that the composition* $\mathbf{g} \circ \mathbf{f}$ *is defined. If* $\mathbf{f}$ *is continuous at* $c \in A$, *and* $\mathbf{g}$ *is continuous at* $\mathbf{f}(c) \in B$, *then* $\mathbf{g} \circ \mathbf{f}$ *is continuous at* c.

Proof. From the continuity of $\mathbf{g}$ at the point $\mathbf{f}(c) \in B$, given $\epsilon > 0$ there is a $\alpha > 0$ neighborhood such that

$$\mathbf{g} : V_\alpha(\mathbf{f}(c)) \to V_\epsilon(\mathbf{g}(\mathbf{f}(c))).$$

From the continuity of $\mathbf{f}$ at the point $c \in A$, for $\alpha > 0$ there is a $\delta > 0$ neighborhood such that

$$V_\delta(c) \xrightarrow{\mathbf{f}} V_\alpha(\mathbf{f}(c)) \xrightarrow{\mathbf{g}} V_\epsilon(\mathbf{g}(\mathbf{f}(c))).$$

Examples

The function $\mathbf{f}(x) = \sqrt{\sin x}$, $0 \leq x \leq \pi$, is continuous: To show this from the Definition would be irritating, but observe that $\mathbf{f}(x)$ is the composite of sine followed by the square root,

$$x \to \sin x \to \sqrt{\sin x}$$

both of which we have proved are continuous.

Continuous Functions and Topology

Let $\mathbf{f} : A \to \mathbb{R}$ be a continuous function on all points of A. We are going to cast this entirely in topological terms.
Recall that an open set O of $\mathbb{R}$ is a set such that for any $a \in O$ there is a neighborhood $V_\epsilon(a) \subset O$. This implies that

$$O = \bigcup (b, d),$$

for all open intervals contained in O. Note that $(b, d) = V_\epsilon(c)$, where c is the center of (b, d) and ϵ is $1/2|b - d|$

We can also consider neighborhoods made up of points of A: $V_\epsilon(a) \cap A$, and corresponding open sets (this is called the induced topology of A).

Theorem

The function $\mathbf{f} : A \to \mathbb{R}$ *is continuous iff for every open set* $O \subset \mathbb{R}$ *the set*

$$\mathbf{f}^{-1}(O) = \{x \in A \mid \mathbf{f}(x) \in O\}$$

is open.

Proof. If O is open, O is a union of neighborhoods, $O = \bigcup(b, d)$, and $\mathbf{f}^{-1}(O) = \bigcup \mathbf{f}^{-1}((b, d))$. If $\mathbf{f}$ is continuous, every $\mathbf{f}^{-1}(b, d)$ contains a neighborhood because of the property

$$|x - c| < \delta \Rightarrow |\mathbf{f}(x) - \mathbf{f}(c)| < \epsilon.$$

We leave the remaining details as a stretching exercise.

Outline

Workshop #5

A: Prove that the function $\mathbf{f}(x) = \sqrt[3]{x}$ is continuous at each $c \in \mathbb{R}$. You can assume, for simplicity, that $c = 2$ and $\epsilon = 1/100$.

B: Decide whether the following statements are true or false. Provide counterexamples to those that are false, and supply proofs for those that are true.

1. If a set has an isolated point, it cannot be open.
2. A set A is closed if and only if $\overline{A} = A$.
3. If A is a bounded set, then $s = \sup A$ is a limit point of A.
4. Every finite set is closed.
5. $\mathbb{R}$ and $\emptyset$ are the only sets that are BOTH open and closed.

Decide whether the following statements are true or false. Provide counterexamples to those that are false, and supply proofs for those that are true.

1. An arbitrary intersection of compact sets is compact.
2. Let $A \subset \mathbb{R}$ be arbitrary, and let $K \subset \mathbb{R}$ be compact. Then the intersection $A \cap K$ is compact.
3. If $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ is a nested sequence of nonempty closed sets, then $\cap F_n \neq \emptyset$.
4. A finite set is always compact.
5. A countable set is always compact.

Let C be the Cantor set. Define

$$\mathbf{f}(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C. \end{cases}$$

Show that **f** is NOT continuous at any $x \in C$, but **f** is continuous at any $x \notin C$.

Outline

Compact Sets

Definition

A set $K \subset \mathbb{R}$ is **compact** if every sequence in K has a subsequence that converges to a limit that is also in K.

Example: A closed interval $[a, b]$. The Bolzano-Weirstrass theorem guarantees that any sequence $(a_n) \subset [a, b]$ admits a convergent subsequence. Because $[a, b]$ is closed, the limit of this subsequence is also in $[a, b]$
This is the fundamental example of a compact subset of $\mathbb{R}$, and the Bolzano-Weierstrass theorem is the key result. It helps if when you read **compact**, these facts jump to the mind.

Characterizing Compact Sets

Definition

A set $K \subset \mathbb{R}$ is **bounded** if there exists $M > 0$ such that $|x| < M$ for all $x \in K$.

Theorem (Heine-Borel Theorem)

A set $K \subset \mathbb{R}$ is compact if and only if it closed and bounded.

Proof. Let K be compact. We first claim K is bounded. Otherwise, for each n there is $x_n \in K$ such that $|x_n| > n$. Since K is compact:

1. (x_n) has a convergent subsequence (x_{n_k}).
2. But convergent sequences are bounded, while $|x_{n_k}| > n_k$, a contradiction as $n_k \to \infty$.

Proof cont'd

Next we show that K is closed. Let $x = \lim x_n$ be a limit point of K, that is $x_n \in K$. We must show $x \in K$. From the compactness assumption, (x_n) admits a convergent subsequence (x_{n_k}) converging to a point $y \in K$. Since (x_n) is convergent, all of its subsequences have the same limit, so $x = y$ as desired.
The converse is left as an exercise.

The following is a super version of the **nested intervals property**:

Theorem

If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ is a nested sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} K_n$ is nonempty.

Proof. The strategy is simple: We pick an element $x_n \in K_n$ (K_n is nonempty) and consider the sequence (x_n). Since $x_n \in K_1$, and K_1 is compact, it admits a convergent subsequence $(x_{n_k}) \to x \in K_1$.

We claim that $x \in K_n$ for every n. Given n_0, the terms in (x_n) are contained in K_{n_0} as long as $n \geq n_0$. This means that the terms of the subsequence (x_{n_k}) are also in K_{n_0} for almost all of them. This implies that its limit lies in K_{n_0}, as desired. $\square$

Outline

Topological Properties of Continuous Functions

- We know what is a continuous fuction $\mathbf{f} : A \to \mathbb{R}$, and understand its various formulations
- (ϵ, δ)-definition and the topological formulation [involves open sets]
- Sequential formulation: If $x_n \to c$, then $\mathbf{f}(x_n) \to \mathbf{f}(c)$

Let $\mathbf{f} : A \to \mathbb{R}$ be a continuous function. Want to know about:

- Let K be a closed, resp. open, compact set.
- Is $\mathbf{f}(K)$ closed, resp. open, compact set?

Studying these will lead to: Extreme Value Theorem, Intermediate Value Theorem, and a bunch of other great stuff.

Preservation of Compact Sets

Theorem

Let $\mathbf{f} : A \to \mathbb{R}$ *be continuous on* A*. If* $K \subset A$ *is compact, then* $\mathbf{f}(K)$ *is compact as well.*

Proof. Recall that a set K is **compact** if it is both bounded and closed–e.g. a set like

$$K = \bigcup_{i=1}^{n} [a_i, b_i].$$

It was proved that a set $K \subset \mathbb{R}$ is **compact** if every sequence (x_n) in K has a subsequence that converges to a limit that is also in K

$$(x_{n_k}) \to x \in K$$

Let us use this criterion.

Proof

1. Let (y_n) be a sequence in $\mathbf{f}(K)$. To prove the assertion we must find a subsequence (y_{n_k}), which converges to a limit also in $\mathbf{f}(K)$.
2. For each y_n, choose $x_n \in K$ so that $y_n = \mathbf{f}(x_n)$. This gives a sequence (x_n) in K
3. Since K is compact, there is a subsequence (x_{n_k}) converging to a limit $x \in K$.
4. Since $\mathbf{f}$ is continuous, $(x_{n_k}) \to x$ implies $(y_{n_k}) = (\mathbf{f}(x_{n_k})) \to \mathbf{f}(x)$, as desired. $\square$

EVT

Theorem (Extreme Value Theorem)

If $\mathbf{f} : K \to \mathbb{R}$ is continuous on a compact set $K \subset \mathbb{R}$, then $\mathbf{f}$ attains a maximum and minimum value. In other words, there exist $x_0, x_1 \in K$ such that $\mathbf{f}(x_0) \leq \mathbf{f}(x) \leq \mathbf{f}(x_1)$ for all $x \in K$.

The function defined by

$$\mathbf{f}(x) = 1/x, \quad A = (0, 1)$$

is continuous, but has no extreme value. Why? The domain A is not compact.

Proof of the Extreme Value Theorem

1. Hypothesis: K compact: (that is **closed** and **bounded**)
2. $\mathbf{f}(K)$ is also compact by a previous theorem
3. Let L be greatest lower bound of $\mathbf{f}(K)$ and let U be the least upper bound of $\mathbf{f}(K)$. Let us show that there is $x_1 \in K$ such that $\mathbf{f}(x_1) = U$ (similarly for L)
4. There is a sequence in $\mathbf{f}(K)$, $(y_n) \to U$. Pick $x_n \in K$ with $\mathbf{f}(x_n) = y_n$. Since K is compact, there is a convergent subsequence $(x_{n_k}) \to x_1$, with $x_1 \in K$ since the set is closed.
5. $\lim x_{n_k} = x_1$ implies $\lim \mathbf{f}(x_{n_k}) = \mathbf{f}(x_1)$, that is $\lim y_n = U = \mathbf{f}(x_1)$

Outline

Uniform Continuity

In the definition of continuous function $\mathbf{f} : A \to \mathbb{R}$ at the point $c \in A$ we have
Given $\epsilon > 0$ there exists $\delta > 0$ such that

$$x \in V_\delta(c) \Rightarrow \mathbf{f}(x) \in V_\epsilon(\mathbf{f}(c))$$

ACHTUNG: δ may depend on c. If we go to another point c', we may have to use a different δ'. An important issue is when we can use the same value of δ at all points.

Examples: $2x + 1$ and x^2: In the first, $\delta = \epsilon/2$ will work for all c. The same will not hold for x^2. Soon we will work out examples in detail.

Limit of Sequences/Series of Functions

Let $\mathbf{f}_n(x) : A \to \mathbb{R}$ be a series of functions. For each x, we have two important mathematical objects, the sequence $(\mathbf{f}_n(x))$ and the series $\sum \mathbf{f}_n(x)$.

Suppose $(\mathbf{f}_n(x)) \to L$: If we change x, we may have a different limit, that is the limit may define a function $L(x)$.

It is clear that the study of $L(x)$ is not going to be easy:Why (discuss)

Recall

$$\lim \mathbf{f}_n(x) = L \quad \text{if} \forall \epsilon > 0 \quad \text{there is } N \text{ such that } |L - \mathbf{f}_n(x)| < \epsilon$$

for $n \geq N$

The difficulty is that N may depend on x.

Uniformly Continuous Function

Definition

A function $\mathbf{f} : A \to \mathbb{R}$ is **uniformly continuous** on A if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|\mathbf{f}(x) - \mathbf{f}(y)| < \epsilon$.

1. In the definition of **continuous at** c, the $|x - c| < \delta$ implies $|\mathbf{f}(x) - \mathbf{f}(c)| < \epsilon$, will require δ depending on ϵ and c.
2. In the uniform version, one must get δ independent of c.

Example: The function $\mathbf{f}(x) = 1/x^2$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

$$|\mathbf{f}(x) - \mathbf{f}(c)| = \left|\frac{(c-x)(c+x)}{c^2x^2}\right| = |c-x|\frac{c+x}{c^2x^2}$$

$$\frac{c+x}{c^2x^2} = \frac{1}{cx^2} + \frac{1}{c^2x} \leq 1 + 1 = 2$$

$$|\mathbf{f}(x) - \mathbf{f}(c)| \leq 2|c-x|$$

$\delta = \epsilon/2$ will work on $[1, \infty)$

Leaving other part of the problem as Exercise.

There is a large number of functions

$$\mathbf{f} : A \to \mathbb{R}$$

that satisfy

$$|\mathbf{f}(x) - \mathbf{f}(y)| \leq M|x - y|$$

for some fixed $M > 0$. This is called Lipschitz's condition. Such functions are obviously uniformly continuous: For $\epsilon > 0$ pick $\delta = \epsilon/M$

$$|x - y| < \delta = \epsilon/M \Rightarrow |\mathbf{f}(x) - \mathbf{f}(y)| < M|x - y| \leq M\delta = M \cdot \epsilon/M = \epsilon$$

Cauchy Sequences and Continuous Functions

1. A sequence (x_n) is called a **Cauchy sequence** if, for every $\alpha > 0$, there is an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that $|x_n - x_m| < \alpha$.
2. If **f** is a continuous function and the x_n lie in the domain of **f**, the sequence $(\mathbf{f}(x_n))$ may fail to be Cauchy.
3. Consider: $x_n = 1/(n\pi + \pi/2)$ is a Cauchy sequence, but $\mathbf{f}(x_n) = \sin(1/x_n)$ is not: $|\mathbf{f}(x_n) - \mathbf{f}(x_{n+1})| = 2$.
4. However, if **f** is uniformly continuous, from the condition

$$|x - y| < \delta \Rightarrow |\mathbf{f}(x) - \mathbf{f}(y)| < \epsilon$$

we get

$$|x_n - x_m| < \delta \Rightarrow |\mathbf{f}(x_n) - \mathbf{f}(y_m)| < \epsilon$$

Contractions

Definition

A function $\mathbf{f} : A \to \mathbb{R}$ is a **contraction** if

$$|\mathbf{f}(x) - \mathbf{f}(y)| \leq c|x - y| < |x - y|,$$

for some fixed c, $0 < c < 1$.

These functions have remarkable properties. Suppose $\mathbf{f}$ actually has range $\subset A$ so that we can iterate it: $\mathbf{f}(x), \mathbf{f}^2(x), \ldots$

Contractions and Fixed Points

Definition

If $\mathbf{f} : A \to A$, a fixed point of $\mathbf{f}$ is an $x_0 \in A$ such that $\mathbf{f}(x_0) = x_0$.

If $\mathbf{f}$ is a contraction

$$|\mathbf{f}(x) - \mathbf{f}(y)| < |x - y|,$$

obviously it cannot have TWO fixed points.

Theorem

If A is closed and if $\mathbf{f} : A \to A$ is a contraction, for any $x \in A$ the sequence of iterates $(\mathbf{f}^n(x))$ converges to a (unique!) fixed point of $\mathbf{f}$.

Proof. We first claim that $(\mathbf{f}^n(x))$ is a Cauchy sequence. For any $n > 1$

$$|\mathbf{f}^n(x) - \mathbf{f}^{n-1}(x)| < c|\mathbf{f}^{n-1}(x) - \mathbf{f}^{n-2}(x)|$$

by the contraction condition.
Now look at the magic: For $n > m$

$$
\begin{aligned}
|\mathbf{f}^n(x) - \mathbf{f}^m(x)| &= |(\mathbf{f}^n(x) - \mathbf{f}^{n-1}(x)) + (\mathbf{f}^{n-1}(x) - \mathbf{f}^{n-2}(x)) \\
&+ \cdots + (\mathbf{f}^{m+1}(x) - \mathbf{f}^m(x))| \\
&\leq |\mathbf{f}^n(x) - \mathbf{f}^{n-1}(x)| + |\mathbf{f}^{n-1}(x) - \mathbf{f}^{n-2}(x)| \\
&+ \cdots + |\mathbf{f}^{m+1}(x) - \mathbf{f}^m(x)| \\
&\leq c^{n-1}|\mathbf{f}(x) - x| + c^{n-2}|\mathbf{f}(x) - x| \\
&+ \cdots + c^m|\mathbf{f}(x) - x| \\
&\leq (c^{n-1} + c^{n-2} + \cdots + c^m)|\mathbf{f}(x) - x| \\
&\leq \left(\frac{c^m}{1-c}\right)|\mathbf{f}(x) - x|
\end{aligned}
$$

Since $\lim c^m = 0$, we can make the RHS $< \epsilon$ arbitrary for $m \geq N$.

We are now in position to prove the theorem:

1. The sequence $(x_n = \mathbf{f}^n(x))$ is Cauchy, by the argument above. Say of limit $x_0 \in A$ (A is closed)
2. If we apply $\mathbf{f}$ we get

$$\lim \mathbf{f}(x_n) = \mathbf{f}(x_0).$$

3. But $(\mathbf{f}^{n+1}(x))$ is a subsequence of $(x_n = \mathbf{f}^n(x))$ with the same limit, so

$$x_0 = \mathbf{f}(x_0)$$

Sequential Criterion for Nonuniform Continuity

Theorem

A function $\mathbf{f} : A \to \mathbb{R}$ *fails to be uniformly continuous on A if and only if there exists a particular* $\epsilon_0 > 0$ *and two sequences* (x_n) *and* (y_n) *in A satisfying*

$$|x_n - y_n| \to 0 \quad \text{but} \quad |\mathbf{f}(x_n) - \mathbf{f}(y_n)| \geq \epsilon_0.$$

Proof. Suppose there exists an ϵ for which for no value of δ $|x - y| < \delta$ implies $|\mathbf{f}(x) - \mathbf{f}(y)| < \epsilon$. Pick $\delta_n = 1/n$ and choose x_n, y_n so that

$$|x_n - y_n| < 1/n \quad \text{but} \quad |\mathbf{f}(x_n) - \mathbf{f}(y_n)| > \epsilon.$$

This defines the required sequences. The converse is clear. □

Example

$$\mathbf{f}(x) = \sin(1/x), \quad x \in (0,1)$$

is continuous on $(0,1)$. However, near $x = 0$ it swings very rapidly between -1 and 1.
For example, consider the two sequences $x_n = 1/(2n\pi + \pi/2)$ and $y_n = 1/(2n\pi - \pi/2)$, both converging to 0, but

$$|\mathbf{f}(x_n) - \mathbf{f}(y_n)| = |1 - (-1)| = 2$$

Theorem

A function that is continuous on a compact set K is uniformly continuous on K.

Proof. Assume $\mathbf{f} : K \to \mathbb{R}$ is continuous on the compact set K. We argue by contradiction.
By the criterion of nonuniform continuity, there exists $\epsilon > 0$ and two sequences (x_n) and (y_n) such that

$$\lim |x_n - y_n| = 0 \quad \text{but} \quad |\mathbf{f}(x_n) - \mathbf{f}(y_n)| > \epsilon.$$

Because K is compact, the sequence (x_n) contains a convergent subsequence (x_{n_k}) with $x = \lim x_{n_k}$ also in K.
Let (y_{n_k}) be the corresponding subsequence of (y_n) and observe

1. By the Algebraic Limit Theorem

$$\lim y_{n_k} = \lim(y_{n_k} - x_{n_k}) + \lim x_{n_k} = x$$

2. Thus both (x_{n_k}) and (y_{n_k}) have the same limit, so by the Sequential Criterion of Continuity

$$\lim \mathbf{f}(x_{n_k}) = \lim \mathbf{f}(y_{n_k}),$$

which contradicts

$$|\mathbf{f}(x_{n_k}) - \mathbf{f}(y_{n_k})| > \epsilon$$

Example

1. $\mathbf{f}(x) = x^3$: continuous (polynomials are put together from the identity function $\mathbf{I}(x) = x$ and the Algebraic Continuity Theorem)
2. It is not uniformly continuous: The sequences $x_n = n + 1/n$ and $y_n = n$ have the properties

$$|x_n - y_n| \to 0 \quad \text{but} \quad |{x_n}^3 - {y_n}^3|$$

$$= |(n + 1/n)^3 - n^3| = 3n + 3/n + 1/n^3 \to \infty$$

3. It is uniformly continuous on any bounded set A: A is a subset of an interval $[a, b]$. Since $[a, b]$ is compact, x^3 is uniformly continuous on it and therefore on any of its subsets.

Outline

Properties of Continuous Functions

Let $\mathbf{f} : A \to \mathbb{R}$ be a continuous function.

1. If A is compact, then $\mathbf{f}(A)$ is compact.
2. **(Extreme Value Theorem)** If A is compact $\mathbf{f}$ attains a maximum and minimum value. In other words, there exist $x_0, x_1 \in A$ such that $\mathbf{f}(x_0) \leq \mathbf{f}(x) \leq \mathbf{f}(x_1)$ for all $x \in A$.
3. If $A = [a, b]$, then $\mathbf{f}$ is uniformly continuous, that is, given $\epsilon > 0$ there is $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |\mathbf{f}(x) - \mathbf{f}(y)| < \epsilon$$

for all $x, y \in [a, b]$.

The Intermediate Value Theorem

Theorem (IVT)

If $\mathbf{f} : [a, b] \to \mathbb{R}$ *is continuous, and if* L *is any real number satisfying* $\mathbf{f}(a) < L < \mathbf{f}(b)$ *or* $\mathbf{f}(a) > L > \mathbf{f}(b)$*, then there exists a point* $c \in (a, b)$ *where* $\mathbf{f}(c) = L$*. In particular, if* $\mathbf{f}(a) < 0$ *and* $\mathbf{f}(b) > 0$*, there exists a point* $c \in (a, b)$ *such that* $\mathbf{f}(c) = 0$*.*

This is due to Bolzano. Before we give a proof, let us explore some of its non-traditional uses. We need two volunteers!

Bisections of Regions

We are going to discuss the following problems:

- Given a **plane region** $\mathcal{R}$, prove that there is a line **L** bisecting it, that is, cutting $\mathcal{R}$ into two regions of equal areas.
- Given 2 **plane regions** $\mathcal{R}_1$ and $\mathcal{R}_1$, prove that there is a line **L** simultaneously cutting $\mathcal{R}_1$ and $\mathcal{R}_2$ into two regions of equal areas.
- Given a **plane region** $\mathcal{R}$, prove that there are two perpendicular lines $\mathbf{L}_1$ and $\mathbf{L}_2$ cutting $\mathcal{R}$ into four regions of equal areas.
- What else can we expect?

Before we prove this theorem, let us highlight an elementary property of continuous functions.

Lemma

Let $\mathbf{f} : [a, b] \to \mathbb{R}$ *be a continuous function and* $c \in [a, b]$ *be a point such that* $\mathbf{f}(c) \neq 0$*. Then there is a subinterval containing* c *throughout which* $\mathbf{f}(x)$ *has the same sign as* $\mathbf{f}(c)$*.*

Proof. Suppose $\mathbf{f}(c) > 0$. The theorem asserts that we can find $\delta > 0$ such that if $|c - x| < \delta$ then $\mathbf{f}(x) > 0$. Let $\epsilon = \mathbf{f}(c)$. Since $\mathbf{f}$ is continuous, there is $\delta > 0$ such that if $|c - x| < \delta$ then

$$|\mathbf{f}(x) - \mathbf{f}(c)| < |\mathbf{f}(c)|$$

Obviously this forces $\mathbf{f}(x) > 0$. The case of $\mathbf{f}(c) < 0$ is similar. (Or simply consider the continuous function $-\mathbf{f}(x)$.)

Here is a simple application:

Corollary

If $\mathbf{f} : [a, b] \to \mathbb{R}$*,* $a < b$*, is continuous and* $\mathbf{f}(x) = 2$ *for all rational numbers (in* $[a, b]$*) then* $\mathbf{f}(x) = 2$ *for all* $x \in [a, b]$*.*

Proof. Consider $\mathbf{f}(x) - 2$...

Proof of IVT

1. Assume that $\mathbf{f}(a) < 0 < \mathbf{f}(b)$. Let us construct $c \in (a, b)$ such that $\mathbf{f}(c) = 0$. We will use the AoC property of $\mathbb{R}$.
2. Define the following set

$$K = \{x \in [a, b] \mid \mathbf{f}(x) < 0\}$$

   $a \in K$ and b bounds all the points of K. By AoC, K has a least upper bound $c \in [a, b]$.
3. We have the possibilities: $\mathbf{f}(c) < 0$, $\mathbf{f}(c) > 0$ or $\mathbf{f}(c) = 0$: Let us exclude the first two possibilities.

Suppose $\mathbf{f}(c) < 0$ This means that $c \in K$. Since $\mathbf{f}$ is continuous at c, by the lemma, there exists a neighborhood of c where $\mathbf{f}(x)$ is negative. But then all of these points would be in K and some of them are larger than c: This means that c is not the least upper bound of K.

A similar argument works if $\mathbf{f}(c) > 0$: Now there would be an entire neighborhood of c made up of upper bounds of K, so c is not the least upper bound.

Another Proof of IVT

- We consider the special case $L = 0$ and $\mathbf{f}(a) < 0 < \mathbf{f}(b)$. **Why can we just consider this case?** Let $I_0 = [a, b]$, and consider its midpoint

$$z = \frac{(a+b)}{2}.$$

If $\mathbf{f}(z) = 0$, we are done.If $\mathbf{f}(z) > 0$, then set $a_1 = a$ and $b_1 = z$. If $\mathbf{f}(z) < 0$, then set $a_1 = z$ and $b_1 = b$, so the interval $I_1 = [a_1, b_1]$ has the property that **f** changes signs at its endpoints. This defines a nested sequence of intervals whose intersection is a unique point $c \in (a, b)$.

- If $\mathbf{f}(c) = 0$, done. We have not used the assumption that $\mathbf{f}$ is continuous, we do so now. Suppose $\mathbf{f}(c) = 2\epsilon > 0$. Since $\mathbf{f}$ is continuous at c, for this ϵ there exists $\delta > 0$ such that for $|x - c| < \delta$

$$|\mathbf{f}(x) - \mathbf{f}(c)| < \epsilon.$$

By the triangle inequality, we have $\mathbf{f}(x) > \epsilon$ for all $x \in V_\delta(c)$. But in the construction of the nested sequence of intervals, the nth interval will have length $(b - a)/2^n$, so it will be contained in the $(c - \delta, c + \delta)$ for large n. But this is a contradiction, since $\mathbf{f}(x)$ changes signs on the interval. $\square$

Outline

Warmups

1. Explain why
$$\frac{x^2+1}{x+2}+\frac{x^4+1}{x-3}=0$$
has at least one root between -2 and 3.

   Look for an interval where function is continuous and changes sign

2. Let
$$\mathbf{f}(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$$
be a real polynomial of odd degree. Prove that $\mathbf{f}(x)$ has a real root.

   Look for an interval where function is continuous and changes sign

Outline

Workshop #6

Need:

1. Continuity: Basic formulation
2. Continuity: Topological formulation
3. Uniform continuity
4. EVT: Extremum Value Theoorem

Workshop #6

1. Show that if $\mathbf{f}(x)$ is continuous on $[a, b]$ with $\mathbf{f}(x) > 0$ for all $x \in [a, b]$ then $1/\mathbf{f}$ is bounded on $[a, b]$.
2. Show that $\mathbf{f}(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$
3. Build a continuous function $\mathbf{f} : A \to \mathbb{R}$, with A an open set, such that $\mathbf{f}(A)$ is not open. Do the similar question, but with the change 'open' $\to$ 'closed'.
4. Given that $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ is continuous at $x = 0$, and that for all x and y, $\mathbf{f}(x + y) = \mathbf{f}(x) + \mathbf{f}(y)$, show that $\mathbf{f}$ is continuous for all values of x. (For fun, prove that $\mathbf{f}(x) = ax$, for some constant a.)

Outline

The derivative of a function

Let $\mathbf{f} : A \to \mathbb{R}$ be a function. The following is one of the two most important instances of the notion of **limit**:

Definition

The derivative of $\mathbf{f}$ at $c \in A$ is the limit (if it exists)

$$\lim_{x \to c} \frac{\mathbf{f}(x) - \mathbf{f}(c)}{x - c}.$$

This value is denoted $\mathbf{f}'(c)$.

The function $\frac{\mathbf{f}(x) - \mathbf{f}(c)}{x - c}$ is the slope of the secant at the graph of $\mathbf{f}(x)$, so its limit is naturally the slope of the tangent of the graph at $x = c$. If the limit exists, we say that $\mathbf{f}$ is differentiable at $x = c$.

Example

$$\mathbf{g}(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

$$\mathbf{g}'(0) = \lim_{x\to 0} \frac{x\sin(1/x) - 0}{x - 0} = \lim_{x\to 0} \sin(1/x)$$

does not exist.

If $\mathbf{g}_2(x) = x\mathbf{g}(x)$,

$$\mathbf{g}_2'(0) = \lim_{x^2\to 0} \frac{x^2\sin(1/x) - 0}{x - 0} = \lim_{x\to 0} x\sin(1/x) = 0$$

Derivative and Continuity

Theorem

If $\mathbf{f} : A \to \mathbb{R}$ is differentiable at $c \in A$, then $\mathbf{f}$ is continuous at c.

Proof. We are assuming that

$$\mathbf{f}'(c) = \lim_{x \to c} \frac{\mathbf{f}(x) - \mathbf{f}(c)}{x - c}$$

exists, and we claim $\lim_{x \to c} \mathbf{f}(x) = \mathbf{f}(c)$. It suffices to apply the Algebraic Limit Theorem

$$\lim_{x \to c} (\mathbf{f}(x) - \mathbf{f}(c)) = \lim_{x \to c} \left(\frac{\mathbf{f}(x) - \mathbf{f}(c)}{x - c} \right) (x - c) = \mathbf{f}'(c) \cdot 0 = 0.$$

It follows $\lim_{x \to c} \mathbf{f}(x) = \mathbf{f}(c)$. □

1. $\mathbf{f}(x)$ differentiable $\Rightarrow \mathbf{f}(x)$ continuous: theorem above
2. $\mathbf{f}(x)$ continuous $\not\Rightarrow \mathbf{f}(x)$ differentiable: $\mathbf{f}(x) = |x|$
3. $\mathbf{f}(x)$ differentiable $\not\Rightarrow \mathbf{f}'(x)$ continuous: $\mathbf{f}(x) = x \sin(1/x)$
4. $\mathbf{f}(x)$ differentiable means really what?
5. What are the properties of a derivative? In other words, if $\mathbf{g} = \mathbf{f}'$ for some $\mathbf{f}$, what can we say about $\mathbf{g}$?

Combinations

Theorem

Let $\mathbf{f}$ *and* $\mathbf{g}$ *be functions defined on an interval A, and assume both are differentiable at a point* $c \in A$. *Then,*

1. $(\mathbf{f}+\mathbf{g})'(c) = \mathbf{f}'(c) + \mathbf{g}'(c)$,
2. $(k\mathbf{f})'(c) = k\mathbf{f}'(c)$, *for all* $k \in \mathbb{R}$,
3. $(\mathbf{f}\mathbf{g})'(c) = \mathbf{f}'(c)\mathbf{g}(c) + \mathbf{f}(c)\mathbf{g}'(c)$, *and*
4. $(\mathbf{f}/\mathbf{g})'(c) = \frac{\mathbf{g}(c)\mathbf{f}'(c) - \mathbf{f}(c)\mathbf{g}'(c)}{[\mathbf{g}(c)]^2}$, *provided* $\mathbf{g}(c) \neq 0$.

The first two rules are easy to prove, and we only deal with rule 3, known as Leibnitz's rule (rule 4 follows from it).

$$\begin{aligned}\frac{(\mathbf{fg})(x)-(\mathbf{fg})(c)}{x-c} &= \frac{\mathbf{f}(x)\mathbf{g}(x)-\mathbf{f}(x)\mathbf{g}(c)+\mathbf{f}(x)\mathbf{g}(c)-\mathbf{f}(c)\mathbf{g}(c)}{x-c} \\ &= \mathbf{f}(x)\left[\frac{\mathbf{g}(x)-\mathbf{g}(c)}{x-c}\right]+\mathbf{g}(c)\left[\frac{\mathbf{f}(x)-\mathbf{f}(c)}{x-c}\right]\end{aligned}$$

Since $\mathbf{f}(x)$ is differentiable at c, $\lim_{x\to c}\mathbf{f}(x)=\mathbf{f}(c)$, so

$$\lim_{x\to c}\frac{(\mathbf{fg})(x)-(\mathbf{fg})(c)}{x-c}=\mathbf{f}(c)\mathbf{g}'(c)+\mathbf{f}'(c)\mathbf{g}(c).$$

The quotient rule can be obtained by applying the product rule to

$$\mathbf{f}'(c) = (\mathbf{g}(\mathbf{f}/\mathbf{g}))'(c) = \mathbf{g}'(c)(\mathbf{f}/\mathbf{g})(c) + \mathbf{g}(c)(\mathbf{f}/\mathbf{g})'(c)$$

and therefore

$$\begin{aligned} (\mathbf{f}/\mathbf{g})'(c) &= \frac{\mathbf{f}'(c) - \mathbf{g}'(c)(\mathbf{f}/\mathbf{g})(c)}{\mathbf{g}(c)} \\ &= \frac{\mathbf{f}'(c)\mathbf{g}(c) - \mathbf{g}'(c)\mathbf{f}(c)}{[\mathbf{g}(c)]^2} \end{aligned}$$

Derivative of Composite Functions

Theorem (Chain Rule)

Let $\mathbf{f} : A \to \mathbb{R}$ and $\mathbf{g} : B \to \mathbb{R}$ satisfy $\mathbf{f}(A) \subset B$ so that the composition $\mathbf{g} \circ \mathbf{f}$ is well-defined. If $\mathbf{f}$ is differentiable at $c \in A$ and if $\mathbf{g}$ is differentiable at $\mathbf{f}(c) \in B$, then $\mathbf{g} \circ \mathbf{f}$ is differentiable at c with $(\mathbf{g} \circ \mathbf{f})'(c) = \mathbf{g}'(\mathbf{f}(c))\mathbf{f}'(c)$.

Proof. Since $\mathbf{g}$ is differentiable at $\mathbf{f}(c)$,

$$\mathbf{g}'(\mathbf{f}(c)) = \lim_{y \to \mathbf{f}(c)} \frac{\mathbf{g}(y) - \mathbf{g}(\mathbf{f}(c))}{y - \mathbf{f}(c)},$$

which we recast as a function of y [not defined at $y = \mathbf{f}(c)$]

$$d(y) = \frac{\mathbf{g}(y) - \mathbf{g}(\mathbf{f}(c))}{y - \mathbf{f}(c)} - \mathbf{g}'(\mathbf{f}(c))\mathbf{f}'(c)$$

which has the property $\lim_{y \to \mathbf{f}(c)} d(y) = 0$.

If we declare $d(\mathbf{f}(c)) = 0$, the extension $d(y)$ becomes a continuous function at $\mathbf{f}(c)$. We rewrite the equation defining $d(y)$ in form of an identity

$$\mathbf{g}(y) - \mathbf{g}(\mathbf{f}(c)) = [\mathbf{g}'(\mathbf{f}(c)) - d(y)](y - \mathbf{f}(c)).$$

Now we replace y by $\mathbf{f}(t)$ for $t \in A$ (it is legitimate). If $t \neq c$, we can divide this equation by $t - c$ to get

$$\frac{\mathbf{g}(\mathbf{f}(t)) - \mathbf{g}(\mathbf{f}(c))}{t - c} = [\mathbf{g}'(\mathbf{f}(c)) - d(\mathbf{f}(t))]\frac{\mathbf{f}(t) - \mathbf{f}(c)}{t - c},$$

for all $t \neq c$. Now apply the Algebraic Limit Theorem, taking into account that $\lim_{t \to c} d(\mathbf{f}(t)) = 0$.

Interior Extremum Theorem

Theorem

Let $\mathbf{f}$ *be differentiable on an open interval* (a, b). *If* $\mathbf{f}$ *attains a maximum value at some point* $c \in (a, b)$ *(i.e.* $\mathbf{f}(c) \geq \mathbf{f}(x)$ *for all* $x \in (a, b)$*), then* $\mathbf{f}'(c) = 0$. *The same is true if* $\mathbf{f}(c)$ *is a minimum value.*

This is the anchor of most methods to find max and min of functions. It is due to Fermat.

Proof. Consider the case of a maximum. Because c is an interior point of (a, b), we can find two sequences (x_n), (y_n), satisfying $x_n < c < y_n$

$$x_n \to c \leftarrow y_n.$$

That is, (x_n) converges to c from the left, the other sequence from the rigtht. By the Order Limit Theorem, since $\mathbf{f}(x_n) \leq \mathbf{f}(c)$ and $x_n < c$

$$f'(c) = \lim_{x_n \to c} \frac{\mathbf{f}(c) - \mathbf{f}(x_n)}{c - x_n} \geq 0$$

Similarly, since $\mathbf{f}(y_n) \leq \mathbf{f}(c)$ and $y_n > c$

$$f'(c) = \lim_{y_n \to c} \frac{\mathbf{f}(c) - \mathbf{f}(y_n)}{c - y_n} \leq 0$$

It follows $\mathbf{f}'(c) = 0$. □

Darboux Theorem: IVT for derivatives

The Intermediate Value Theorem asserts that if $\mathbf{f}(x)$ is continuous on $[a, b]$, then it assumes all values between $\mathbf{f}(a)$ and $\mathbf{f}(b)$: If

$$\mathbf{f}(a) < \alpha < \mathbf{f}(b)$$

there is

$$a < c < b, \quad \mathbf{f}(c) = \alpha$$

There are other functions with this property, noteworthy those which are derivatives (which may be not continuous):

Theorem

If $\mathbf{f}$ *is differentiable on the interval* $[a, b]$*, and* α *satisfies* $\mathbf{f}'(a) < \alpha < \mathbf{f}'(b)$ *(or* $\mathbf{f}'(a) > \alpha > \mathbf{f}'(b)$*), then there is* $c \in (a, b)$ *where* $\mathbf{f}'(c) = \alpha$*.*

That is, any $\mathbf{f}'$ on $[a, b]$ satisfies the **[IVT]**.

Proof. Let

$$\mathbf{g}(x) = \mathbf{f}(x) - \alpha x$$

This is a continuous function on $[a, b]$, and therefore attains its infimum at some $c \in [a, b]$.
We claim that $c \in (a, b)$, that is, it is an interior point, so that we may be able to apply the IVT (**Interior Extremum Theorem**) that asserts: There is $c \in (a, b)$ such that

$$0 = \mathbf{g}'(c) = \mathbf{f}'(c) - \alpha$$

1. $\mathbf{g}'(a) = \mathbf{f}'(a) - \alpha < 0$ and $\mathbf{g}'(b) = \mathbf{f}'(b) - \alpha > 0$
2. We claim that there is $x \in (a, b)$ such that $\mathbf{g}(a) > \mathbf{g}(x)$: If not
$$\mathbf{g}'(a) = \lim_{x \to a} \frac{\mathbf{g}(a) - \mathbf{g}(x)}{a - x} \geq 0.$$
3. Similarly there is $y \in (a, b)$ such that $\mathbf{g}(b) > \mathbf{g}(y)$
4. These two observations prove that neither $\mathbf{g}(a)$ nor $\mathbf{g}(b)$ is the minimum of $\mathbf{g}(x)$.
5. Thus $c \in (a, b)$, that is $\mathbf{g}'(c) = 0$, which means
$$\mathbf{f}'(c) = \alpha.$$

Outline

Last Time & Today

- The Derivative: continuity, chain rule
- IET: Interior Extremum Theorem–the heart of max/min problems
- Darboux Theorem: IVT for derivatives
- MVT: Mean Value Theorem

Mean Value Theorem

Let **f** be a differentiable function on the interval $[a, b]$. Probably the most useful assertion of the differential calculus is the relationship between the value of the slope of the secant to the graph of $\mathbf{f}(x)$,

$$\frac{\mathbf{f}(b) - \mathbf{f}(a)}{b - a},$$

and values of the derivative. Even the so-called Fundamental Theorem of Calculus will be seen as one of its consequences.

Quantum Derivative

One issues that arises in the application of derivatives to Physics is the following: Suppose $\mathbf{f}(x)$ is a function of the variable x which is either length or time.In the definition

$$\mathbf{f}'(x) = \lim_{h \to 0} \frac{\mathbf{f}(x+h) - \mathbf{f}(x)}{h},$$

if h is length (or time), there is a minimal value below which it is not 'relevant' (? measurable), so the limit makes no sense! One could take $h =$ Planck's length, and define the h-derivative of $\mathbf{f}(x)$ as

$$D_h\mathbf{f}(x) = \frac{\mathbf{f}(x+h) - \mathbf{f}(x)}{h},$$

Rolle's Theorem

Theorem

Let $\mathbf{f} : [a, b] \to \mathbb{R}$ *be a continuous function on* $[a, b]$ *and differentiable on* (a, b)*. If* $\mathbf{f}(a) = \mathbf{f}(b)$*, then there exists a point* $c \in (a, b)$ *where* $\mathbf{f}'(c) = 0$*.*

Proof. Since **f** is continuous on a compact set, it attains a maximum and a minimum. If both are reached at the endpoints, $\mathbf{f}(x)$ must be constant, and $\mathbf{f}'(x) = 0$ on (a, b). On the other hand, if a minimum or a maximum occur at an interior point c, we would have $\mathbf{f}'(c) = 0$ by the IET. □

Mean Value Theorem

Theorem

Let $\mathbf{f} : [a, b] \to \mathbb{R}$ *be a continuous function on* $[a, b]$ *and differentiable on* (a, b)*. If* $\mathbf{f}(a) \neq \mathbf{f}(b)$*, then there exists a point* $c \in (a, b)$

$$\mathbf{f}'(c) = \frac{\mathbf{f}(b) - \mathbf{f}(a)}{b - a}.$$

Proof. We reduce the proof to Rolle's Theorem: Let $\mathbf{g}(x)$ be the equation of the line through the endpoints of the graph of $\mathbf{f}(x)$:

$$\mathbf{g}(x) = \mathbf{f}(a) + \frac{\mathbf{f}(b) - \mathbf{f}(a)}{b - a}(x - a).$$

Consider

$$d(x) = \mathbf{f}(x) - \mathbf{g}(x).$$

Observe that

$$d(a) = d(b) = 0$$

Note

$$d'(x) = \mathbf{f}'(x) - \frac{\mathbf{f}(b) - \mathbf{f}(a)}{b - a}$$

and by Rolle's theorem $d'(c) = 0$ for some $c \in (a, b)$,

$$\mathbf{f}'(c) = \frac{\mathbf{f}(b) - \mathbf{f}(a)}{b - a}$$

Outline

Workshop #7

Need:

- **IVT**: Intermediate Value Theorem
- **Rolle**: Special case of MVT
- **MVT**: Mean Value Theorem
- **Darboux**: IVT for derivatives

Workshop #7

Let $\mathbf{h}$ be a differentiable function on the interval $[0, 3]$, an assume $\mathbf{h}(0) = 1$, $\mathbf{h}(1) = 2$, and $\mathbf{h}(3) = 2$.

1. Argue that there is a point $d \in [0, 3]$ where $\mathbf{h}(d) = d$.
   Apply IVT to the continuous function $\mathbf{f}(x) = \mathbf{h}(x) - x$:
   $\mathbf{f}(0) = 1 - 0 = 1$, $\mathbf{f}(3) = 2 - 3 = -1$.
2. Argue that there is a point $c \in [0, 3]$ where $\mathbf{h}'(c) = 1/3$.
   $(\mathbf{h}(3) - \mathbf{h}(0))(3 - 0) = (2 - 1)/3$: Apply MVT
3. Argue that $\mathbf{h}'(x) = 1/4$ at some point in the domain.
   Since $\mathbf{h}(1) = \mathbf{h}(3) = 2$, there must be $d \in (2, 3)$ such that $\mathbf{h}'(d) = 0$ by Rolle's Theorem. Since $\mathbf{h}'(d) = 0 < 1/4 < 1/3 = \mathbf{h}'(c)$,by Darboux's Theorem there is x such that $\mathbf{h}'(x) = 1/4$.

Workshop #7, Cont'd

4: If $\mathbf{h} : [a, b] \to \mathbb{R}$ is a continuous function, prove that $\mathbf{h}([a, b]) = [c, d]$. Begin by describing c and d. If $\mathbf{h}'(x) \neq 0$, what are the possible values of c, d?

5: If $\mathbf{h}'$ is not constant, then $\mathbf{h}'$ must take some irrational values.

Outline

Naive but useful fact

Theorem

Let $\mathbf{f}(x)$ be differentiable at any open interval (a, b), and suppose that $\mathbf{f}'(x) = 0$ at each such point. Then the value of the function is constant in the interval.

Proof. We know that $\mathbf{f}(x)$ is continuous at each point in (a, b). Consider any pair of distinct points x_0 and x_1 of (a, b). We are going to prove that

$$\mathbf{f}(x_0) = \mathbf{f}(x_1).$$

Let us apply the **MVT** to the points:

$$\mathbf{f}(x_1) - \mathbf{f}(x_0) = \mathbf{f}'(c)(x_1 - x_0) = 0,$$

since $c \in (a, b)$.

Exercise

Use the **MVT** to prove that

$$\frac{1}{9} < \sqrt{66} - 8 < \frac{1}{8}$$

Let us apply the **MVT** to the function $\mathbf{f}(x) = \sqrt{x}$ on the interval $[64, 66]$:

$$\frac{\sqrt{66} - \sqrt{64}}{66 - 64} = \frac{1}{2\sqrt{c}}$$

for $c \in (64, 66)$. But the function $\frac{1}{\sqrt{x}}$ is decreasing so

$$\frac{1}{\sqrt{81}} < \frac{1}{\sqrt{c}} < \frac{1}{\sqrt{64}}$$

Exercise

Prove that $\mathbf{f}(x) = x^n + ax + b = 0$ $(a, b \in \mathbb{R})$ has at most two distinct real roots if n is even, and at most three if n is odd.

Between any pair of roots, $\mathbf{f}(x_0) = \mathbf{f}(x_1)$, by the **MVT**,

$$0 = \mathbf{f}(x_1) - \mathbf{f}(x_0) = \mathbf{f}'(c)(x_1 - x_0),$$

we must have a root for $\mathbf{f}'(x) = nx^{n-1} + a$,

$$c = \sqrt[n-1]{\frac{-a}{n}}$$

But if n is even, there is at most one such c. If n is odd, $n - 1$ is even and we shall have at most 2 such c.

Uniform Continuity

We proved that if $\mathbf{f} : A \to \mathbb{R}$ is continuous and A is compact, then $\mathbf{f}$ is uniformly continuous. Let us prove a more precise version of this result if $A = [a, b]$ and $\mathbf{f}'(x)$ is continuous. Let M be the maximum value of the the continuous function $|\mathbf{f}'(x)|$ on the interval $[a, b]$. If x, y are any two points in the interval, by the MVT

$$\mathbf{f}(x) - \mathbf{f}(y) = \mathbf{f}'(c)(x - y)$$

for some c in $[x, y] \subset [a, b]$. Taking absolute values, since $|\mathbf{f}'(c)| \leq M$, we get

$$|\mathbf{f}(x) - \mathbf{f}(y)| \leq M|x - y|,$$

that is, $\mathbf{f}(x)$ has Lipschitz's condition and therefore is uniformly continuous.

Higher Degrees Mean Value Theorems

If $\mathbf{f} : [a, b] \to \mathbb{R}$ is continuous and $\mathbf{f}'(x)$ exists in (a, b), the MVT says that

$$\mathbf{f}(b) = \mathbf{f}(a) + (b - a)\mathbf{f}'(c),$$

for some $c \in (a, b)$.

Let us assume more: Suppose $\mathbf{f}'(x)$ is continuous on $[a, b]$ and $\mathbf{f}''(x)$ exists in (a, b). We claim that

$$\mathbf{f}(b) = \mathbf{f}(a) + (b - a)\mathbf{f}'(a) + \frac{(b - a)^2}{2}\mathbf{f}''(c),$$

for some $c \in (a, b)$.

To prove this, consider the function

$$\mathbf{g}(x) = \mathbf{f}(b) - \mathbf{f}(x) - (b-x)\mathbf{f}'(x) - \frac{(b-x)^2}{(b-a)^2}(\mathbf{f}(b) - \mathbf{f}(a) - (b-a)\mathbf{f}'(a)).$$

Note that it vanishes for $x = a$ and $x = b$. Since it is differentiable, by Rolle's Theorem

$$\mathbf{g}'(c) = 0$$

for some $c \in (a, b)$. Since

$$\mathbf{g}'(x) = -(b-x)\mathbf{f}''(x) - \frac{2(b-x)}{(b-a)^2}(\mathbf{f}(b) - \mathbf{f}(a) - (b-a)\mathbf{f}'(a)),$$

we get the desired formula.

An application of the formula is the **second derivative** test in the theory of max/min. Apply the formula to two points, a, b, in its domain where a is an **interior extremum point**:

$$\mathbf{f}(b) = \mathbf{f}(a) + (b-a)\mathbf{f}'(a) + \frac{(b-a)^2}{2}\mathbf{f}''(c), \quad c \in (a,b).$$

Since $\mathbf{f}'(a) = 0$, this gives

$$\mathbf{f}(b) = \mathbf{f}(a) + \frac{(b-a)^2}{2}\mathbf{f}''(c)$$

and therefore

$$\begin{aligned} \mathbf{f}(b) &\geq \mathbf{f}(a), \quad \text{if } \mathbf{f}''(c) > 0 \\ \mathbf{f}(b) &\leq \mathbf{f}(a), \quad \text{if } \mathbf{f}''(c) < 0 \end{aligned}$$

If $\mathbf{f}''$ is continuous at a, $\mathbf{f}''(c)$ has the same sign as $\mathbf{f}''(a)$ for c near a.

Second derivative test for max/min

Test: If $\mathbf{f}(x)$ is a function with an extremum interior point at $x = a$ and $\mathbf{f}''(x)$ is continuous at this point, then:

- If $\mathbf{f}''(a) < 0$, $x = a$ is a local maximum of $\mathbf{f}(x)$;
- If $\mathbf{f}''(a) > 0$, $x = a$ is a local minimum of $\mathbf{f}(x)$;
- If $\mathbf{f}''(a) = 0$, no conclusion.

Question: Are with stuck in the third possibility? Not realy, we need more information from the **MVT**. Let us get it.

There are versions of the **MVT** in all degrees:

$$
\begin{aligned}
\mathbf{f}(b) &= \mathbf{f}(a) + (b-a)\mathbf{f}'(a) + \frac{(b-a)^2}{2}\mathbf{f}''(a) \\
&+ \cdots + \frac{(b-a)^n}{n!}\mathbf{f}^{(n)}(a) + \frac{(b-a)^{n+1}}{(n+1)!}\mathbf{f}^{(n+1)}(c),
\end{aligned}
$$

for some $c \in (a, b)$.
The proof is analogous to the degree 2 case.

Third derivative test for max/min

Let us use the degree 3 of this formula to study the case of **extremum** point $x = a$, when $\mathbf{f}'(a) = \mathbf{f}''(a) = 0$:

$$\mathbf{f}(b) \; = \; \mathbf{f}(a) + (b-a)\mathbf{f}'(a) + \frac{(b-a)^2}{2}\mathbf{f}''(a) + \frac{(b-a)^3}{3!}\mathbf{f}^{(3)}(c)$$

$$\mathbf{f}(b) \; = \; \mathbf{f}(a) + \frac{(b-a)^3}{3!}\mathbf{f}^{(3)}(c)$$

Note now that the factor $(b-a)^3$ changes sign, depending on the side of a that b lies. This means that if $\mathbf{f}^{(3)}(c) \neq 0$, then $\mathbf{f}(b)$ is larger that $\mathbf{f}(a)$ on one side and smaller on the other: $x = a$ is an **inflection point**.

If $\mathbf{f}^{(3)}(a) = 0$, we go to the next formula of the **MVT** and develop a fourth derivative test for max/min. And so on **...**

Generalized Mean Value Theorem

Theorem

Let $\mathbf{f}, \mathbf{g} : [a, b] \to \mathbb{R}$ *are continuous functions on* $[a, b]$ *and differentiable on* (a, b). *If* $\mathbf{f}(a) \neq \mathbf{f}(b)$, *then there exists a point* $c \in (a, b)$ *where*

$$[\mathbf{f}(b) - \mathbf{f}(a)]\mathbf{g}'(c) = [\mathbf{g}(b) - \mathbf{g}(a)]\mathbf{f}'(c).$$

If $\mathbf{g}'$ *is never zero on* (a, b), *the assertion can be stated as*

$$\frac{\mathbf{f}'(c)}{\mathbf{g}'(c)} = \frac{\mathbf{f}(b) - \mathbf{f}(a)}{\mathbf{g}(b) - \mathbf{g}(a)}.$$

Proof. It is enough to apply the MVT to the function

$$\mathbf{h}(x) = [\mathbf{f}(b) - \mathbf{f}(a)]\mathbf{g}(x) - [\mathbf{g}(b) - \mathbf{g}(a)]\mathbf{f}(x)$$

$$\begin{aligned}
\mathbf{h}(x) &= [\mathbf{f}(b) - \mathbf{f}(a)]\mathbf{g}(x) - [\mathbf{g}(b) - \mathbf{g}(a)]\mathbf{f}(x) \\
\mathbf{h}(a) &= [\mathbf{f}(b) - \mathbf{f}(a)]\mathbf{g}(a) - [\mathbf{g}(b) - \mathbf{g}(a)]\mathbf{f}(a) = \mathbf{f}(b)\mathbf{g}(a) - \mathbf{f}(a)\mathbf{g}(b) \\
\mathbf{h}(b) &= [\mathbf{f}(b) - \mathbf{f}(a)]\mathbf{g}(b) - [\mathbf{g}(b) - \mathbf{g}(a)]\mathbf{f}(b) = \mathbf{f}(b)\mathbf{g}(a) - \mathbf{f}(a)\mathbf{g}(b) \\
\mathbf{h}(a) &= \mathbf{h}(b)
\end{aligned}$$

Hence by the **Mean Value Theorem** there is $c \in [a, b]$

$$\begin{aligned}
\mathbf{h}'(c) &= [\mathbf{f}(b) - \mathbf{f}(a)]\mathbf{g}'(c) - [\mathbf{g}(b) - \mathbf{g}(a)]\mathbf{f}'(c) = 0 \\
\frac{\mathbf{f}'(c)}{\mathbf{g}'(c)} &= \frac{\mathbf{f}(b) - \mathbf{f}(a)}{\mathbf{g}(b) - \mathbf{g}(a)}
\end{aligned}$$

Improper limits

1. What is $\lim_{x\to 0} \frac{\sin 2x}{\sin x}$? If we remember the double angle formula $\sin 2x = 2\sin x \cos x$, the answer is easy

$$\frac{\sin 2x}{\sin x} = \frac{2\sin x \cos x}{\sin x} = 2\cos x \to 2$$

as $x \to 0$. Another way?

2. How about $\lim_{x\to 0} x \ln x$? The answer is going to be 0. Why?

3. How about

$$\lim_{x\to\infty} \frac{e^x}{P(x)}, \quad P(x) \text{ some nonzero polynomial}$$

Different animal.

L'Hospital Rule: $0/0$

Theorem

Assume $\mathbf{f}$ *and* $\mathbf{g}$ *are continuous functions defined on an interval containing a, and assume that* $\mathbf{f}$ *and* $\mathbf{g}$ *are differentiable on this interval, with the possible exception of the point a. If* $\mathbf{f}(a) = 0$ *and* $\mathbf{g}(a) = 0$*, then*

$$\lim_{x \to a} \frac{\mathbf{f}'(x)}{\mathbf{g}'(x)} = L \quad \Rightarrow \quad \lim_{x \to a} \frac{\mathbf{f}(x)}{\mathbf{g}(x)} = L.$$

Proof. By the Generalized MVT, for $x \neq a$ in the interval,

$$\frac{\mathbf{f}(x)}{\mathbf{g}(x)} = \frac{\mathbf{f}(x) - \mathbf{f}(a)}{\mathbf{g}(x) - \mathbf{g}(a)} = \frac{\mathbf{f}'(c)}{\mathbf{g}'(c)}$$

for some $c \in (a, x)$.

If (x_n) is a sequence in the interval converging to a, the corresponding sequence (c_n) will converge to a as well. We then have

$$\lim_{x_n \to a} \frac{\mathbf{f}(x_n)}{\mathbf{g}(x_n)} = \lim_{c_n \to a} \frac{\mathbf{f}'(c_n)}{\mathbf{g}'(c_n)} = L.$$

Example:

$$\lim_{x \to 0} \frac{x^2 \sin(1/x)}{\sin x} = \lim_{x \to 0} \frac{x \sin(1/x)}{\sin x / x} = 0/1 = 0,$$

since

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1.$$

Comment: Circular argument!

Examples

1. What is $\lim_{x\to 0} \frac{\sin 2x}{\sin x}$?

$$\lim_{x\to 0} \frac{\sin 2x}{\sin x} = \lim_{x\to 0} \frac{2\cos 2x}{\cos x} = \frac{2}{1} = 2.$$

2. How about $\lim_{x\to 1} \frac{\ln x}{\log x}$?

$$\lim_{x\to 1} \frac{\ln x}{\log x} = \lim_{x\to 1} \frac{1/x}{1/(x \ln 10)} = \ln 10,$$

since $\log x = \frac{\ln x}{\ln 10}$. (So we would not need l'Hospital's here.)

3. How about

$$\lim_{x\to\infty} \frac{e^x}{P(x)}, \quad P(x) \text{ some nonzero polynomial}$$

Need a different technique.

L'Hospital Rule: ∞/∞

Theorem

Assume $\mathbf{f}$ *and* $\mathbf{g}$ *are differentiable on* (a, b)*, and that* $\lim_{x\to a} \mathbf{g}(x) = \infty$ *(or* $-\infty$*). Then*

$$\lim_{x\to a} \frac{\mathbf{f}'(x)}{\mathbf{g}'(x)} = L \quad \Rightarrow \quad \lim_{x\to a} \frac{\mathbf{f}(x)}{\mathbf{g}(x)} = L.$$

The proof is very different in this case. To begin, one must clarify the meaning of $\lim_{x\to a} \mathbf{g}(x) = \infty$:

Definition

Given $\mathbf{g} : A \to \mathbb{R}$ and a limit point a of A, we say $\lim_{x\to a} \mathbf{g}(x) = \infty$ if, for every $M > 0$, there exists a $\delta > 0$ such that whenever $0 < |x - a| < \delta$ it follows that $\mathbf{g}(x) > M$.

Proof. Because $\lim_{x \to a} \frac{\mathbf{f}'(x)}{\mathbf{g}'(x)} = L$, for any $\epsilon > 0$ there is δ_1 so that

$$L - \epsilon/2 < \frac{\mathbf{f}'(x)}{\mathbf{g}'(x)} < L + \epsilon/2$$

for all $a < x < a + \delta_1 = t$.
On the other hand, by Generalized MVT on $[a, t]$

$$\frac{\mathbf{f}(x) - \mathbf{f}(t)}{\mathbf{g}(x) - \mathbf{g}(t)} = \frac{\mathbf{f}'(c)}{\mathbf{g}'(c)}$$

for $c \in (a, t)$.

We must isolate $\mathbf{f}(x)/\mathbf{g}(x)$ in first expression. To begin we multiply it by $(\mathbf{g}(x) - \mathbf{g}(t))/\mathbf{g}(x)$. To preserve inequalities this factor must be positive, which is assured as follows.

Because $\lim_{x\to a} \mathbf{g}(x) = \infty$, there is a $\delta_2 > 0$ such that for $a < x < a + \delta_2$ $\mathbf{g}(x) \geq \mathbf{g}(t)$.

$$\begin{aligned}(L - \epsilon/2)\left(1 - \frac{\mathbf{g}(t)}{\mathbf{g}(x)}\right) &< \frac{\mathbf{f}(x) - \mathbf{f}(t)}{\mathbf{g}(x)} \\ &< (L + \epsilon/2)\left(1 - \frac{\mathbf{g}(t)}{\mathbf{g}(x)}\right)\end{aligned}$$

Recall that t is fixed and we could here $x \to a$, but to get a more honest proof we clean out:

$$L - \epsilon/2 + A(x) < \frac{\mathbf{f}(x)}{\mathbf{g}(x)} < L + \epsilon/2 + B(x)$$

where

$$\begin{aligned} A(x) &= \frac{-L\mathbf{g}(t) + \epsilon/2\mathbf{g}(t) + \mathbf{f}(t)}{\mathbf{g}(x)} \\ B(x) &= \frac{L\mathbf{g}(t) - \epsilon/2\mathbf{g}(t) + \mathbf{f}(t)}{\mathbf{g}(x)} \end{aligned}$$

Since $\lim_{x\to a} \mathbf{g}(x) = \infty$, we can find $\delta_3 > 0$ so that both $A(x)$ and $B(x)$ are less than $\epsilon/2$ for $a < x < a + \delta_3$. Now we pick $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and get

$$L - \epsilon < \frac{\mathbf{f}(x)}{\mathbf{g}(x)} < L + \epsilon,$$

for $a < x < a + \delta$.

Example

$$\lim_{x \to \pi/2} (x - \pi/2) \tan x$$

The function can be written in several ways as a quotient

$$(x - \pi/2) \tan x = \frac{\tan x}{1/(x - \pi/2)} = \frac{\sin x}{\cos x/(x - \pi/2)}$$

Take your choice.

Examples

- Evaluate the limit $\lim_{x\to\infty} \frac{(\ln x)^n}{x}$: $\frac{\infty}{\infty}$ case

$$\begin{aligned}\lim_{x\to\infty} \frac{(\ln x)^n}{x} &= \lim_{x\to\infty} \frac{(n(\ln x)^{n-1}/x}{x'} \\ &= n \lim_{x\to\infty} \frac{(\ln x)^{n-1}}{x} \\ &= 0\end{aligned}$$

using induction on n will get the limit is 0.

Example

Sometimes the function has the form $\mathbf{f}(x) - \mathbf{g}(x)$ and both become ∞ as x approaches c. A good practice is to try to convert the difference into a quotient:

$$
\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right) &= \lim_{x\to 0}\frac{\sin x - x}{x\sin x} \\
&= \lim_{x\to 0}\frac{\cos x - 1}{x\cos x + \sin x} \\
&= \lim_{x\to 0}\frac{-\sin x}{x\sin x + 2\cos x} \\
&= \frac{0}{2} = 0
\end{aligned}
$$

Example

Sometimes the function has the form $y = (\mathbf{f}(x))^{\mathbf{g}(x)}$ and one or the other approaches 0 or ∞ as x approaches c. For instance,

- $\mathbf{f}(x) \to 1$ and $\mathbf{g}(x) \to \infty$;
- $\mathbf{f}(x) \to \infty$ and $\mathbf{g}(x) \to 0$;
- $\mathbf{f}(x) \to 0$ and $\mathbf{g}(x) \to 0$;

The trick is usually to look for the limit of $\ln y$

$$\ln y = \mathbf{g}(x) \ln \mathbf{f}(x) = \frac{\ln \mathbf{f}(x)}{\frac{1}{\mathbf{g}(x}}$$

Consider $\lim_{x\to 0} x^x$:

$$\begin{aligned}\lim_{x\to 0} \ln x^x = \lim_{x\to 0} x \ln x &= \lim_{x\to 0} \frac{\ln x}{1/x} \\ &= \lim_{x\to 0} \frac{1/x}{-1/x^2} \\ &= \lim_{x\to 0} -x = 0\end{aligned}$$

This implies

$$\lim_{x\to 0} x^x = 1$$

Exercises

- Show that if $\mathbf{f}$ is a function that is differentiable on an interval with $\mathbf{f}'(x) \neq 1$, then there exists at most one point where $\mathbf{f}(c) = c$.

  Solution: If there are two such points (called **fixed points**) x_0, x_1, **MVT** requires $c \in [x_0, x_1]$

  $$\frac{\mathbf{f}(x_1) - \mathbf{f}(x_0)}{x_1 - x_0} = \frac{x_1 - x_0}{x_1 - x_0} = \mathbf{f}'(c) = 1$$

  which is impossible since $\mathbf{f}'(c) \neq 1$.

- Evaluate $\lim_{x\to\infty} \frac{e^x}{x^{10}}$ and $\lim_{x\to\infty} x \ln \frac{x+1}{x-1}$.

- Let $\mathbf{f} : [0, 1] \to \mathbb{R}$ be a continuous function. If $\mathbf{f}(0) \neq \mathbf{f}(1)$, prove that the image of **f**, $\mathbf{f}([0, 1])$, is uncountable.

  Solution: By the **Intermediate Value Theorem**, any number $y \in [\mathbf{f}(0), \mathbf{f}(1)]$ is equal to some $x \in [0, 1]$. Since $[\mathbf{f}(0), \mathbf{f}(1)]$ is a non-trivial interval, it is uncountable.

- Let **f** be a nontrivia real valued function (i.e. **f** is not the null function) such that $\mathbf{f}(xy) = \mathbf{f}(x) + \mathbf{f}(y)$ for all $x, y \in \mathbb{R}^+$. Prove the following properties of **f**: (i) If **f** is continuous at $c = 1$, then it is continuous at all $c \in \mathbb{R}^+$. (ii) Describe **all** possible **f**'s.

  Solution: (i) Let c be a point of $\mathbb{R}^+ = (0, \infty)$. From $\mathbf{f}(x \cdot 1) = \mathbf{f}(x) + \mathbf{f}(1)$, we get $\mathbf{f}(1) = 0$. We can write any other point x as $x = c \cdot h$. Note that $x \to c$ is equivalent to $h \to 1$. We have

  $$\lim_{x \to c} \mathbf{f}(x) = \lim_{h \to 1} \mathbf{f}(ch) = \mathbf{f}(c) + \lim_{h \to 1} \mathbf{f}(h) = \mathbf{f}(c),$$

  since **f** is continuous at $x = 1$ and $\mathbf{f}(1) = 0$.

  By assumption, there is x_0 with $\mathbf{f}(x_0) \neq 0$. Note that because

  $$\mathbf{f}(1) = \mathbf{f}(x_0 \cdot \frac{1}{x_0}) = \mathbf{f}(x_0) + \mathbf{f}(\frac{1}{x_0}),$$

  either $\mathbf{f}(x_0)$ or $\mathbf{f}(1/x_0)$ is > 0.

We may assume $\mathbf{f}(x_0) > 0$ (otherwise use $1/x_0$). Since $\mathbf{f}(x_0^n) = n\mathbf{f}(x_0)$, $\mathbf{f}(x)$ is not bounded so its image contains, by the **IVT**, the interval $[0, \infty]$.

Let $a > 0$ be such that $\mathbf{f}(a) = 1$.

Claim: $\mathbf{f}(x) = \log_a x$

1. Prove that $\mathbf{f}(a^{p/q}) = p/q$; for that check $\mathbf{f}(a^p) = p$ and $q\mathbf{f}(a^{1/q}) = 1$.
2. Argue that $\mathbf{f}(a^x) = x$ for all $x \in \mathbb{R}^+$ by considering the continuous function $\mathbf{f}(a^x) - x$, where the first is a composite of a^x and $\mathbf{f}(x)$.

Outline

Inverse Function Theorem

If $\mathbf{f} : A \to \mathbb{R}$ is a 1–1 function, one can define the inverse function

$$\mathbf{f}^{-1} : \mathbf{f}(A) \to A \subset \mathbb{R}$$

by the rule:

$$\mathbf{f}^{-1}(y) = x \Leftrightarrow \mathbf{f}(x) = y.$$

Theorem

If $\mathbf{f} : A \to \mathbf{f}(A)$ is continuous and 1–1 with A compact, then $\mathbf{f}^{-1} : \mathbf{f}(A) \to A$ is continuous.

Proof. Let (y_n) be any sequence in $\mathbf{f}(A)$ converging to $y_0 \in \mathbf{f}(A)$. We will show that $(\mathbf{f}^{-1}(y_n))$ converges to $\mathbf{f}^{-1}(y_0)$. Write $x_n = \mathbf{f}^{-1}(y_n)$ for convenience. Since A is compact, (x_n) is bounded and by Bolzano-Weierstrass admits convergent subsequences. Consider a convergent subsequence (x_{n_k}), of limit, say, z_0.

Now $z_0 \in A$ since A is closed, and so $\mathbf{f}$ is continuous at z_0. Thus $(\mathbf{f}(x_{n_k}))$ converges to $\mathbf{f}(z_0)$. But $(\mathbf{f}(x_{n_k})) = (y_{n_k})$ is a subsequence of (y_n), hence it converges to $y_0 = \mathbf{f}(x_0)$. Since $\mathbf{f}$ is 1–1 and $\mathbf{f}(z_0) = \mathbf{f}(x_0)$, $z_0 = x_0$.

This says that all convergent subsequences of (x_n) have the same limit, from which we get that (x_n) has for limit x_0, in other words $(\mathbf{f}^{-1}(y_n))$ converges to $\mathbf{f}^{-1}(y_0)$. Thus $\mathbf{f}^{-1}$ is continuous at y_0.

Inverse Function Theorem

Theorem

Suppose $\mathbf{f} : [a, b] \to \mathbb{R}$ *is continuous and differentiable with* $\mathbf{f}'(x) \neq 0$ *for all* $x \in [a, b]$*. Then* $\mathbf{f}$ *is* $1-1$, $\mathbf{f}^{-1}$ *is continuous and differentiable on* $\mathbf{f}([a, b])$ *and*

$$(\mathbf{f}^{-1})'(\mathbf{f}(x)) = \frac{1}{\mathbf{f}'(x)}$$

for all $x \in [a, b]$*.*

Proof. Since $\mathbf{f}'(x) \neq 0$, by the **MVT f** is 1–1: For any distinct points x_0, x_1, recall the formula

$$\mathbf{f}(x_1) - \mathbf{f}(x_0) = \mathbf{f}'(c)(x_1 - x_0), \quad \mathbf{f}'(c) \neq 0.$$

Assume that $\mathbf{f}([a, b]) = [c, d]$ (This is actually the case!). Choose $y_0 \in [c, d]$ and let (y_n) be a sequence in $[c, d]$, $y_n \neq y_0$, converging to y_0. Let $x_n = \mathbf{f}^{-1}(y_n)$; note that $x_n \neq x_0$ since $\mathbf{f}$ is 1–1.

By the differentiability of $\mathbf{f}$, the sequence of terms

$$\frac{\mathbf{f}(x_n) - \mathbf{f}(x_0)}{x_n - x_0}$$

converges to $\mathbf{f}'(x_0)$. By hypothesis $\mathbf{f}'(x_0) \neq 0$. Since $\mathbf{f}(x_n) - \mathbf{f}(x_0) \neq 0$, we can flip the limit and write

$$\lim \frac{\mathbf{f}^{-1}(y_n) - \mathbf{f}^{-1}(y_0)}{y_n - y_0} = \frac{1}{\mathbf{f}'(x_0)}.$$

Thus $\mathbf{f}^{-1}$ is differentiable and

$$(\mathbf{f}^{-1})'(\mathbf{f}(x)) = \frac{1}{\mathbf{f}'(x)}.$$

Example

Let $\mathbf{f}(x) = \tan x, \quad x \in (-\pi/2, \pi/2)$. Using the definition, $\tan x = \frac{\sin x}{\cos x}$, a quick calculation gives $(\tan x)' = \sec^2 x$, a function that is always ≥ 1. We can apply the theorem to find the derivative of its inverse function,

$$\mathbf{f}^{-1}(x) = \tan^{-1}(x) :$$

Write $\mathbf{g}(x) = \tan^{-1}(x)$.

$$\begin{aligned}
\mathbf{g}'(x) = \frac{1}{\mathbf{f}'(\mathbf{g}(x))} &= \frac{1}{\sec^2(\mathbf{g}(x))} \\
&= \frac{1}{1 + \tan^2(\mathbf{g}(x))} \\
&= \frac{1}{1 + (\mathbf{f}(\mathbf{g}(x)))^2} \\
&= \frac{1}{1 + x^2}
\end{aligned}$$

Outline

Workshop #8

For each statement below give a proof or provide a counterexample.

1. If $\mathbf{f}'$ exists on an open interval, then both $\mathbf{f}$ and $\mathbf{f}'$ are continuous on that interval.
2. If $\mathbf{f}'$ exists on an open interval, and there is some point c where $\mathbf{f}'(c) > 0$, then there exists a δ-neighborhood $V_\delta(c)$ around c in which $\mathbf{f}'(x) > 0$ for all $x \in V_\delta(c)$.
3. If $\mathbf{f}$ is differentiable on an interval containing zero and $\lim_{x\to 0} \mathbf{f}'(x) = L$ then it must be that $L = \mathbf{f}'(0)$.
4. Repeat (3) but drop the assumption that $\mathbf{f}'(0)$ necessarily exists. If $\mathbf{f}'(x)$ exists for all $x \neq 0$ and $\lim_{x\to 0} \mathbf{f}'(x) = L$, then $\mathbf{f}'(0)$ exists and equals L. (Hint: Use the MVT)
5. If $\mathbf{f}$ and $\mathbf{h}$ are real continuous functions on $[0, 1]$, then $\{x \in [0, 1] : \mathbf{f}(x) = \mathbf{h}(x)\}$ is a closed set.