Math 311: Advanced Calculus

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Set 3

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Outline

Real Functions

Our main aim is to study interesting functions of the kind

$$\mathbf{X} \rightarrow \boxed{\quad \mathbf{f} \quad} \rightarrow \mathbf{Y}$$

where **X** and **Y** are subsets of $\mathbb{R}$.

If **f** is a function and the sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

lies in the domain of **f**, then the sequence

$$\mathbf{f}(a_1), \mathbf{f}(a_2), \mathbf{f}(a_3), \ldots, \mathbf{f}(a_n), \ldots$$

is contained in **Y**.

We want **f** to have the following property:

- If (a_n) is convergent then $(\mathbf{f}(a_n))$ convergent.

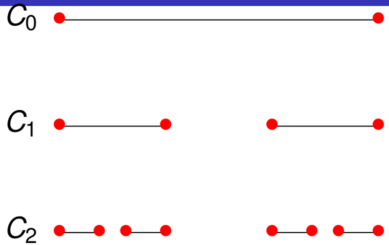
This requires us to examine some sets of subsets of $\mathbb{R}$:

- Open Sets
- Closed Sets
- Compact Sets
- Connected Sets
- Strange Sets

These subsets have properties that will explain why *continuous* functions act as they do.

Outline

Cantor Set



Rule: From each subinterval of C_n remove the inner third, to obtain C_{n+1}

Cantor Set: $C = \bigcap C_{n\geq 0}$

Building the Cantor set in detail

1. $C_0 = [0, 1]$, $C_1 = C_0 \setminus (1/3, 2/3)$, that is C_1 is obtained by removing from the interval C_0 its mid third (leaving the endpoints):
$$C_1 = [0, 1/3] \cup [2/3, 1]$$
2. Iterate by removing from each closed subinterval above its mid third (and so on)
$$C_2 = ([0, 1/9] \cup [2/9, 1/3]) \cup ([2/3, 7/9] \cup [8/9, 1])$$
3. This leads to a nested sequence of sets $C_0 \supset C_1 \supset C_2 \supset \cdots C_n \supset \cdots$.
4. $C = \bigcap C_{n \geq 0}$ is called the **Cantor** set.

Building the Cantor set–cont'd

Note that C is obtained from $[0, 1]$ by repeatedly carving out the heart. At least, the endpoints of the various subintervals belong to C. What else?

1. We are going to argue C is very thin by adding the lengths of the intervals that were removed:

$$\frac{1}{3} + 2\frac{1}{3^2} + 2^2\frac{1}{3^3} + \cdots ,$$

   a geometric series whose first term is $1/3$ and whose ratio is $2/3$, so it has for sum

$$\frac{1/3}{1 - 2/3} = 1!$$

So from $[0, 1]$ we took away a subset of measure 1!

$\mathbb{Q}$ is very thin

Exercise Given $\epsilon > 0$, argue that any countable set A is contained in a countable union $\bigcup_{n\geq 1}[a_n, b_n]$, such that

$$\sum_{n\geq 1} |b_n - a_n| < \epsilon.$$

Cardinality of C

If C only contained the endpoints [all rational points] of the subintervals of its construction, it would be countable. Let us show otherwise:

1. We are going to code the elements of C by infinite strings of $\{0, 1\}$ as follows: If $a \in C$, we set $a_1 = 0$ if a belongs to the leftmost subinterval of C_1, otherwise we set $a_1 = 1$.
2. Once a_1 is assigned, we consider the subinterval of C_2 that contains x, and apply the same rule. In this we get a unique address for x as the string $(a_1, a_2, a_3, \ldots)$.
3. Conversely, given any such string we build a nested sequence of closed intervals $I_1 \supset I_2 \supset I_3 \supset \cdots$: By **NIP** there is a point in the intersection. Actually unique **why?**

Thinness and Fractal Nature of C

1. We observed two contrasting things about C: (i) it is very thin, since $[0,1] \setminus C$ has length 1. (ii) it is uncountable. Can one compare it in other ways to the unit interval $U = [0,1]$?
2. Observe that if we expand $[0,1]$ by multiplying each number in it by 3, we obtain the interval $[0,3]$, that is we get 3 copies of U. However, if we do the same operation on C, we only get 2 copies of C! Care to visualize?

Dimension of a Set

- One way to define **dimension** of subset S of $\mathbb{R}^n$ is to compare S with the set obtained by expanding all points in it by a scale, say 3.
- For example, the dimension of $[0, 1]$ is 1, because we got $3U = [0, 3]$, while the dimension of a unit square is 2 [9 new squares], of the unit cube is 3 [27 new cubes].
- In all of these examples, we say that the dimension is d if 3^d is the size relative of the new set obtained by scaling the set by 3: $3 = 3^1$ for the unit interval, $9 = 3^2$ for the unit square, and $27 = 3^3$ for the unit cube. So they have dimensions $1, 2, 3$ respectively.

Dimension of C

1. For the Cantor set C, if we scale scale the set by 3 we get the union of two Cantor sets
2. This means that

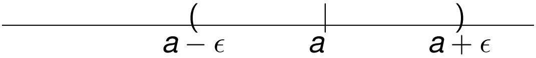
$$2 = 3^d,$$

so

$$\dim C = \frac{\ln 2}{\ln 3}!$$

Outline

Neighborhoods



Definition

Given a real number $a \in \mathbb{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbb{R} \colon |x - a| < \epsilon\}$$

is called the ϵ-**neighborhood** of a.

Thus a neighborhood of a point $a \in \mathbb{R}$ is just an **open** interval centered at a.

Open Sets

Definition (Open Set)

A set O of $\mathbb{R}$ is open if for all points $a \in O$ there exists an ϵ-neighborhood $V_\epsilon(a) \subset O$.

1. The entire $\mathbb{R}$ is an open set. Th definition also fits the empty subset $\emptyset$ of $\mathbb{R}$.
2. Any interval
$$(c, d) = \{x \in \mathbb{R} \mid c < x < d\}$$
is open. For any $a \in (c, d)$, if we pick $\epsilon = \min\{a - c, d - a\}$, then the interval $V_\epsilon(a) \subset (c, d)$.
3. The subsets $(c, d]$, $[c, d)$ or $[c, d]$ are NOT open: at least one of the endpoints do not pass the neighborhood test.

Theorem (Template for a Topology)

1. *The union of an arbitrary collection of open sets is open.*
2. *The intersection of a finite collection of open sets is open.*

Proof.

1. Let $\{O_\lambda \mid \lambda \in \Lambda\}$ be a collection of open sets of $\mathbb{R}$, and O its union. If $a \in O$, $a \in O_\lambda$ for some λ. Since O_λ is open, there exists an ϵ-neighborhood $V_\epsilon(a) \subset O_\lambda \subset O$.
2. Let $\{O_1, O_2, \ldots, O_n\}$ be a finite collection of open subsets of $\mathbb{R}$. If $a \in O = \bigcap O_i$, for every open set O_i pick an ϵ_i-neighborhoods $V_{\epsilon_i}(a) \subset O_i$. Choosing $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\}$, we get $V_\epsilon(a) \subset O_i$ for each O_i, and therefore $V_\epsilon(a) \subset O$. □

Limit Point of a Set

Definition

A point x is a **limit** point of a set A if every ϵ-neighborhood $V_\epsilon(x)$ of x intersects A in some point other than x.

Other terminology for **limit** point: **accumulation** point, or **cluster** point. It is important to note that a limit point of A does not have to be a point of A.

Theorem

A point x is a limit point of a set A iff $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.

Proof. By considering a values $\epsilon = 1/n$, to a limit point x of A, we select $a_n \in A \cap V_{1/n}(x)$, $a_n \neq x$. Note that $a_n \in V_{1/N}(x)$, for $n \geq N$. This means $(a_n) \to x$. The converse is clear. $\square$

Isolated Point

Definition

A point $x \in A$ is an **isolated** point of A if it is not a limit point of A.

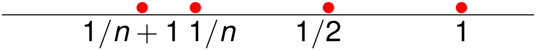
This essentially means that we have an ϵ-neighborhood $V_\epsilon(x)$ that contains no other point of A.
For example, let

$$A = \{1/n \mid n \in \mathbb{N}\}$$

The sequence of points of A, $(1/n) \to 0$, so 0 is a limit point.

Any point of A is isolated: For example, if $x = 1/3$, the closest other point in A is $1/4$, so if we choose $\epsilon < 1/3 - 1/4 = 1/12$, $V_\epsilon(1/3) \cap A = \{1/3\}$.

Example



Let $A = \{1/n \mid n \in \mathbb{N}\}$. Note that the closest point to $1/n$ is $1/(n+1)$: So if $\epsilon < 1/n - 1/(n+1)$

$$V_\epsilon(1/n) \cap A = \{1/n\}$$

Closed Set

Definition

A set $F \subset \mathbb{R}$ is **closed** if it contains (all) its limit points.

In other words, for any convergent sequence $(a_n) \to x$ of distinct points $a_n \in F$, $x \in F$ also.

Closed sets are ubiquotous.

Plenty of Closed Sets

Theorem

Let A be a subset of $\mathbb{R}$. The set L of limit points of A is closed.

Proof.

1. Let x be a limit point of L. To show that $x \in L$ we must show that x is a limit point of A.
2. Let $V_\epsilon(x)$ be a neighborhood of x. It contains some $y \in L$. Pick a (possibly) smaller neighborhood $V_{\epsilon'}(y) \subset V_\epsilon(x)$.
3. Since $y \in L$, $V_{\epsilon'}(y)$ contains some $z \in A$, as desired.

Examples

1. The interval $A = [c, d]$ is a closed set: If x is a limit point of A there is a sequence (x_n) of points of A with $(x_n) \to x$. Applying Order Theorem to
$$c \leq x_n \leq d,$$
we get $c \leq \lim x_n \leq d$, so $x \in A$.
2. Consider the rational numbers: $\mathbb{Q} \subset \mathbb{R}$. The set of limit points of $\mathbb{Q}$ is $\mathbb{R}$: Given any element $y \in \mathbb{R}$, by the Density Theorem there exists a rational number $r \neq y$ in $V_\epsilon(y)$. This can be reformulated as:

Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$)

Given any $y \in \mathbb{R}$, there is a sequence of rational numbers that converges to y.

Closure of a Set

Definition

Given a set $A \subset \mathbb{R}$, let L be the set of all limit points of A. The **closure** of A is the set $\overline{A} = L \cup A$.

1. $\overline{A}$ consists of A plus its accumulation points.
2. If $A = (0, 1)$, its closure $\overline{A}$ is $[0, 1]$.
3. If $A = \{1/n \mid n \in \mathbb{N}\}$, its limit set is $L = \{0\}$, so

$$\overline{A} = A \cup \{0\}.$$

4. $\overline{\mathbb{Q}} = \mathbb{R}$

Closure of a Set

Theorem

For any $A \subset \mathbb{R}$, the closure $\overline{A}$ is a closed set and is the smallest closed set containing A.

Proof.

1. Let x be a limit point of $\overline{A}$, which we assume does not lie in $\overline{A}$. Note that any neighborhood of x must contain an element $x \neq y \in \overline{A}$.
2. We will show that x is a limit point of L, and since we have already proved that L is closed this would imply $x \in L$.
3. Let $V_\epsilon(x)$ be a neighborhood of x. We want to argue that it contains some element of A. If not, it would have to contain an element $y \in L$.
4. Let $V_{\epsilon'}(y) \subset V_\epsilon(x)$. With $y \in L$, $V_{\epsilon'}(y)$ contains an element of A, as desired. □

Closed versus Open

Theorem

A set O is open if and only if its complement O^c is closed. Likewise, a set F is closed if and only if F^c is open.

Proof. Let O be an open subset of $\mathbb{R}$. To show that O^c is closed, we must show that it contains all of its limit points. If x is a limit point of O^c, then every neighborhood of x contains some point of O^c. If $x \notin O^c$, $x \in O$ and since O is open there is a neighborhood of x contained in O. This contradiction shows that $x \in O^c$.

For the converse, assume O^c is closed and we argue that O is open. This means that for every point $x \in O$ there must be a neighborhood $V_\epsilon(x) \subset O$. If not, each such neighborhood would intersect O^c, which is closed. In this case, x would be a limit point of O^c, and thus $x \in O^c$, which is a contradiction.

For the second part, just note that for any subset $E \subset \mathbb{R}$, $(E^c)^c = E$. $\square$

Theorem (Template for a Topology)

1. *The intersection of an arbitrary collection of closed sets is closed.*
2. *The union of a finite collection of closed sets is closed.*

Corollary

The Cantor set C is closed.

Outline

Definition

A set $K \subset \mathbb{R}$ is **compact** if every sequence in K has a subsequence that converges to a limit that is also in K.

Example: A closed interval $[a, b]$. The Bolzano-Weirstrass theorem guarantees that any sequence $(a_n) \subset [a, b]$ admits a convergent subsequence. Because $[a, b]$ is closed, the limit of this subsequence is also in $[a, b]$

Heine-Borel Theorem

Definition

A set $K \subset \mathbb{R}$ is **bounded** if there exists $M > 0$ such that $|x| < M$ for all $x \in K$.

Theorem

A set $K \subset \mathbb{R}$ is compact if and only if it closed and bounded.

Proof. Let K be compact. We first claim K is bounded. Otherwise, for each n there is $x_n \in K$ such that $|x_n| > n$. Since K is compact:

1. (x_n) has a convergent subsequence (x_{n_k}).
2. But convergent sequences are bounded, while $|x_{n_k}| > n_k$, a contradiction as $n_k \to \infty$.

Next we show that K is closed. Let $x = \lim x_n$ be a limit point of K, that is $x_n \in K$. We must show $x \in K$. From the compactness assumption, (x_n) admits a convergent subsequence (x_{n_k}) converging to a point $y \in K$. Since (x_n) is convergent, all of its subsequences have the same limit, so $x = y$ as desired.
The converse is left as an exercise.

Theorem

If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ is a nested sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} K_n$ is nonempty.

Proof. The strategy is simple: We pick an element $x_n \in K_n$ (K_n is nonempty) and consider the sequence (x_n). Since $x_n \in K_1$, and K_1 is compact, it admits a convergent subsequence $(x_{n_k}) \to x \in K_1$.
We claim that $x \in K_n$ for every n. Given n_0, the terms in (x_n) are contained in K_{n_0} as long as $n \geq n_0$. This means that the terms of the subsequence (x_{n_k}) are also in K_{n_0} for almost all of them. This implies that its limit lies in K_{n_0}, as desired. □