Math 311: Advanced Calculus

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Set 2

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Outline

Some Goals

Understand mathematical objects such as

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots = ?$$

$$\prod_{n=0}^{\infty} a_n = a_0 \cdot a_1 \cdot a_2 \cdot a_3 + \cdots = ?$$

The building blocks of these objects are

$$\underbrace{a_1, a_2, a_3, \ldots, a_n, \ldots}$$

Outline

Sequences of real numbers

Definition

A sequence is a function **f** whose domain is $\mathbb{N}$.

It can be represented as

$$\{\mathbf{f}(1), \mathbf{f}(2), \mathbf{f}(3), \ldots\}$$

$$\{\mathbf{f}(0), \mathbf{f}(1), \mathbf{f}(2), \mathbf{f}(3), \ldots\}$$

or

$$\{\mathbf{f}(n), \ldots, \quad n \geq n_0\}$$

We will first examine sequences of real numbers, $\mathbf{f} : \mathbb{N} \to \mathbb{R}$. Later we will study sequences of functions.

It allows us to look at real numbers in a concrete manner: If

$$x = A.a_1 a_2 \cdots a_n \cdots ,$$

where a_i are the decimal digits, we form the sequence of rational numbers

$$\begin{aligned} x_0 &= A \\ x_1 &= A.a_1 \\ x_2 &= A.a_1 a_2 \\ x_n &= A.a_1 a_2 \cdots a_n, \quad \text{and so on} \end{aligned}$$

Examples

We will look for features such as **clustering**

1. $(1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots)$
2. $(c, c, c, c, \ldots)$
3. $(1, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \ldots)$
4. $(\frac{1}{2^n})_{n=1}^{\infty} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$
5. (a_n), $a_1 = 1$, and $a_{n+1} = \frac{a_n}{2} + 1$
6. (a_n), a_n is the nth digit in the decimal expansion of π.
7. (a_n), $a_n = (1 + 1/n)^n$

Why Sequences?

We use sequences to make sense of:

- $\sum_{n\geq 1} a_n$: Series

$$1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 + \cdots$$

  Question: How to handle

$$(a_0 + a_1 + \cdots + a_n + \cdots)(b_0 + b_1 + \cdots + b_n + \cdots)$$

- $\sum_{m,n\geq 1} a_{m,n}$: Double [multiple] Series

$$\sum_{m,n} \frac{1}{m^2+n^2}$$

- $\prod_{n\geq 1} a_n$: Infinite Products

$$\prod_p \left(\frac{1}{1-p}\right), \quad p \text{ prime number}$$

Convergence of a Sequence

Sequences are wonderful ways to represent data, but we are mostly interested is one of its aspects:

Definition

A sequence (a_n) converges to a real number a if, for every positive real number ϵ, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

One notation: $\lim a_n = a$, or $(a_n) \to a$. To understand this we introduce the notion of a **neighborhood** of a real number a.

Example

Consider the sequence (a_n), $a_n = \frac{n+1}{n}$. It is natural to expect that $\lim a_n = 1$. Let us follow the template:

- Given $\epsilon > 0$, to determine N we solve

$$\left| \frac{n+1}{n} - 1 \right| < \epsilon$$

- That is

$$\left| \frac{1}{n} \right| < \epsilon \quad \Rightarrow \quad n > \frac{1}{\epsilon}$$

- Thus if $\epsilon = 1/100$, $N = 101$ will work.

Neighborhoods

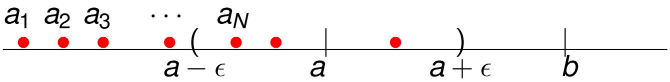
Definition

Given a real number $a \in \mathbb{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbb{R} \colon |x - a| < \epsilon\}$$

is called the ϵ-**neighborhood** of a.

Limit and Neighborhoods



a is the limit of (a_n) if once a_N **enters** the neigbhorhood $V_\epsilon(a)$, all a_n that follow will stay in it. That is, the a_n cluster around a in a very specific manner.

Note that this implies that if (a_n) converges, its limit is unique: the a_n cannot be in both $V_\epsilon(a)$ and $V_\epsilon(b)$ if $\epsilon < 1/2|a-b|$.

Exercise

Let $a_n = \frac{2n^2+n+1}{n^2}$. It can be written as

$$a_n = 2 + \frac{1}{n} + \frac{1}{n^2}$$

It is now easy to see that $\lim a_n = 2$: Just notice that

$$|a_n - 2| = \frac{1}{n} + \frac{1}{n^2} \leq 2\frac{1}{n}$$

and we can use the argument of the previous Example to finish.
Exercise: For every real number $x \in \mathbb{R}$, there exists a sequence (a_n) of rational numbers such that $(a_n) \to x$.

Limit Template

Let us summarize the procedure to compute the limit of a sequence:

$(a_n) \to a$ involves all the following steps:

1. Let $\epsilon > 0$ be arbitrary
2. Demonstrate a choice for $N \in \mathbb{N}$: hard work here often
3. Assume $n \geq N$
4. Check that

$$|a - a_n| < \epsilon$$

Example

Define the sequence

$$a_1 = \sqrt{2}, \quad a_2 = \sqrt{2\sqrt{2}}, \quad a_3 = \sqrt{2\sqrt{2\sqrt{2}}}, \cdots$$

Question: $(a_n) \to ?$ Note

$$a_1 = \sqrt{2}, \quad a_2 = a_1 \sqrt[4]{2}, \quad a_3 = a_2 \sqrt[8]{2}, \cdots$$

$$a_n = 2^{1/2+1/4+\cdots+1/2^n} < 2$$

So this sequence is bounded [and increasing]. Show that its least upper bound is 2.

Infinity as the limit of a sequence

If a sequence (a_n) is not **convergent**, we say that it is **divergent**. We also use the following terminology for some divergent sequences:

Definition

The sequence (a_n) converges to ∞, $\lim a_n = \infty$, if given any positive number b, there is an $N \in \mathbb{N}$ such that $a_n \geq b$ for $n \geq N$.

Example: $\{1, 2, 3, \ldots, n, \ldots\}$

Some sequences don't make up their minds:

1. $1, -1, 1, \ldots, \pm 1, \ldots$
2. one gets a very complicated sequence by glueing two unrelated sequences (a_n), (b_n), as in

$$a_0, b_0, a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots,$$

Boundedness of Convergent Sequences

Definition

A sequence (a_n) is bounded if there exists a number $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem

Every convergent sequence is bounded.

Proof. Suppose $(a_n) \to \ell$. For $\epsilon = 1$ let $N \in \mathbb{N}$ be such that $|a_n - \ell| < 1$ for $n \geq N$.
We claim that $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |\ell| + 1\}$ satisfies

$$|a_n| \leq M$$

Converse?

The sequence $(1, -1, \ldots, (-1)^n, \ldots)$ is bounded but not convergent.

Many sequences are put together from two or more sequences: Say start with

$$\{a_1, a_2, a_3, \ldots\} \quad \{b_1, b_2, b_3, \ldots\}$$

$$\{a_1, b_1, a_2, b_2, a_3, b_3, \ldots\}$$

Outline

Algebraic Limit Theorem

Theorem

Let $\lim a_n = a$ *and* $\lim b_n = b$. *Then*

(i) $\lim ca_n = ca$, *for all* $c \in \mathbb{R}$;

(ii) $\lim(a_n + b_n) = a + b$;

(iii) $\lim(a_n b_n) = ab$;

(iv) $\lim(a_n/b_n) = a/b$ *provided* $b_n \neq 0$ *and* $b \neq 0$.

Note an important consequence: Since we can view real numbers as limits of rational numbers, we can carry out the desired field operations

$$\begin{aligned} x &= X.x_1x_2\ldots x_n|\ldots \\ y &= Y.y_1y_2\ldots y_n|\ldots \end{aligned}$$

Proof. (i) [If $\lim a_n = a$, then $\lim ca_n = ca$] Consider the case $c \neq 0$. To prove $(ca_n) \to ca$, we use the proof template. Let $\epsilon > 0$. We want to argue that $|ca_n - ca| < \epsilon$ from some term of the sequence (ca_n) on. Since $(a_n) \to a$, given $\epsilon/|c|$, there is $N \in \mathbb{N}$ such that for $n \geq N$
$|a_n - a| < \epsilon/|c|$.
This leads to

$$|ca_n - ca| = |c||a_n - a| < \epsilon, \quad n \geq N,$$

as desired. This proves (i) for $c \neq 0$. The case $c = 0$ is trivial.

(ii) [If $\lim a_n = a$, $\lim b_n = b$, then $\lim(a_n + b_n) = a + b$] Given $\epsilon > 0$, pick N_1 and N_2 so that

$$|a_n - a| < \epsilon/2, \quad \& \quad |b_n - b| < \epsilon/2$$

for $n \geq N_1$ and $n \geq N_2$, respectively. Thus $n \geq N = \max\{N_1, N_2\}$

$$\begin{aligned} |(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| \\ &\leq \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

(iii) [If $\lim a_n = a$, $\lim b_n = b$, then $\lim a_n b_n = ab$] If $\lim a_n = a$, $\lim b_n = b$, we know that $|a_n|$ and $|b_n|$ are bounded, that is $|a_n| < M_1$ and $|b_n| < M_2$ for all n. Let $M = \max\{M_1, M_2\}$. Given $\epsilon > 0$, pick N_1 and N_2 so that

$$|a_n - a| < \epsilon/2M, \quad \& \quad |b_n - b| < \epsilon/2M$$

for $n \geq N_1$ and $n \geq N_2$, respectively.
This leads to: for all $n \geq N = \max\{N_1, N_2\}$

$$\begin{aligned} |a_n b_n - ab| &= |(a_n b_n - a_n b) + (a_n b - ab)| \\ &\leq |(a_n b_n - a_n b)| + |(a_n b - ab)| \\ &= |a_n||b_n - b| + |b||a_n - a| \leq M_1|b_n - b| + M_2|a_n - a| \\ &\leq \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

which completes the proof.

(iv) [If $\lim a_n = a$, $\lim b_n = b$, $b_n, b \neq 0$, then $\lim a_n/b_n = a/b$]. In the case of a_n/b_n, we are going to apply the product rule to the product $a_n \frac{1}{b_n}$. This requires

Lemma

If the sequence $(b_n) \to b$ and $b_n, b \neq 0$, then $(\frac{1}{b_n}) \to \frac{1}{b}$.

Proof. Let $\epsilon_0 = |b|/2$. Pick N_1 large enough so that for $n \geq N_1$ $|b_n - b| < \epsilon_0 = |b|/2$. This shows that in this range $|b_n| > |b|/2$. Next, given $\epsilon > 0$, choose N_2 so that for $n \geq N_2$

$$|b_n - b| < \frac{\epsilon b^2}{2}$$

Finally, if we let $N = \max\{N_1, N_2\}$,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{bb_n}\right| \leq \frac{\epsilon b^2}{2} \frac{1}{|b||b|/2} = \epsilon$$

We examine in detail this important sequence. Two cases are easy: $x = 1$, when the sequence is constant (so $\lim x^n = 1$), and $x = -1$ (when it alternates between 1 and -1) when it does not converge. Let us next examine the case $|x| < 1$, that is $-1 < x < 1$. We make a series of technical observations.

A useful limit calculation

Lemma

For any $p > -1$ and all $n \in \mathbb{N}$, $(1+p)^n \geq 1 + pn$.

Proof. We prove this by induction. It is true for $n = 1$. Now consider

$$\begin{aligned}(1+p)^{n+1} &= (1+p)^n(1+p) \geq (1+pn)(1+p) \\ &= 1 + p(n+1) + p^2 n \geq 1 + p(n+1).\end{aligned}$$

- Back to our limit. If $|x| < 1$, $\frac{1}{|x|} = 1 + p$, $p > 0$ and thus

$$\frac{1}{|x^n|} = (1+p)^n \geq 1 + pn > pn$$

- Therefore

$$|x^n| < \frac{1}{pn}$$

- Which shows that for $|x| < 1$ $\lim |x^n| = 0$ and $\lim x^n = 0$ as well.
- The case $|x| > 1$. Apply the algebraic limit theorem: By the case above, $\lim \frac{1}{x^n} = 0$, which shows (x^n) does not converge.

Limits and Order

Theorem (Order Limit Theorem)

Assume $\lim a_n = a$ *and* $\lim b_n = b$. *Then*

1. *If* $a_n \geq 0$ *for all* $n \in \mathbb{N}$, *then* $a \geq 0$.
2. *If* $a_n \leq b_n$ *for all* $n \in \mathbb{N}$, *then* $a \leq b$.
3. *If there exists* $c \in \mathbb{R}$ *for which* $c \leq b_n$ *for all* $n \in \mathbb{N}$, *then* $c \leq b$. *Similarly, if* $a_n \leq c$ *for all* $n \in \mathbb{N}$, *then* $a \leq c$.

Proof. (i) Assume, by way of contradiction, that $a < 0$. Let us show that this produces some $a_n < 0$. Let $\epsilon = |a|$. There exists N such that

$$|a_n - a| < \epsilon, \quad n \geq N$$

If $a_n \geq 0$ for $n \geq N$,

$$|a_n - a| = |a_n + (-a)| = a_n + |a| \geq \epsilon,$$

a contradiction.

(ii) The Algebraic Limit Theorem guarantees that the sequence $(b_n - a_n)$ converges to $b - a$. Because $b_n - a_n \geq 0$, by Part (i), $b \geq a$.

(iii) Take $a_n = c$ (or $b_n = c$) for all $n \in \mathbb{N}$ and apply (ii). $\square$

Examples

- The constant sequence $(c, c, c, \ldots)$ converges to c:

  $x_n = c$ for all n, so for $\epsilon > 0$, $|x_n - c| = 0 < \epsilon$
- Let $x_n \geq 0$ for all $n \in \mathbb{N}$.
  1. If $(x_n) \to 0$, show that $(\sqrt{x_n}) \to 0$: Given $\epsilon > 0$ we can find N such that $|x_n| < \epsilon^2$ for $n \geq N$. It follows that $|\sqrt{x_n}| < \epsilon$ for $n \geq N$.
  2. If $(x_n) \to x$, show that $(\sqrt{x_n}) \to \sqrt{x}$: We already know that $x \geq 0$ and that the sequence is bounded, that is $L < x_n < U$.In particular $\sqrt{x_n} \geq \sqrt{L}$ and $x \geq \sqrt{L}$. Given $\epsilon > 0$ pick N so that $|x_n - x| < \epsilon 2\sqrt{L}$ for $n \geq N$. Then

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x}| &\leq |\sqrt{x_n} - \sqrt{x}| \frac{|\sqrt{x_n} + \sqrt{x}|}{2\sqrt{L}} \\ &= \frac{|x_n - x|}{2\sqrt{L}} < \epsilon \end{aligned}$$

Exercises

1. (i) Show that if $(b_n) \to b$, then the sequence $(|b_n|)$ converges to $|b|$.(ii) Converse?
2. Let (a_n) be a bounded (not necessarilly convergent) sequence, and assume $(b_n) \to 0$. Show that $(a_n b_n) \to 0$. Why we are not allowed to use the Algebraic Limit theorem?
3. Exercises 32(a,c,e) in page 56 of Textbook.

Outline

Monotone Sequences

Definition

A sequence (a_n) is **increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$, and **decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is **monotone** if it is either increasing or decreasing.

Theorem (Monotone Convergence Theorem)

If the sequence (a_n) is monotone and bounded, then it converges.

Proof. The assumption is that there is a B such that $a_n \leq B$ for all $n \in \mathbb{N}$. We are going to 'build' $\lim a_n$. For that we are going to use the decimal representation of the a_n.

Visual Proof

$$\begin{aligned}
a_1 &= A_1.a_{11}a_{12}a_{13}a_{14}\cdots \\
a_2 &= A_2.a_{21}a_{22}a_{23}a_{24}\cdots \\
a_3 &= A_3.a_{31}a_{32}a_{33}a_{34}\cdots \\
\vdots &\quad \vdots \\
a_N &= A_N.a_{N1}a_{N2}a_{N3}a_{N4}\cdots \\
\vdots &\quad \vdots \\
a_n &= A_n.a_{n1}a_{n2}a_{n3}a_{n4}\cdots
\end{aligned}$$

Since the a_n are bounded, its integral parts A_n are also bounded and non-increasing. Thus, there is an N such that $A_n = A_N$ for all $n \geq N$.

Let us scan the first decimal digits from a_N on:

$$\begin{aligned} a_{N1} &= A_N.a_{N1}a_{N2}a_{N3}a_{N4}\cdots \\ \vdots & \quad \vdots \\ a_n &= A_n.a_{n1}a_{n2}a_{n3}a_{n4}\cdots \end{aligned}$$

Since $A_n = A_N$, and a_n are increasing, the digits a_{n1} must be increasing so once it hits its maximal value, say at $n = N_1$, it must stay there, i.e. $a_{n1} = a_{N_1 1}$ for $n \geq N_1$.

We move over the second decimal place, and so on. In this manner we build the element $a = A_N.b_1 b_2 b_3 b_4 \ldots$ with the property $|a - a_n| < 10^{-N_r}$ for $n \geq N_{r+1}$. This shows that $a = \lim a_n$. Note that a is the least upper bound of the set $\{a_n\}$.

'Abstract' Proof

Let (a_n) be a **bounded monotone increasing sequence**,

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots \leq B$$

Because the set of terms $\{a_n, n \geq 1\}$ is bounded, by the **Axiom of Completeness** the set has a **least upper bound** B_0. Now we verify that $a_n \to B_0$. We use the limit template:

- Given $\epsilon > 0$, $B_0 - \epsilon$ is not an upper bound so there is N such that $a_N > B_0 - \epsilon$. Since a_n is increasing, we have

$$B_0 \geq a_n \geq a_N > B_0 - \epsilon, \quad n \geq N.$$

- This means that $|a_n - B_0| < \epsilon$ for $n \geq N$, thus proving that $\lim a_n = B_0$.

Example

A sequence we met already was (x_n), where $x_1 = 1$ and

$$x_{n+1} = \frac{x_n}{2} + 1$$

We proved that $x_n < x_{n+1} < 2$, so this is a monotone bounded sequence. Let $a = \lim x_n$. If we delete x_1, we obtain the sequence $(x_{n+1}, n \geq 1)$ which obviously is monotone, and has the same limit. Thus

$$\lim x_{n+1} = a = \frac{\lim x_n}{2} + 1 = \frac{a}{2} + 1$$

and therefore

$$a = 2$$

Calculating Square Roots

Let $x_1 = 2$, and define

$$x_{n+1} = 1/2\left(x_n + \frac{2}{x_n}\right)$$

- Show that $x_n^2 \geq 2$, and then prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.
  We use induction. Squaring we have $x_{n+1}^2 = 1/4(x_n^2 + 4 + 4/x_n^2)$. To show that $x_{n+1}^2 > 2$, it suffices to show that if $x_n^2 > 2$, then $x_n^2 + 4/x_n^2 > 4$. But

  $$x_n^2 + 4/x_n^2 - 4 = (x_n - \frac{2}{x_n})^2 > 0$$

  Note also $x_n - x_{n+1} = 1/2(x_n - 2/x_n) > 0$, since $x_n^2 > 2$. Thus the sequence (x_n) is bounded and decreasing. Its limit a satisfies $a = 1/2(a + 2/a)$, i.e. $a = \sqrt{2}$.

- Modify the sequence so that it converges to $\sqrt{c}$:

$$x_{n+1} = 1/2\left(x_n + \frac{c}{x_n}\right)$$

We again check that the sequence (x_n) is monotone and bounded. When solving for the limit, we get $a = 1/2(a + c/a)$, i.e. $a = \sqrt{c}$.

- Many other equations $\mathbf{f}(x) = 0$ can be set up as

$$x = \frac{\mathbf{g}(x)}{\mathbf{h}(x)}$$

which we turn into a dynamical scheme

$$x_{n+1} = \frac{\mathbf{g}(x_n)}{\mathbf{h}(x_n)}$$

If (x_n) is monotone and bounded, the limit is a root.

Outline

Subsequences

Definition

Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < \cdots$ be an increasing sequence of natural numbers. Then the sequence

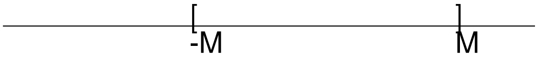
$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \ldots$$

is called a **subsequence** of (a_n) and is denoted by (a_{n_j}), where $j \in \mathbb{N}$ indexes the subsequence.

Theorem

Subsequences of a convergent sequence converge to the same limit as the original sequence.

About bounded sequences: Bolzano-Weierstrass



Theorem (Bolzano-Weierstrass Theorem)

Every bounded sequence (a_n) contains a convergent subsequence.

Proof. The assumption is that all a_n lie in some closed interval $I_1 = [-M, M]$. (Note that we allow repetitions.) Since the sequence is infinite, an infinite subset of terms lies in either $[-M, 0]$ or in $[0, M]$. We pick one of the subintervals with an infinite number of terms and call it I_2.

We continue the process: bisect I_2 pick I_3 one of its two halfs that contain an infinite number of terms. In this manner we get a decreasing sequence of closed intervals

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

If in each subset I_k we pick an element a_{n_k} of the sequence in it, we obtain a subsequence

$$\{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}$$

We claim this (sub)sequence converges.

By he Nested Interval Property there exists at least one point $x \in \mathbb{R}$ contained in every I_k.
We claim $(a_{n_k}) \to x$. Note that the length of I_k is $M\frac{1}{2^{k-1}}$, which converges to 0 (discussed in Workshop #3).
Choose N so that $k \geq N$ implies that the length of I_k is less than ϵ.
Because x and a_{n_k} are both in i_k, $|x - a_{n_k}| < \epsilon$. $\square$

Exercise

Let (a_n) be a bounded sequence, and define the set

$$S = \{x \in \mathbb{R} \mid x < a_n \quad \text{for infinitely many } a_n\}$$

Show that there exists a subsequence (a_{n_k}) converging to $s = \sup S$. (This is a direct proof of the BW Theorem using AoC.)

Examples

Give an example of each of the following, or argue that such a request is impossible.

1. A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.
2. A monotone sequence that diverges but has a convergent subsequence.
3. A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, \ldots\}$.
4. An unbounded sequence with a convergent subsequence.
5. A sequence that has a subsequence that is bounded but contains no subsequence that converges.

Outline

Warming Up ...

Thus far we have two basic results about convergence of sequences:

Theorem (Monotone Convergence Theorem)

If the sequence (a_n) is monotone and bounded, then it converges.

Essentially, if

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq B,$$

then $a_n \rightarrow B_0$, **least upper bound** of the a_n

Theorem (Bolzano-Weierstrass Theorem)

Every bounded sequence (a_n) contains a convergent subsequence.

Essentially, if the sequence (a_n) is bounded, that is there is $M > 0$ such that $-M \leq a_n \leq M$ for all n, then there is a subsequence

$$a_{n_1}, a_{n_2}, a_{n_3}, \ldots$$

that is convergent.

The notion of convergence of a sequence that we are using is:

Definition (Convergence of a Sequence)

A sequence (a_n) converges to the real number a if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

$$\lim a_n = a \text{ if} \rightarrow \text{given } \epsilon > 0 \rightarrow \text{find } N \rightarrow \text{for } n \geq N \rightarrow |a_n - a| < \epsilon$$

Cèsaro Means

There are other ways of defining **convergence** of sequences. Today we study a powerful notion, but first we do warm ups.

Let (a_n) be a sequence and define the sequence of its **means**,

$$c_n = \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad n \geq 1$$

thus forming the sequence (c_n) of averages. For example, the sequence $(1, 0, 1, 0, 1, 0, \ldots)$ has sequence of means

$$(1, 1/2, 2/3, 1/2, 3/5, 1/2, 5/7, \ldots, 1/2, (n+2)/(2n+1), \ldots) \to 1/2$$

Theorem (Cèsaro Means)

If $(a_n) \to a$, then $(c_n) \to a$ also.

Proof.

- Given $\epsilon > 0$ we will find N such that $|c_n - a| < \epsilon$ for $n \geq N$.Since $(a_n) \to a$, we know that (a_n) is bounded, say $|a_n| < M$ for some M, and for $\epsilon' = \epsilon/2$ there is N_0 such that

$$|a_n - a| < \epsilon' \quad n \geq N_0$$

- Now consider $|c_n - a|$

$$\begin{aligned} |c_n - a| &= \left| \frac{a_1 + \cdots + a_n}{n} - a \right| = \left| \frac{(a_1 - a) + \cdots + (a_n - a)}{n} \right| \\ &\leq \frac{|a_1 - a| + \cdots + |a_n - a|}{n} \end{aligned}$$

We are going to split the numerator of

$$\frac{|a_1 - a| + \cdots + |a_n - a|}{n}$$

into two summands, up to N_0 and from there to n: Note that $|a_n - a| \leq |a_n| + |a| \leq 2M$ by the triangle inequality. Choosing

$$N = \max\{N_0, 4N_0M/\epsilon\}$$

$$\frac{2N_0M}{n} + \frac{(n - N_0)\epsilon/2}{n} \leq \epsilon/2 + \epsilon/2 = \epsilon$$

for $n \geq N$, as desired. □

Cauchy Sequence

Definition

A sequence (a_n) is called a **Cauchy sequence** if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \epsilon$.

Compare to the standard definition of convergence:

Definition (Convergence of a Sequence)

A sequence (a_n) converges to the real number a if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

Comment on the differences!

Exercise

Prove that $a_n = \frac{2n+1}{n}$ is Cauchy

1. We estimate $|a_n - a_m|$: For $n < m$

$$|\frac{2n+1}{n} - \frac{2m+1}{m}| = |\frac{1}{n} - \frac{1}{m}| = |\frac{m-n}{mn}|$$

2. Note that $|\frac{m-n}{mn}| \leq \frac{1}{n}$.
3. If $\epsilon > 0$ and N is chosen so that $\epsilon > \frac{1}{N}$, we have

$$|a_n - a_m| < \epsilon, \quad n, m \geq N$$

More Interesting Example

Let a sequence be defined as follows: $x_1 = 1$, $x_2 = 2$, $x_3 = 1/2(x_1 + x_2)$ and in general $x_{n+1} = 1/2(x_{n-1} + x_n)$. Show that

$$|x_n - x_m| \leq \frac{1}{2^{N-1}}, \quad \forall n, m \geq N,$$

so Cauchy's condition is fulfilled.

Hint: Note that each term is midway between the two preceding ones.

Theorem

Every convergent sequence is a Cauchy sequence.

Proof. Assume (x_n) converges to x. To prove (x_n) is Cauchy, we must find N such that $|x_n - x_m| < \epsilon$ for $n, m \geq N$. This is easily done: given $\epsilon/2$ find N such that

$$|x - x_n| < \epsilon/2, \quad n \geq N.$$

By the triangle inequality,

$$|x_n - x_m| \leq |x_n - x| + |x - x_m| \leq \epsilon/2 + \epsilon/2 = \epsilon, \quad n, m \geq N.$$

Cauchy Criterion

Theorem

A sequence converges if and only if it is a Cauchy sequence.

While the definition of convergence requires a candidate for the limit, Cauchy's Criterion is a softer requirement. [**Discuss**]

Proof. The preceding theorem showed that every convergent sequence is a Cauchy sequence. To prove the converse, we first show that every Cauchy sequence is bounded, apply Bolzano- Weierstrass, and then complete proof.

Boundedness of Cauchy sequences

Lemma

Cauchy sequences are bounded.

Proof. Given $\epsilon = 1$, there exists an N such that $|x_n - x_m| < 1$ for all $m, n \geq N$. Thus, making $m = N$, we must have $|x_n| \leq |x_N| + 1$ for all $n \geq N$. It follows that

$$M = \max\{|x_1|, |x_2|, |x_3|, \ldots, |x_{N-1}|, |x_N| + 1\}$$

is a bound for (x_n).

Cauchy Criterion

Theorem

A sequence converges if and only if it is a Cauchy sequence.

Proof. By the Bolzano-Weierstrass theorem, since (x_n) is bounded, it has a convergent subsequence (x_{n_k}) of limit, say, x. We want to argue that x is the limit of (x_n) also.

Let $\epsilon > 0$. Because (x_n) is Cauchy, there exists N such that

$$|x_n - x_m| < \epsilon/2, \quad m, n \geq N.$$

Because $(x_{n_k}) \to x$, choose a term x_{N_K}, with $N_K \geq N$ such that

$$|x_{N_K} - x| < \epsilon/2.$$

Now observe: If $n \geq N_K$,

$$\begin{aligned} |x_n - x| &= |x_n - x_{N_K} + x_{N_K} - x| \\ &\leq |x_n - x_{N_K}| + |x_{N_K} - x| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

This shows that $(x_n) \to x$ $\square$

Outline

Workshop #3

Review the following concepts/techniques:

- Algebraic and order limit theorems
- Your favorite limit tricks [see two slides down for one useful tool]

Warmups

This uses only the cute lemma and some of the algebraic limits theorems.

1. Let $a_n = q^n$. If $q > 1$, prove that $\lim a_n = \infty$: Set $q = 1 + p$, $p > 0$. By the Lemma, $(1 + p)^n \geq 1 + np$, which clearly converges to ∞.

2. Let $a_n = q^n$. If $0 < q < 1$, prove that $\lim a_n = 0$. [*Hint:* work with $1/q$.] This means $(1/q)^n \to \infty$, hence $q^n \to 0$.

Workshop #3: there is a second page

1. If $q > 0$, show that $\lim \sqrt[n]{q} = 1$. [*Hint:* Use the technique above. First assume $q > 1$. Then set $\sqrt[n]{q} = 1 + p_n$, $p_n > 0$. Now $q = (1 + p_n)^n \geq 1 + np_n$. In case $0 < q < 1$, use $\frac{1}{\sqrt[n]{q}}$.]
2. Show that $\lim \sqrt[n]{n} = 1$. [*Hint:* Work with $\sqrt[n]{\sqrt{n}} = 1 + k_n$.]Explain why setting $\sqrt[n]{n} = 1 + a_n$ will not work.
3. Find the limit of $\sqrt[n]{a^n b^n + b^n c^n + a^n c^n}$ if $a > b > c > 0$.
4. Find the limit of $\sqrt{n^2 + an + b} - n$.

5 Give an example or argue request is impossible.

(i) A Cauchy sequence that is not monotone.

(ii) A monotone sequence that is not Cauchy.

(iii) A Cauchy sequence with a divergent subsequence.

(iv) An unbounded sequence containing a subsequence that is Cauchy.

The following lemma discussed in class is helpful.

Lemma

If $p > -1$, $(1+p)^n \geq 1 + pn$ for all $n \in \mathbb{N}$.

Proof. We prove this by induction.

- Base Case: It is true for $n = 1$.
- Induction Step: Now consider

$$\begin{aligned} (1+p)^{n+1} &= (1+p)^n(1+p) \geq (1+pn)(1+p) \\ &= 1 + p(n+1) + p^2 n \geq 1 + p(n+1). \end{aligned}$$

Comment on a Limit

In the Workshop #3 Problem like

$$\lim \sqrt[n]{a^n + b^n + c^n}, \quad a > b > c > 0$$

can [?] be argued as follows

$$\begin{aligned} \lim \sqrt[n]{a^n + b^n + c^n} &= \lim a \sqrt[n]{1 + (b/a)^n + (c/a)^n} \\ &= a \lim \sqrt[n]{1 + (b/a)^n + (c/a)^n} \end{aligned}$$

which is fine but then argued wrongly [why?]

$$\begin{aligned} \lim \sqrt[n]{1 + (b/a)^n + (c/a)^n} &= \sqrt[n]{1 + \lim(b/a)^n + \lim(c/a)^n} \\ &= \sqrt[n]{1 + 0 + 0} = 1 \end{aligned}$$

One of the proper ways to argue

$$a = \sqrt[n]{a^n} \leq \sqrt[n]{a^n + b^n + c^n} \leq \sqrt[n]{3a^n} = a\sqrt[n]{3}$$

and then use Problem #4 that shows

$$\lim \sqrt[n]{3} = 1$$

$\lim(1+1/n)^n$

$$
\begin{aligned}
(1+\frac{1}{n})^n &= 1 + n\frac{1}{n} + \frac{n(n-1)}{1\cdot 2}1/n^2 + \cdots + \frac{n(n-1)\ldots(n-n+1)}{1\cdots n}1/n^n \\
&= 1 + 1 + \frac{1}{1\cdot 2}(1-\frac{1}{n}) + \frac{1}{1\cdot 2\cdot}(1-\frac{1}{n})(1-\frac{2}{n}) + \cdots \\
&+ \frac{1}{1\cdot 2\cdots n}(1-\frac{1}{n})\cdots(1-\frac{n-1}{n})
\end{aligned}
$$

Note that

$$\frac{1}{1\cdot 2\cdots n}(1-\frac{1}{n})\cdots(1-\frac{n-1}{n}) < \frac{1}{n!}$$

This shows that

$$2 < (1+\frac{1}{n})^n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots < 3$$

Outline

Intro to Infinite Series

Question: What do we see in the Infinite Series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots = ?$$

Answer: At least two things

- The sequence of terms, (a_n) and
- The sequence of partial sums, (s_n),

$$s_n = a_0 + a_1 + \cdots + a_n$$

- We say the **series converges** to $S \in \mathbb{R}$ if $\lim s_n = S$. By abuse of notation, we then replace the ? by S.

Backbone Examples

The perspective we use is to view a series as the **pair** of related sequences:

$$a_n, \quad s_n = a_0 + a_1 + \cdots + a_n$$

with emphasis on the question:

What should the sequence (a_n) be like so that the sequence of partial sums (s_n) converges?

We need to look close at some important series.

The Geometric Series

For $q \in \mathbb{R}$, the **geometric series of ratio** q is

$$1 + q + q^2 + q^3 + \cdots + q^n + \cdots$$

Sometimes, all terms are multiplied by a same constant, that instead of the sequence of terms (q^n), one has (aq^n). Let us examine when it converges and find the corresponding limit.

- We need an expression for the partial sum $s_n = 1 + q + \cdots + q^n$.
- If we multiply s_n by q and subtract s_n we get

$$\begin{aligned} qs_n - s_n &= q(1 + q + \cdots + q^n) - (1 + q + \cdots + q^n) \\ &= q^{n+1} - 1 \end{aligned}$$

- We get an explicit expression for s_n

$$s_n = \frac{1}{1-q} - \frac{q^{n+1}}{1-q}$$

- According to the value of q, we conclude: If $|q| < 1$, since $q^n \to 0$,

$$1 + q + q^2 + q^3 + \cdots + q^n + \cdots = \frac{1}{1-q}$$

- Otherwise the series diverges. If $q \geq 1$, it converges to infinity. [Note the confusing language.]

The Harmonic Series

This is the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

This series diverges: It suffices to organize its partial sums in groups that add to at least $1/2$:

$$\begin{aligned} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots &= 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + \cdots \\ &\geq 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + \cdots \\ &= 1 + 1/2 + 1/2 + 1/2 + \cdots \end{aligned}$$

Zeta Function

The series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots ,$$

for $p > 1$ will always converge. Its sum is denoted by $\zeta(p)$.

For example, $\zeta(2) = \frac{\pi^2}{6}$.
This function is actually defined for all complex numbers p whose real part is > 1. It is known as **Riemann zeta function**.It is probably the most famous function of Mathematics.

Let us show that

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots ,$$

for $p > 1$ will always converge.

We are going to bound each term $1/n^p$ by the terms of another series, and then argue the new series converges.

Consider the function $\mathbf{f}(x) = 1/x^p$, $x \geq 2$. This is a decreasing function (draw the graph).
Observe

$$1/n^p \leq \int_{x=n-1}^{n} 1/x^p dx$$

Therefore its partial sums are bounded by

$$s_n \leq 1 + \int_{x=1}^{n} \frac{dx}{x^p} = 1 + \frac{1}{p-1}\left[1 - \frac{1}{n^{p-1}}\right] < 1 + \frac{1}{p-1}$$

Alternating the Harmonic Series

This is the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1}\frac{1}{n} + \cdots$$

- Its even partial sums, $s_0 = 1$, $s_2 = 1 - 1/2 + 1/3$, ... are decreasing
- Its odd partial sums, $s_1 = 1 - 1/2$, $s_3 = 1 - 1/2 + 1/3 - 1/4$, ... are increasing
- The nested intervals $[s_1, s_0] \supset [s_3, s_2] \supset [s_5, s_4] \supset \cdots$ will define the limit 0.69... [actually $\ln 2$]

Exponential Series

We claim that the series

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

convergent.

Note that the sequence of its partial sums is monotone but it is bounded by the partial sums of a geometric series

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$

a series that converges to 3. We can refine the comparison.

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{12!} = 2.71828183$$

with error

$$
\begin{aligned}
\frac{1}{13!}+\frac{1}{14!}+\cdots &< \frac{1}{13!}(1+\frac{1}{13}+\frac{1}{13^2}+\cdots \\
&= \frac{1}{13!}\frac{1}{1-\frac{1}{13}}=\frac{1}{12}\frac{1}{12!}
\end{aligned}
$$

a number that does not affect the 8th decimal place.
The limit of this famous series is denoted *e*, after Euler.

Irrationality of e

We claim that the series

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

is not a rational number. We already know that $2 < e < 3$, in particular e is not an integer. Suppose $e = \frac{p}{q}$, with $q \geq 2$ since e is not an integer. Multiplying the equality by $q!$, we have

$$\begin{aligned} eq! &= p(q-1)! = \left[q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{q!}\right] \\ &+ \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \cdots \end{aligned}$$

Note that $p(q-1)!$ and

$$\left[q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{q!}\right]$$

are integers, so that its difference

$$\frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \cdots$$

must also be an integer. But this series is smaller than the geometric series

$$\frac{1}{q+1} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \cdots$$

whose sum is

$$\frac{1}{q+1}\frac{1}{1-\frac{1}{q+1}} = \frac{1}{q} < 1$$

Exercises

Is the series

$$(1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots$$

convergent or divergent? Justify answer.
Is the series

$$\frac{1^1}{(101)!} + \frac{2^2}{(100+2)!} + \cdots + \frac{n^n}{(100+n)!} + \cdots$$

convergent or divergent? Justify answer.

Outline

Convergence of Series

Given the series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots \quad ?$$

there are two sequences associated to it

- The sequence of **terms**, (a_n) and
- The sequence of **partial sums**, (s_n),

$$s_n = a_0 + a_1 + \cdots + a_n$$

- We say the **series converges** to $A \in \mathbb{R}$ if $\lim s_n = A$.We write this as

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots = A$$

A cautionary tale

We pick the alternating harmonic series–which we know to be convergent–and carry out arithmetic operations: See what happens

$$\begin{aligned} S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \\ \frac{1}{2}S &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots \\ S + \frac{1}{2}S &= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \cdots \end{aligned}$$

Thus $S + \frac{1}{2}S = \frac{3}{2}S$ is just a rearrangement of S! The arithmetic is saying instead that

$$\frac{3}{2}S = S!$$

Algebraic Limit Theorem for Series

Theorem

If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then:

1. $\sum_{k=1}^{\infty} ca_k = cA$ *for all* $c \in \mathbb{R}$ *and*
2. $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$.

Proof. (i) To show $\sum_{k=1}^{\infty} ca_k = cA$, we consider the sequence of partial sums

$$t_n = ca_1 + ca_2 + \cdots + ca_n.$$

Since $\sum_{k=1}^{\infty} a_k = A$, its sequence of partial sums

$$s_n = a_1 + a_2 + \cdots + a_n$$

converges to A. By the Algebraic Limit Theorem for Sequences, $\lim t_n = c \lim s_n = cA$.

(ii) To show that $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$, let $r_n = a_1 + \cdots + a_n$, $s_n = b_1 + \cdots + b_n$ be the partial sum terms of the series. The partial sum term of the addition of the two series is

$$t_n = (a_1 + b_1) + \cdots + (a_n + b_n) = (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n) = r_n + s_n.$$

By the Algebraic Limit Theorem for Sequences,

$$\lim t_n = \lim r_n + \lim s_n = A + B.$$

Product of Series

Other operations are harder:
Question: Given two series, $a_0 + a_1 + a_2 + \cdots + a_n + \cdots$ and $b_0 + b_1 + b_2 + \cdots + b_n + \cdots$, what is

$$(a_0 + a_1 + a_2 + \cdots + a_n + \cdots)(b_0 + b_1 + b_2 + \cdots + b_n + \cdots) = ?$$

Part of the issue arises from the **distributive rule**. We will offer a partial fix later.

Cauchy Criterion for Series

Definition

A sequence (a_n) is called a **Cauchy sequence** if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that whenever $m, n \geq N$ it follows that $|a_n - a_m| < \epsilon$.

Recall:

Theorem

A sequence converges if and only if it is a Cauchy sequence.

We apply this criterion to the sequence (s_n) of partial sums of a series $\sum_{k=1}^{\infty} a_k$. Note that

$$|s_m - s_n| = |a_{m+1} + \cdots + a_n|$$

Cauchy Test for Series

Theorem

The series $\sum_{k=1}^{\infty} a_k$ *converges if and only if given* $\epsilon > 0$, *there exists an* $N \in \mathbb{N}$ *such that whenever* $n > m \geq N$ *it follows that*

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.$$

Proof. Just observe

$$|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon,$$

and apply the Cauchy's Criterion for sequences. □

Corollary

If the series $\sum_{k=1}^{\infty} a_k$ *converges, then* $(a_k) \to 0$.

Proof. Set $n = m + 1$, then $|s_n - s_m| = |a_n|$.

Converse?

Question: Is a series whose sequence of terms a_n converges to 0 convergent? This one is easy:

Answer: No. The (harmonic) series

$$1 + 1/2 + 1/3 + \cdots + 1/n + \cdots$$

has $1/n \to 0$ but it is divergent.

Comparisons

Given two series $\sum_{k\geq 1} a_k$ and $\sum_{k\geq 1} b_k$ that loosely connected we seek to link their convergence/divergence:

Theorem (Comparison Test)

Assume $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are series satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$.

1. *If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.*
2. *If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.*

Proof. Both follow from Cauchy's Criterion applied to the partial sums

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |b_{m+1} + b_{m+2} + \cdots + b_n|$$

If, for instance, given $\epsilon > 0$ we can find N so that for $n, m > N$ $|b_{m+1} + a_{m+2} + \cdots + b_n| < \epsilon$, then the same condition will apply to the a_n.

Example

1. We know that the **harmonic series,** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. It is clear that the same happens if we form the series $\sum_{n=N}^{\infty} \frac{1}{n}$ where N is some fixed number $N \geq 1$.
2. If a and b are positive numbers, consider the series [called generalized harmonic series] whose terms are given by the rule:

$$\frac{1}{a}, \frac{1}{a+b}, \frac{1}{a+2b}, \ldots, \frac{1}{a+nb}, \cdots$$

3. We claim that this series is also divergent: We compare the terms to a multiple of the harmonic series

$$\frac{1}{a+bn} \geq \frac{1}{n+bn} = \frac{1}{b+1}\frac{1}{n}, \quad n \geq a$$

Absolute Convergence Test

If $\sum_{n=1}^{\infty} a_n$ is a series of non-negative terms, its partial sums

$$s_n = a_1 + a_2 + \cdots + a_n, \quad s_{n+1} = s_n + a_n$$

is a monotone sequence. Therefore, by the criterion, the series converges exactly when the sequence (s_n) is bounded.

We make use of this:

Theorem (Absolute Convergence Test)

If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges as well.

Proof of the Absolute Convergence Test

1. We make use of Cauchy criterion for series: Let $\epsilon > 0$. Since the series $\sum_{k=1}^{\infty} |a_k|$ converges, there exists N so that

$$|a_{n+1}| + |a_{n+1}| + \cdots + |a_m| < \epsilon \quad m \geq n > N$$

2. By the **triangle inequality** (one that say $|a + b| \leq |a| + |b|$), we get

$$|a_{n+1} + a_{n+1} + \cdots + a_m| < \epsilon \quad m \geq n > N$$

3. Therefore the series $\sum_{k=1}^{\infty} a_k$ satisfies the Cauchy condition and therefore converges.

Converse?

The series

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots (-1)^{n-1}\frac{1}{n} \cdots$$

is convergent (alternating harmonic series) (the one that won a Grammy's Award), but the series of the absolute values is

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \cdots ,$$

is divergent.

Alternating Series

An alternating series is one with consecutive terms have opposite signs. One group of them is easy to study:

Theorem (Alternating Series Test)

Let (a_n) be a sequence satisfying

1. $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$, *and*
2. $(a_n) \to 0$.

Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

In other words: If (a_n) is a decreasing sequence of positive terms then

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n \quad \text{converges if and only if} \quad \lim a_n = 0$$

Proof. Observe the odd and even sequences of partial sums

$$s_1 = a_1 \geq s_3 = a_1 - (a_2 - a_3) \geq s_5 = s_3 - (a_4 - a_5), \ldots$$

$$s_2 = a_1 - a_2 \leq s_4 = s_2 + (a_3 - a_4) \leq s_5 = s_3 + (a_5 - a_6), \ldots$$

They are monotone and bounded: Since $(a_n) \to 0$, there exists $a_n \leq K$, $s_{2n} = s_{2n-1} + a_{2n} \leq s_{2n-1} + K \leq a_1 + K$, therefore the even sequence is increasing and bounded. Thus it has a limit ℓ_1. Similarly, the other sequence is decreasing and with a lower bound, so it has a limit ℓ_2. Since $\pm a_n = s_n - s_{n-1}$ converges to 0, $\ell_1 = \ell_2$.

Rearrangements

Definition

Let $\sum_{k\geq 1} a_k$ be a series. A series $\sum_{k\geq 1} b_k$ is said to be a **rearrangement** of $\sum_{k\geq 1} a_k$ if there exists a 1–1, onto function $\mathbf{f}: \mathbb{N} \to \mathbb{N}$ such that $b_{\mathbf{f}(k)} = a_k$ for all $k \in \mathbb{N}$.

Consider the geometric series of ratio q

$$1 + q + q^2 + q^3 + \cdots + q^n + \cdots$$

Now we shuffle the terms

$$q + 1 + q^3 + q^2 + q^5 + q^4 + \cdots$$

This is not a geometric series, but we should expect its fate linked to the first series. The next result says this.

A cautionary tale

$$\begin{aligned} S &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \\ \frac{1}{2}S &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots \\ S + \frac{1}{2}S &= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \cdots \end{aligned}$$

Thus $S + \frac{1}{2}S = \frac{3}{2}S$ is just a rearrangement of S! The arithmetic is saying instead that

$$\frac{3}{2}S = S!$$

Series of Positive Terms

Theorem (Dirichlet)

The sum of a series of positive terms [convergence/divergence] is the same in whatever order [rearrangement] the terms are taken.

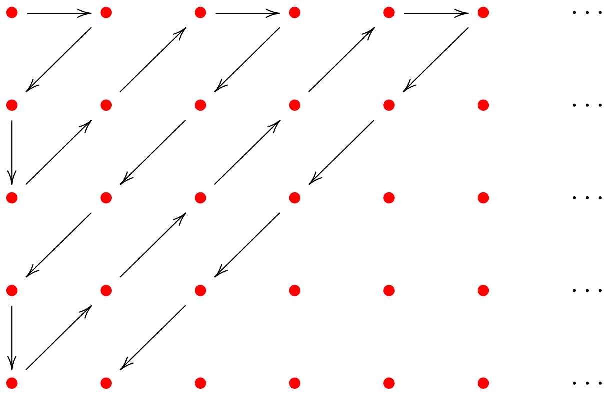
Proof. Let $a_0 + a_1 + a_2 + \cdots + a_n + \cdots$ be a series of positive terms of sum s. Then any partial sum of rearrangement $b_0 + b_1 + b_2 + \cdots + b_n + \cdots$ is bounded by s. Thus the second is convergent and its sum t is bound by s.
We reverse the roles to obtain $s \leq t$. □

Product of Series

Question: Given two series, $a_0 + a_1 + a_2 + \cdots + a_n + \cdots$ and $b_0 + b_1 + b_2 + \cdots + b_n + \cdots$, what is

$$(a_0 + a_1 + a_2 + \cdots + a_n + \cdots)(b_0 + b_1 + b_2 + \cdots + b_n + \cdots) = ?$$

The issue is: we have all the poducts $a_m b_n$ that can be organized into many different series, and then grouped. For instance, if we list the $a_m b_n$ as the double array, we



We could try the following: **Define** the product as the series

$$a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots$$

Makes sense? [Discuss] Will see another rearrangement soon.

$$\begin{array}{ccccc}
a_0 b_0 & a_1 b_0 & a_2 b_0 & a_3 b_0 & \ldots \\
a_0 b_1 & a_1 b_1 & a_2 b_1 & a_3 b_1 & \ldots \\
a_0 b_2 & a_1 b_2 & a_2 b_2 & a_3 b_2 & \ldots \\
a_0 b_3 & a_1 b_3 & a_2 b_3 & a_3 b_3 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

The partial sums remind us how polynomials are multiplied

$$(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)(b_0 + b_1x + b_2x^2 + \cdots + b_mx^m)$$

$$= \sum_{k=0}^{m+n} \Big(\sum_{0\leq i\leq k} a_i b_{k-i} \Big) x^k$$

$a_0b_0, a_0b_1 + a_1b_0$, $a_0b_2 + a_1b_1 + a_2b_2, \ldots$
Another aspect of this definition is:

Theorem

If $\sum_{n\geq 0} a_n$ and $\sum_{n\geq 0} b_n$ are two convergent series of positive terms, and s and t are their respective sums, then the third series is convergent and has the sum st.

Out of all products $a_m b_n$, the 'product' above is given in terms of the diagonals

$$\begin{array}{ccccc} a_0 b_0 & a_1 b_0 & a_2 b_0 & a_3 b_0 & \ldots \\ a_0 b_1 & a_1 b_1 & a_2 b_1 & a_3 b_1 & \ldots \\ a_0 b_2 & a_1 b_2 & a_2 b_2 & a_3 b_2 & \ldots \\ a_0 b_3 & a_1 b_3 & a_2 b_3 & a_3 b_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

$a_0 b_0, a_0 b_1 + a_1 b_0$, $a_0 b_2 + a_1 b_1 + a_2 b_2, \ldots$ whose partial sums don't write conveniently.

We want to re-write the terms of the product series differently:

$$\begin{array}{ccccc} a_0b_0 & a_1b_0 & a_2b_0 & a_3b_0 & \dots \\ a_0b_1 & a_1b_1 & a_2b_1 & a_3b_1 & \dots \\ a_0b_2 & a_1b_2 & a_2b_2 & a_3b_2 & \dots \\ a_0b_3 & a_1b_3 & a_2b_3 & a_3b_3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{array}$$

$a_0b_0, (a_0 + a_1)(a_0 + a_1) - a_0b_0$,
$(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) - (a_0 + a_1)(b_0 + b_1), \dots$ whose nth partial sum is

$$(a_0 + a_1 + \cdots + a_n)(b_0 + b_1 + \cdots + b_n),$$

a sequence that converges to st by the Algebraic Limit Theorem.

Theorem

If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then any rearrangement of this series converges to the same limit.

Proof. Assume $\sum_{k\geq 1} a_k$ converges absolutely to A, and let $\sum_{k\geq 1} b_k$ be an rearrangement of $\sum_{k\geq 1} a_k$. Let

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

and

$$t_n = \sum_{k=1}^{n} b_k = b_1 + b_2 + \cdots + b_n$$

be the corresponding partial sums.
Let $\epsilon > 0$. Since $(s_n) \to A$, choose N_1 such that

$$|s_n - A| < \epsilon/2$$

for all $n \geq N_1$.

Because the convergence is absolute, we can choose N_2 so that

$$\sum_{k=m+1}^{n} |b_k| < \epsilon/2$$

for all $n > m \geq N_2$. Take $N = \max\{N_1, N_2\}$. We know that the terms $\{a_1, a_2, \ldots, a_N\}$ must all appear in the rearranged series, and we move far out enough in the series $\sum_{k\geq 1} b_k$ that these terms are all included. Thus, choose $M = \max\{f(k) \mid 1 \leq k \leq N\}$.
It is clear that if $m \geq M$, then $(t_m - s_N)$ consists of a finite number of terms, the absolute values of which appear in the tail of $\sum_{k=N+1}^{\infty} |a_k|$. The earlier choice of N_2 guarantees $|t_m - s_N| < \epsilon/2$, and so

$$\begin{aligned} |t_m - A| &= |t_m - s_N + s_N - A| \\ &\leq |t_m - s_N| + |s_N - A| \leq \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

Outline

Convergence Tests for Series

3 elementary tests of convergene

- Integral Test
- Ratio Test
- Root Test

Integral Test

Theorem (Integral Test)

Let $\sum_{n\geq 0} a_n$ be a series of positive terms. If there is a decreasing function $\mathbf{f}(x)$ such that $a_n \leq \mathbf{f}(n)$ for large n and

$$\int_{x=1}^{\infty} \mathbf{f}(x)dx < \infty,$$

then $\sum_{n\geq 0} a_n$ converges.

Proof. If $a_n \leq \mathbf{f}(n)$ for $n \geq n_0$, since $\mathbf{f}(x)$ is decreasing,

$$a_n \leq \int_{n-1}^{n} \mathbf{f}(x)dx, \quad n > n_0.$$

From this, and the assumption that $\int_1^\infty \mathbf{f}(x)dx < \infty$, we get that the partial sums of the series $\sum_{n\geq 0} a_n$ are bounded, and therefore converge by the theorem on bounded monotone sequences. ☐

Consider the function $\mathbf{f}(x) = 1/x^p$, $x \geq 2$. This is a decreasing function (draw the graph).
Observe

$$1/n^p \leq \int_{x=n-1}^{n} 1/x^p dx$$

Therefore its partial sums are bounded by

$$s_n \leq 1 + \int_{x=1}^{n} \frac{dx}{x^p} = 1 + \frac{1}{p-1}\left[1 - \frac{1}{n^{p-1}}\right] < 1 + \frac{1}{p-1}$$

Convergence

Let us show that

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots ,$$

for $p > 1$ will always converge.

We are going to bound each term $1/n^p$ by the terms of another series, and then argue the new series converges.

Examples

Comparison gives

$$\sum_{n\geq 1} \frac{1}{n(n+1)} \leq \sum_{n\geq 1} \frac{1}{n^2}$$

which is convergent.
In the same manner, if

$$\sum_{n\geq 1} \frac{p(n)}{q(n)},$$

where $p(n)$ and $q(n)$ are positive polynomial expressions with $\deg q \geq 2 + \deg p$, then the series converges by the same reason. Do it!

Ratio Tests

There are very useful tests involving the ratio a_{n+1}/a_n of two successive terms of a series. Sometimes we compare the ratio a_{n+1}/a_n to another b_{n+1}/b_n. In these we suppose that a_n and b_n are strictly positive.

Suppose $a_n, b_n > 0$ and that

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$$

for sufficiently large n, that is for $n \geq n_0$.

Then

$$\begin{aligned} a_n &= \frac{a_{n_0+1}}{a_{n_0}} \frac{a_{n_0+2}}{a_{n_0+1}} \cdots \frac{a_n}{a_{n-1}} a_{n_0} \\ &\leq \frac{b_{n_0+1}}{b_{n_0}} \frac{b_{n_0+2}}{b_{n_0+1}} \cdots \frac{b_n}{b_{n-1}} a_{n_0} = \frac{a_{n_0}}{b_{n_0}} b_n \\ &= Cb_n, \quad C = a_{n_0}/bn_0. \end{aligned}$$

Here are some applications:

Theorem

Let $\sum a_n$ and $\sum b_n$ be series of positive terms.

1. *If for $n \geq n_0$*
$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n},$$
and the series $\sum b_n$ converges, then $\sum a_n$ converges also.
2. *If for $n \geq n_0$*
$$\frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n},$$
and the series $\sum a_n$ diverges, then $\sum b_n$ diverges also.

Theorem (d'Alambert Test)

The series $\sum a_n$ is convergent if $a_{n+1}/a_n \leq r$, where $r < 1$, for all sufficiently large n.

Theorem

Given a series $\sum_{n\geq 1} a_n$ with $a_n \neq 0$, if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1,$$

then the series converges absolutely.

Proof.

1. Let r' satisfy $r < r' < 1$. For $\epsilon = r' - r$, there is N such that for $n \geq N$ $||a_{n+1}/a_n| - r| < \epsilon$, and therefore

$$|a_{n+1}/a_n| - r \leq ||a_{n+1}/a_n| - r| < \epsilon = r' - r,$$

giving $|a_{n+1}| \leq r'|a_n|$ for $n \geq N$.

2. The above shows that the series $\sum_{n=N}^{\infty} |a_n|$ satisfies $|a_n| \leq |a_N|(r')^{n-N}$, a geometric series of ratio $r' < 1$, which converges.

Exponential

A quick application of the ratio test:
We claim that the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

converges for all values of x.

For the ratio of consecutive terms

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}/(n+1)!}{x^n/n+!} = \frac{x}{n+1}$$

so that for any x, $\lim a_{n+1}/a_n = 0$.

This is a well used technique for power series.

Examples

1. For the series $\sum_{n\geq 1} \frac{n}{2^n}$ we invoke the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} / \frac{n}{2^n} = \frac{n+1}{n}\frac{1}{2}$$

which has limit $1/2 < 1$. So the series converges.

2. Decide [with justification] whether the series

$$\sum_{n\geq 1} \frac{n!}{n^n},$$

is convergent or divergent?

Exercises

1. Show that if $a_n > 0$ and $\lim na_n = L$, with $L \neq 0$, then the series $\sum a_n$ diverges.
2. Show that if $a_n > 0$ and $\lim n^2 a_n = L$, with $L \neq 0$, then the series $\sum a_n$ converges.
3. Find examples of two series $\sum a_n$ and $\sum b_n$ both of which diverge but for which $\sum \min\{a_n, b_n\}$ converges. To make it more difficult, choose examples where (a_n) and (b_n) are positive and decreasing.

Root Test

Let $\sum_{n\geq 1} a_n$ be a series of positive terms. We are going to examine how the limit

$$\lim_{n\to\infty} \sqrt[n]{a_n}$$

is used to decide convergence. We recall one special calculation of these limits: If $x > 0$

$$\lim_{n\to\infty} \sqrt[n]{x} = 1$$

Recall another limit: $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Root Test

Theorem

If $\sum_{n\geq 1} a_n$ is a series of positive terms and $\lim_{n\to\infty} \sqrt[n]{a_n} = r < 1$, then the series converges.

Proof. Let $r < r' < 1$ and pick $\epsilon = r' - r$. This is the same subtle point we used above.

1. There is N so that for $n > N$
$$|\sqrt[n]{a_n} - r| < \epsilon$$
2. This implies that $\sqrt[n]{a_n} < r + \epsilon = r' < 1$ for $n > N$. As a consequence
$$a_n < (r')^n$$
3. We now compare the series $\sum n \geq 1 a_n$ to the geometric series $\sum_{n\geq 1}(r')^n$ of ratio $r' < 1$. Thus both series converge.

Example

Consider the series (for $q > 0$)

$$1 + q + 2q^2 + \cdots + nq^n + \cdots$$

We invoke the **root test**

$$\lim_{n\to\infty} \sqrt[n]{nq^n} = q \lim_{n\to\infty} \sqrt[n]{n} = q$$

Therefore it converges if $q < 1$
Let us calculate the sum of tbe series. For that we must have an inkling on how the series arose from the geometric series. At these times we replace q by x and recall:

Nice calculation

1. Differentiate the 'equality'

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

2. To get almost our series

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots$$

3. Now multiply by x and add 1

$$1 + \frac{x}{(1-x)^2} = 1 + x + 2x^2 + \cdots + nx^n + \cdots$$

4. Thus for $0 < q < 1$ the series sums to

$$1 + \frac{q}{(1-q)^2}$$

Exercises

- Show that the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

converges. (*Hint*: Look up one of the class examples)
To find the limit, sum the geometric series

$$1 - x^2 + x^4 - x^6 + \cdots ,$$

and integrate over $[0, 1]$. Indicate what steps will have to be properly justified.

- Is the series

$$\frac{1^1}{(101)!} + \frac{2^2}{(100+2)!} + \cdots + \frac{n^n}{(100+n)!} + \cdots$$

convergent or divergent? Justify answer.

More Exercises

1. Show that
$$\sum_{n\geq 0}(-1)^n \frac{2n+3}{(n+1)(n+2)} = 1.$$
2. Determine the values of q for which the series
$$q + 2q^2 + 3q^3 + \cdots + nq^n + \cdots$$
is converget.
3. Show that $\sum_{n\geq 2} \frac{1}{n(\ln n)^p}$ converges if $p > 1$, and diverges if $p \leq 1$.

Outline

Workshop #4

Think/Do next 4 Questions [in 2 frames]

1. Find the sum of the series
$$\sum_{n\geq 1} \frac{1}{n(n+4)}.$$
As a warmup, find the sum of the series
$$\sum_{n\geq 1} \frac{1}{n(n+1)}.$$
2. Show that if $a_n > 0$ and $\lim n^p a_n = L$, with $L \neq 0$ for some integer $p > 1$, then the series $\sum a_n$ converges. An application: If
$$\sum_{n\geq 1} \frac{p(n)}{q(n)},$$
where $p(n)$ and $q(n)$ are positive polynomial expressions with $\deg q \geq 2 + \deg p$, then the series converges.

Workshop #4, Cont'd

3 Determine the values of $q > 0$ for which the following series converges and find its sum

$$1 + q + \frac{q^2}{2} + \cdots + \frac{q^n}{n} + \cdots .$$

Calculate the sum of the series.

4 Is the following series

$$\sum_{n \geq 0} e^{-n^2}$$

convergent or divergent? Try all [ratio, root, and integral tests]

Exercises

1. Show that the sequence

$$\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+2}-\sqrt{n+1}}, \quad n \in \mathbb{N}$$

   converges. As a challenge, find also a bound for it.

2. Let $0 \leq a, b \in \mathbb{R}$ and define recursively $a_0 = a$, $b_0 = b$, $a_{n+1} = \sqrt{a_n b_n}$ and $b_{n+1} = (a_n + b_n)/2$. Show that $[a_n, b_n]$ form a nested sequence of intervals. Prove that the intersection of these intervals is a single point.

3. If the series $\sum_{n\geq 1} a_n^2$ and $\sum_{n\geq 1} b_n^2$ are convergent, prove that $\sum_{n\geq 1} a_n b_n$ is convergent.

$\lim \sqrt[n]{n}$

1. Write $\sqrt[n]{\sqrt{n}} = 1 + a_n$, so that $\sqrt[n]{n} = (1 + a_n)^2$ and $\sqrt{n} = (1 + a_n)^n$
2. By a Lemma we have used often, $\sqrt{n} = (1 + a_n)^n \geq 1 + na_n > na_n$,

$$\frac{1}{\sqrt{n}} = \frac{\sqrt{n}}{n} > a_n$$

3. Thus

$$1 \leq \sqrt[n]{n} = (1 + a_n)^2 = 1 + 2a_n + a_n^2 < 1 + \frac{2}{\sqrt{n}} + \frac{1}{n}$$

4. Therefore, by the Squeeze Theorem, $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Outline

Typical E-Questions

- Prove that bounded monotone sequences are convergent.
- Why the cardinalities of $\mathbb{N}$ and of $\mathbb{N}^4$ are the same?
- If $(a_n) \to a$ and $(b_n) \to b$, with $b_n, b \neq 0$, prove that $\lim(a_n/b_n) = a/b$.
- What is the **nested interval property** of $\mathbb{R}$? Give an interesting example and sketch the proof.
- If (a_n) and (b_n) are sequences such that $\lim a_n + b_n = 5$ and $\lim a_n = 2$, must (b_n) be convergent? Explain or give counter-example.
- If $(a_n) \to 5$, $a_n \geq 0$, prove with full details that $\lim \sqrt{a_n} = \sqrt{5}$. [You may use $\epsilon = 1/10$.]
- Find $\lim \sqrt[n]{a^{n+1}b^n + b^{n+1}c^n + c^{n+1}a^n}$, with $a > b > c > 0$

• Do all sequences have a convergent subsequence? If not, when? Explain.

• Let (a_n) and (b_n) be two Cauchy sequences. Prove directly that $(a_n b_n)$ is a Cauchy sequence.

• If a is a positive integer, give a formula for the sum of the series

$$\sum_{n\geq 1} \frac{1}{n(n+a)}.$$

A beautiful limit

• Prove that $\lim n(\sqrt[n]{x} - 1)$, $x > 0$, exists. [Not easy, not in exam, just tossed as a challenge.]
The limit defines a function $\mathbf{f}(x)$. Observe the property

$$n(\sqrt[n]{xy} - 1) = n(\sqrt[n]{x} - 1)\sqrt[n]{y} + n(\sqrt[n]{y} - 1)$$

Taking into account $\lim \sqrt[n]{y} = 1$ from a Workshop, we get

$$\mathbf{f}(xy) = \mathbf{f}(x) + \mathbf{f}(y),$$

a defining property of Logs. [? Maybe $\mathbf{f}(x) = e^x$]

An old First Hourly

- (15 pts)
  1. What is a countable set?
  2. Why is $\mathbb{Q}$ countable?
  3. Prove that $\mathbb{N}$ and $\mathbb{N}^2$ have the same cardinality.
- (10 pts) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

  converges.
- (15 pts) Describe very carefully and in full the following terms:
  1. **lower bound** of a subset $A \subset \mathbb{R}$
  2. **Nested Interval Property**
  3. give an example for each term.

- (15 pts)
  1. Define precisely the notion of a **convergent** sequence.
  2. What is a **subsequence** of a sequence?
  3. Prove that all subsequences of a convergent sequence have the same limit.
- (15 pts)
  1. What is a **monotone** sequence? Give an example.
     If a monotone sequence (a_n) is bounded, prove that it is convergent.
- (15 pts) Find (with proof!) the limit of the sequence

$$\sqrt[n]{a^n b^n + b^n c^n + c^n a^n}, \quad a > b > c > 0.$$

- (15 pts)
  1. What is a **Cauchy** sequence?
  2. If (a_n) and (b_n) are Cauchy sequences, prove directly that $(a_n b_n)$ is a Cauchy sequence.

Exercise

The equation $x^3 - 3x + 1 = 0$ has a root α between 0 and 1. To find it, define the sequence

$$x_1 = 0, \quad x_{n+1} = \frac{1}{3 - x_n^2}$$

Show that the sequence is monotone and converges to α.

Exercises

1. Show that if $a_n > 0$ and $\lim na_n = L$, with $L \neq 0$, then the series $\sum a_n$ diverges.
2. Show that if $a_n > 0$ and $\lim n^2 a_n = L$, with $L \neq 0$, then the series $\sum a_n$ converges.
3. Find examples of two series $\sum a_n$ and $\sum b_n$ both of which diverge but for which $\sum \min\{a_n, b_n\}$ converges. To make it more difficult, choose examples where (a_n) and (b_n) are positive and decreasing.

Outline

Hourly #1 Review

Important Topics

- Least Upper Bound
- Axiom of Completeness
- Cardinality: Countable and Uncountable Sets, Power Sets
- Sequences, Convergence/Divergence
- Monotone Sequences
- Bolzano-Weirstrauss Theorem
- Cauchy Sequences
- Series: Backbone Examples
- Convergence of Series: Meaning
- Tests of Convergence: Integral, Ratio, Root