Math 311–03: Advanced Calculus

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Set 1

Spring 2010

Outline

General Orientation

- Pre-requisites: Calc 4, Math 300
- web:www.math.rutgers.edu/(tilde)vasconce
- Meetings: MWTh4 1:40-3:00 SEC-205
- Office Hours [Hill 228]: MTh3, or by arrangement
- Textbook: **Introduction to Analysis**, 5th Ed., by E. D. Gaughan
- All this detailed in General Info page: Look over

Scoring Info

- Quizzes Total: 50
- Workshops Total: 100
- 2 Midterms Total: 2 x 100 = 200
- Final: 200
- Total: 550 pts

Course Symbol

$$\mathbb{R}$$

Some Goals

- What is $\mathbb{R}$, and what are some of its important properties?
- Topology of $\mathbb{R}$: continuous functions
- Really Understand objects such

$$\int_a^b \mathbf{f}(x)dx$$

$$a_1 + a_2 + a_3 + \cdots$$

Theorem (FTC)

Let $\mathbf{f} : [a, b] \to \mathbb{R}$ *be a function such that* $\int_a^b \mathbf{f}$ *exists. If* $\mathbf{F}$ *is a function such that*

$$\mathbf{F}'(c) = \mathbf{f}(c)$$

for all $c \in [a, b]$*, then*

$$\int_a^b \mathbf{f}(x)dx = \mathbf{F}(b) - \mathbf{F}(a).$$

Outline

Rational Numbers

At the outset of our journey are the **natural** numbers

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

Its 'modern' construction [e.g. Peano's] is a paradigm of beauty. It is enlarged by the **integers**

$$\mathbb{N} \subset \mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

and the **rational** numbers

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} = \left\{\frac{m}{n}, \quad m, n \in \mathbb{Z}, n \neq 0\right\}$$

These sets exhibit different **structures**: of a monoid, of a ring and of a field, respectively.

Peano

The construction by Peano of the set $\mathbb{N}$ is grounded on two ingredients: The set $\mathbb{N}$ contains a particular element 1.

- **[Successor Function]** There is a function $s : \mathbb{N} \to \mathbb{N}$ that is injective, and for every $n \in \mathbb{N}$ $s(n) \neq 1$.
- **[Induction Axiom]** If the subset $S \subset \mathbb{N}$ has the properties

$$1 \in S \quad \& \quad \text{whenever} \quad n \in S \Rightarrow s(n) \in S$$

then $S = \mathbb{N}$

Given these definitions, we can define several operations/compositions and structures on $\mathbb{N}$:

- $a + b :=?$

$$\begin{aligned} a + 1 &:= s(a) \\ a + s(n) &:= s(a + n) \end{aligned}$$

- $a \times b :=?$

$$\begin{aligned} a \times 1 &:= a \\ a \times s(n) &:= a \times n + a \end{aligned}$$

Example

Theorem

Suppose $a \geq -1$. Then for all $n \in \mathbb{N}$, $(1+a)^n \geq 1 + na$.

Proof.

We shall prove the statement by **induction**:

- **(base case):** If $n = 1$, $(1+a)^1 = 1 + a \geq 1 + a$ is true
- **(induction step)**: Suppose $(1+a)^n \geq 1 + na$. Then, since $1 + a \geq 0$ by hypothesis,

$$
\begin{aligned}
(1+a)^{n+1} &= (1+a)^n(1+a) \geq (1+na)(1+a) \\
&= 1 + na + a + na^2 = 1 + (n+1)a + na^2 \\
&\geq 1 + (n+1)a
\end{aligned}
$$

□

Ordering

Out of these notions, addition and multiplication are defined in $\mathbb{N}$, and then extended to $\mathbb{Z}$ and $\mathbb{Q}$. An interesting consequence that arises is a notion of **order**: $\forall a, b \in \mathbb{Q}$, exactly one of the following holds

$$a < b, \quad a > b, \quad a = b$$

It has the properties: If $a > b$ then

$$\begin{aligned} \forall c \quad &\Rightarrow \quad a + c > b + c \\ \forall c > 0 \quad &\Rightarrow \quad ac > bc \end{aligned}$$

Significance: This leads to metric properties: lengths, angles, etc.

Fields

A composition on a set **X** is a function assigning to pairs of elements of **X** an element of **X**,

$$(a, b) \mapsto \mathbf{f}(a, b).$$

That is a function of two variables on **X** with values in **X**.
It is nicely represented in a composition table

$\mathbf{f}$	$*$	b	$*$
$*$	$*$	$*$	$*$
a	$*$	$\mathbf{f}(a, b)$	$*$
$*$	$*$	$*$	$*$

We represent it also as

$$\mathbf{X} \times \mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{X}$$

An abelian group is a set **G** with a composition law denoted ‘$+$’

$$\mathbf{G} \times \mathbf{G} \to \mathbf{G},$$

$$a, b \in \mathbf{G}, \quad a + b \in \mathbf{G}$$

satisfying the axioms

- associative $\forall a, b, c \in \mathbf{G}, \quad (a + b) + c = a + (b + c)$
- commutative $\forall a, b \in \mathbf{G}, \quad a + b = b + a$
- existence of O

$$\exists O \in \mathbf{G} \quad \text{such that } \forall a \quad a + O = a$$

- existence of inverses

$$\forall a \in \mathbf{G} \quad \exists b \in \mathbf{G} \quad \text{such that } a + b = O$$

  This element is unique and denoted $-a$.

Field

A field $\mathbf{F}$ is a set with two composition laws, called 'addition' and 'multiplication', say $+$ and $\times$: $\forall a, b \in \mathbf{F}$ have compositions $a + b$ and $a \times b$. (The second composition is also written $a \cdot b$, or simply ab.)

- $(\mathbf{F}, +)$ is an abelian group
- $(\mathbf{F}, \times)$: multiplication is associative, commutative and distributive over $+$, that is $\forall a, b, c \in \mathbf{F}$,

$$(ab)c = a(bc), \quad ab = ba, \quad a(b + c) = ab + ac$$

- existence of identity $\exists e \in \mathbf{F}$ such that

$$\forall a \in \mathbf{F} \quad a \times e = a$$

- existence of inverses For every $a \neq 0$, there is $b \in \mathbf{F}$

$$a \times b = e.$$

There is a unique element e, usually we denote it by 1. For $a \neq 0$, the element b such that $ab = 1$ is unique; it is often denoted by $1/a$ or a^{-1}.

We can now define scalars: the elements of a field.

Another noteworthy example is $\mathbb{F}_2$, the set made up by two elements $\{0, 1\}$ (or (even, odd))with addition defined by the table

$+$	0	1
0	0	1
1	1	0

$1 + 1 = 0!$

and multiplication by

$\times$	0	1
0	0	0
1	0	1

A field is the mathematical structure of choice to do arithmetic. Given a field $\mathbf{F}$, fractions can defined as follows: If $a, b \in \mathbf{F}, \quad b \neq 0$,

$$\frac{a}{b} := ab^{-1}.$$

The usual calculus of fractions then follows, for instance

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Example: The field of constructable numbers

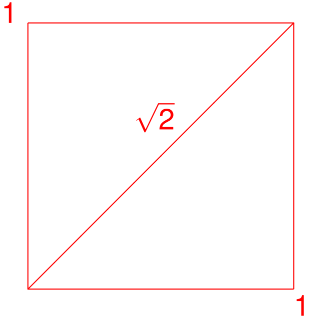
Class Discussion: Volunteers!

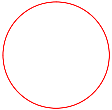
Rational Numbers: Counting and Measuring

- Counting
- Measuring by Counting

Irrationality of $\sqrt{2}$

The arrival of new numbers:





The construction of an irrational number

Example

Theorem

$\sqrt{2} \notin \mathbb{Q}$.

Proof.

- We are going to argue by contradiction: Suppose

$$\sqrt{2} = \frac{m}{n}$$

- We may assume that m and n have no common factor.
- Squaring both sides of the equality, we obtain $m^2 = 2n^2$
- This implies that m is even, as the square of an odd number, say $m = 2p + 1$, is odd

$$(2p+1)^2 = 4p^2 + 4p + 1 = 4(p^2 + p) + 1$$

- We may then assume that m is even. In $m^2 = 2n^2$, set $m = 2p$ to get

$$4p^2 = 2n^2$$

  and therefore
- $n^2 = 2p^2$, which implies that n is also even.
- This contradicts our assumption that m and n have no common factors. $\square$

This will also work with $\sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{8}$ and many other cases. Obviously, these **numbers** need a **home**.

Exercise

Exercise: Show that $z = \sqrt{2} + \sqrt{3}$ is not a rational number.

- Will argue by contradiction. If z is a rational, then $z^2 = 2 + 2\sqrt{6} + 3$ is also a rational number.
- From $\sqrt{6} = 1/2(z^2 - 5)$, it follows that $\sqrt{6} = m/n$ for $m, n \in \mathbb{N}$. Assume m, n have no common factors.
- This gives $m^2 = 6n^2 = 2 \times 3 \times n^2$. Thus 2 must divide m and therefore 2^2 divides m^2, $m^2 = 2^2 p = 2 \times 3 \times n^2$. This shows that 2 divides n, a contradiction.

Exercise: Show that $x = 3.1212...$ (repeating 12's) is a rational number.

- Note that $100x = 312.1212....$
- $100x - x = 99x = 312 - 3 = 309$
- Thus $x = \frac{309}{99}$
- Same trick works for any repeating decimal.

Outline

Basic Set Theory

- How to specify a set: listing its elements, membership test, etc

$$\{x, P(x)\}, \quad P(x) \text{ is test}$$

$$\{x \in \mathbb{N}, \quad 6|x\} = \{6, 12, 18, \ldots\}$$

- Pair (x, y): $\{\{x\}, \{x, y\}\}$? What is a triple?
- Product of sets **A** and **B**: $\{(a, b), \quad a \in \mathbf{A}, b \in \mathbf{B}\}$
- Relation: subset of a product of sets

In order to deal with real numbers, we are going to use the language of set theory: If $A, B, C \ldots$ are subsets of the set $\mathbf{X}$, will assume familiarity with the following notions and notation:

- **union:** $A \cup B$
- **intersection:** $A \cap B$
- **complement:** $A^c = \{x \in \mathbf{X} \mid x \notin A\}$ = $\mathbf{X} \setminus A$
- **Morgan's laws:**

$$\begin{aligned} C \cap (A \cup B) &= (C \cap A) \cup (C \cap B) \\ C \cup (A \cap B) &= (C \cup A) \cap (C \cup B) \\ (A \cap B)^c &= A^c \cup B^c \quad \& \quad (A \cup B)^c = A^c \cap B^c \end{aligned}$$

- There are various 'infinite' versions of some of these.

Ordered Pair

Definition

Let A and B be sets. For $a \in A$ and $b \in B$, the **ordered pair** (a, b) is the set

$$\{\{a\}, \{a, b\}\}.$$

a is called the first coordinate of the pair, and b the second coordinate.

Note that (a, b) may be different from (b, a):

$$\{\{a\}, \{a, b\}\} \neq \{\{b\}, \{a, b\}\},$$

if $a \neq b$.

Definition

Let A and B be sets. The set of all ordered pairs having first coordinate in A and second coordinate in B is called the **Cartesian product** of A and B and written $A \times B$. Thus

$$A \times B = \{(a, b) : a \in A \quad \text{and} \quad b \in B.\}$$

Functions

Let $\mathbf{X}$ and $\mathbf{Y}$ be two sets. The general way to define a function of source $\mathbf{X}$ and target $\mathbf{Y}$, $\mathbf{F} : \mathbf{X} \to \mathbf{Y}$, is the following:

A function is a subset $\mathbf{F}$ of $\mathbf{X} \times \mathbf{Y}$ with the properties

- $\forall x \in \mathbf{X}$,
$$\text{there is } y \in \mathbf{Y} \text{ such that} \quad (x, y) \in \mathbf{F}$$
- If
$$(x, y) \quad \& \quad (x, y') \in \mathbf{F} \quad \Rightarrow y = y'$$

Sets of Rational Numbers

Define a set of rational numbers

$$A = \{x_1, x_2, \ldots, x_n, \ldots\}$$

by the rules: $x_1 = 1$,

$$\forall n \quad x_{n+1} = \frac{x_n}{2} + 1$$

Let us prove

$$x_n < x_{n+1} < 2$$

We are going to argue by induction. The assertion is true for $n = 1$, as $x_1 = 1 < x_2 = 3/2 < 2$.

Suppose it holds for n, that is

$$x_n < x_{n+1} < 2.$$

If we divide these inequalities by 2 and add 1, we have

$$x_{n+1} < x_{n+2} < 2$$

We further claim that there is no rational number $q < 2$ such that $x_n < q$ for all n.

Outline

Upper and Lower Bounds

- The number b is said to be an **upper bound** of the set $A \subset \mathbb{R}$ if

$$a \leq b \mid \forall a \in A$$

- A number ℓ is said to be a **lower bound** of the set $A \subset \mathbb{R}$ if

$$a \geq \ell \mid \forall a \in A$$

- Consider the set $A = \{q \in \mathbb{Q} \mid q^2 < 2\}$. -2 is a lower bound of A, while $3/2$ is an upper bound. Clearly there are many other bounds.

Least Upper and Greatest Lower Bounds

- A number b is said to be a **least upper bound** of the set $A \subset \mathbb{R}$ if b is an upper bound of A and $b \leq b'$ for any other upper bound b'. Least upper bounds are also known as the **supremum** of A. If $b \in A$, it is called the **maximum** of A.

$$A = \{x_1 = 1, \forall n \quad x_{n+1} = \frac{x_n}{2} + 1\}$$

  has 2 for supremum [needs a proof, as we only proved that 2 is an upper bound]
- Similarly we define **greatest lower bound** [and of **infimum/minimum**].

Example

Define the set $\mathbf{A} = \{a_1, a_2, a_3, \cdots\}$ by the rule

$$a_1 = \sqrt{2}, \quad a_2 = \sqrt{2\sqrt{2}}, \quad a_3 = \sqrt{2\sqrt{2\sqrt{2}}}, \cdots$$

Let us show that $\sup \mathbf{A} = 2$:

$$a_1 = \sqrt{2}, \quad a_2 = a_1\sqrt[4]{2}, \quad a_3 = a_2\sqrt[8]{2}, \cdots$$

$$a_n = 2^{1/2+1/4+\cdots+1/2^n} < 2$$

$$a_n = 2^r, \quad r = \frac{1/2 - 1/2^{n+1}}{1/2} = 1 - 1/2^n$$

Exercise

For any number $1 < a < 2$, we can show that there is $n \in \mathbb{N}$ such that

$$2^{1-1/2^n} > a$$

You may need help, try this lemma:

Lemma

For any $p \geq 1$ and all $n \in \mathbb{N}$, $p^{1-1/2^n} \geq p - p/n$.

Proof. ?

David Hilbert (1862-1943)

David Hilbert

David Hilbert
(1862 - 1943)
Mathematician
Algebraist
Topologist
Geometrist
Number Theorist
Physicist
Analyst
Philosopher
Genius
And modest too...

"Physics is much too hard for physicists." - Hilbert, 1912

This site is dedicated to David Hilbert, the funkiest mathematician alive.

Axiom of Completeness

Axiom: Every set *A* of real numbers with an upper bound has a least upper bound.

This is a defining property of $\mathbb{R}$. A lot flows out of it. We will explore some of it in the next lectures **[Discuss]**.

Nested Interval Property

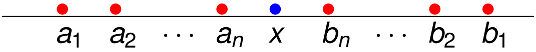
Theorem

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} \mid a_n \leq x \leq b_n\}$. Assume that each I_n contains I_{n+1}. Then the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

has a nonempty intersection, that is

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$



Proof. We plotted the ends of the intervals $I_n = [a_n, b_n]$. We will use the axiom of completeness to the set A of left ends a_n of the intervals. Note that each b_n is an upper bound for A.

Let $x = \sup A$. Consider a particular interval $I_n = [a_n, b_n]$ Since $a_n \leq x$ and each b_n is an upper bound of A, $x \leq b_n$. Thus $x \in I_n$, for each n as desired.

Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem (Archimedean Property)

(i) *Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.*

(ii) *Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$.*

Proof. (i) Assume, by contradiction, that $\mathbb{N}$ is bounded above. By (AoC), $\mathbb{N}$ should have a least upper bound, set $\alpha = \sup \mathbb{N}$. $\alpha - 1$ is not an upper bound, so there is an $n \in \mathbb{N}$ such that $\alpha - 1 < n$. Thus $\alpha < n + 1$.

Part (ii) follows from (i) by letting $x = 1/y$. □

Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$)

For every two real numbers a and b with $a < b$, there is a rational number r satisfying $a < r < b$.

Proof. To simplify matters a little, we assume $0 \leq a < b$. We must find $m, n \in \mathbb{N}$ such that

$$a < \frac{m}{n} < b.$$

First, we use the archimedean property to pick $n \in \mathbb{N}$ so that

$$\frac{1}{n} < b - a$$

With n chosen, we must find m so that $na < m < nb$. Pick m the smallest natural number greater than na. That is

$$m - 1 \leq na < m$$

Note that this already gives $a < m/n$. Writing $1/n < b - a$ as $a < b - 1/n$, we can write

$$m \quad \leq \quad na + 1 < n(b - \frac{1}{n}) + 1 = nb$$

which gives $m/n < b$, to complete the proof. □

Existence of $\sqrt{2}$

Theorem

The least upper bound α of the set $A = \{q \in \mathbb{R} \mid q^2 < 2\}$ satisfies $\alpha^2 = 2$.

Proof. We are going to argue that $\alpha^2 < 2$ and $\alpha^2 > 2$ violate the assumption on α.

Let us first show if $\alpha^2 < 2$ then α is not an upper bound of A. We will find elements in A larger than α.

$$\begin{aligned}(\alpha + \frac{1}{n})^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \\ &< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} \\ &= \alpha^2 + \frac{2\alpha + 1}{n}\end{aligned}$$

If $\alpha^2 < 2$, choose $n_0 \in \mathbb{N}$ large enough so that

$$\frac{1}{n_0} < \frac{2-\alpha^2}{2\alpha+1}$$

This implies $(2\alpha + 1)/n_0 < 2 - \alpha^2$, and consequently

$$\begin{aligned}(\alpha + \frac{1}{n_0})^2 &< \alpha^2 + \frac{2\alpha+1}{n_0} \\ &< \alpha^2 + (2-\alpha^2) = 2\end{aligned}$$

Thus $\alpha + 1/n_0 \in A$, so α is not an upper bound of A.

Suppose $\alpha^2 > 2$: Write

$$\begin{aligned}(\alpha - \frac{1}{n})^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n}\end{aligned}$$

As in the previous case, pick n_0 large enough so that

$$\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha + 1}$$

This implies $(2\alpha + 1)/n_0 < \alpha^2 - 2$, and consequently

$$\begin{aligned}(\alpha - \frac{1}{n_0})^2 &> \alpha^2 - \frac{2\alpha + 1}{n_0} \\ &> \alpha^2 - (\alpha^2 - 2) = 2\end{aligned}$$

Thus $\alpha - 1/n_0$ is an upper bound of A, so α is not the least upper bound.

Outline

Last Time & Today

- Bounded sets, least upper bounds
- The Axiom of Completeness
- ◦ Cardinality of Sets
- ◦ Countable Sets, including a Cool Proof

Wednesday: Workshop #1

Outline

Cardinality

The Axiom of Completeness creates a lot of order but it is also a burst dam of new relationships and problems. It becomes intimately related to logic and the foundations of mathematics.

Let us begin by introducing a method to size sets. If A and B are two sets, a function

$$\mathbf{f} : A \to B$$

is **one-one** if $\mathbf{f}(x) = \mathbf{f}(y)$ implies $x = y$, and it is **onto** if B is the image of **f**.

On a nice collection of sets, e.g. the subsets of $\mathbb{R}$, we can define a relation $A \sim B$ by requiring a function $\mathbf{f} : A \to B$ as above. This is obviously an **equivalence relation**. The **equivalence class** of A is called the **cardinality** of A, $\mathrm{card}\,(A)$.

A set A is said to be **countable** if $\mathrm{card}\,(A) = \mathrm{card}\,(\mathbb{N})$:

$$\mathbf{f} : \mathbb{N} \to A$$

$$A = \{\mathbf{f}(1), \mathbf{f}(2), \ldots, \}.$$

If $A = \{1, 2, \ldots, n\} \subset \mathbb{N}$, we write $\mathrm{card}\,(A) = n$.

If the set $B \sim \{1, 2, \ldots, n\}$ we say that B is finite and has n elements.

Exercise: If $\mathrm{card}\,(A)$ is countable and $B \subset A$, then B is countable or finite.

It is obviously a tricky thing to determine the cardinality of sets, particularly of infinite sets. Let us get our hands busy!

Exercise

The set $\mathbb{N} \times \mathbb{N}$ is countable: Let define a one-one function

$$\mathbf{f} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

Define

$$\mathbf{f}(m, n) = 2^m 3^n$$

By the unique factorization on integers,

$$2^m 3^n = 2^p 3^q \Rightarrow m = p \quad n = q,$$

which proves the claim that **f** is injective.

Exercise: Use the infinity of prime numbers to show that the set **X** of all infinite tuples $(x_1, x_2, x_3, \ldots)$ such that all $x_i = 0$ except for finitely many exceptions is countable.

Theorem

The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countable.

We must establish one-one & onto correspondences between $\mathbb{N}$ and each of these sets. In other words, we must describe $\mathbb{Z}$ and $\mathbb{Q}$ as long lists

$$\{\mathbf{f}(1), \mathbf{f}(2), \ldots, \}.$$

For $\mathbb{Z}$, this is very easy

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots, \pm n, \ldots\}$$

for example, $0 = \mathbf{f}(1)$, $23 = \mathbf{f}(46)$, $-55 = \mathbf{f}(111)$. If we cared, **f** can even be made explicit.

A list description of $\mathbb{Q}$ is not much different. Each $x \in \mathbb{Q}$, can be written uniquely as

$$x = \pm\frac{p}{q} \quad | \quad p \geq 0, q > 0$$

$\gcd(p, q) = 1$ when $q \neq 0$. Define the finite subsets of $\mathbb{Q}$, $A_0 = \{0\}$, for $n \geq 1$

$$A_n = \left\{\pm\frac{p}{q} \quad | \quad p + q = n\right\}.$$

$$A_{10} = \{\pm 1/9, \pm 9/1, \pm 7/3, \pm 3/7\}$$

$$\mathbb{Q} = A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots$$

is a disjoint union of finite sets. Listing the elements of each A_n gives a desired listing for $\mathbb{Q}$. □

A more general argument is the following:

Theorem

If the sets A_i, $i \geq 1$, are countable, then $A = \bigcup_{i=1}^{\infty} A_i$ is countable.

Proof. Here is a way to list the elements of A. Since the A_i are countable, each comes with an injective mapping $\mathbf{f}_i : A_i \to \mathbb{N}$. We are going to define an injective mapping from A into the set $\mathbb{N} \times \mathbb{N}$. (By a previous exercise $\mathbb{N} \times \mathbb{N}$ is countable.) If $x \in A$, x belongs to some A_i and thus there exists an integer m such that

$$x \in A_m, \quad x \notin A_i, \quad i < m$$

Define $\mathbf{f} : A \to \mathbb{N} \times \mathbb{N}$ by the rule:

$$\mathbf{f}(x) = (m, \mathbf{f}_m(x)).$$

To verify that $\mathbf{f}$ is one-one we check:

$$\mathbf{f}(x) = \mathbf{f}(y)$$

means

$$(m, \mathbf{f}_m(x)) = (n, \mathbf{f}_n(y))$$

and thus

$$x \& y \in A_m = A_n$$

and therefore

$$\mathbf{f}_m(x) = \mathbf{f}_m(y)$$

implies that

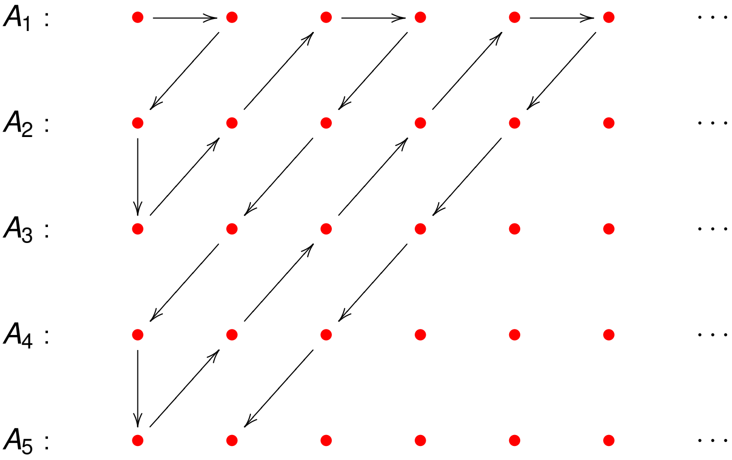
$$x = y$$

since $\mathbf{f}_m$ is one-one. □

Theorem

If the sets A_i, $i \geq 1$, are countable, then $A = \bigcup_{i=1}^{\infty} A_i$ is countable.

Proof. Here is a beautiful way to list the elements of A:



Exercise: Prove that the set $\mathbf{A}$ of finite subsets of $\mathbb{N}$ is countable.

Solution: Let $\mathbf{A}_n$ be the subset of $\mathbf{A}$ made up of subsets of $\mathbb{N}$ with n elements. Note that $\mathbf{A}_0 = \{\emptyset\}$ is not the empty set! and that

$$\mathbf{A} = \bigcup_{n \geq 0} \mathbf{A}_n.$$

To apply the theorem above, we prove that each $\mathbf{A}_n$ is countable. There are various ways to do it.

- The set of n-tuples of natural numbers

$$\mathbb{N}^n = \{(a_1, \ldots, a_n) \mid a_i \in \mathbb{N}\}$$

is countable, by the theorem.

- The set $\mathbf{A}_n$ is on a 1-1 correspondence with the n-tuples

$$\{(a_1, \ldots, a_n) \mid a_1 < a_2 < \cdots < a_n\}$$

so $\mathbf{A}_n$ is countable.

Algebraic Numbers

An (real) **algebraic** number is a real number $x \in \mathbb{R}$ that satisfies an equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0, \quad a_n \neq 0, a_i \in \mathbb{Q}$$

$\sqrt{2}$, $\sqrt[n]{3}$ are examples. Clear denominators, we may assume that all a_i are integers.

Exercise: The set of algebraic numbers is a field.

Theorem

*The set **A** of algebraic numbers is countable.*

Proof. For an integer m, let A_m be the set of all algebraic real numbers which are roots of equations such that

$$n + |a_n| + |a_{n-1}| + \cdots + |a_0| \leq m.$$

The number of such polynomials is finite, so the number of its roots is also finite. Since

$$\mathbf{A} = \bigcup_{m=1}^{\infty} A_m,$$

A is countable. □

Outline

Concepts Needed for Workshop

Draw pictures [whenever possible] of the following notions:

- Bounded sets
- Least Upper Bound: **LUB**
- Axiom of Completeness: Recall what it says
- Consider example:

$$\mathbf{A} = \{x_1, x_2, \ldots, \}, \quad x_1 = 1, \quad x_n = x_{n-1}/2 + 1, n \geq 2$$

$$\mathbf{A} = \{1, 3/2, 7/4, 15/8, \ldots\}$$

- Function

$$\mathbf{f}: \mathbf{X} \to \mathbf{Y}$$

  Subset $\mathbf{f}$ of $\mathbf{X} \times \mathbf{Y}$ with the properties:

  1. For each $x \in \mathbf{X}$ there is $y \in \mathbf{Y}$ such that $(x, y) \in \mathbf{f}$
  2. If $(x, y)\&(x, y') \in \mathbf{f}$, then $y = y'$

- Cardinality The sets **X** and **Y** have the same cardinality if there is a function

$$\mathbf{f}: \mathbf{X} \to \mathbf{Y}$$

  that is **one-one** and **onto.** Other terminology: injective and surjective

Workshop # 1

Problem 1: Find the least upper bound [if it exists] for the set of numbers $\mathbf{A} = \{x_1, \ldots, x_n, \ldots\}$, $x_n = \sqrt{2 + \sqrt{2 + \sqrt{\cdots + \sqrt{2}}}}$, n square roots.

- Label the terms of the sequence and look for relationships
- Prove the set is bounded
- Find the LUB
- Write solution

Workshop # 1, cont'd

Problem 2: The goal is to show that given $a < b$, there is a 1-1 onto function $\mathbf{f} : (a, b) \to [a, b]$, that is, these intervals are equivalent. We shall begin with the cases $a = 0$, $b = 1$.

Define $\mathbf{f} : (0, 1) \to \mathbb{R}$ as follows

$$\begin{aligned} \mathbf{f}(1/n) &= \frac{1}{n-1}, \quad n \in \mathbb{N}, n \geq 2 \\ \mathbf{f}(x) &= x \quad \text{otherwise} \end{aligned}$$

- Prove that **f** is 1-1 onto $(0, 1]$
- Find a 1-1 function from $[0, 1)$ onto $[0, 1]$
- Prove that $[0, 1)$ is equivalent to $(0, 1]$
- Prove that $(0, 1)$ is equivalent to $[0, 1]$
- For $a < b$, prove that (a, b) is equivalent to $[a, b]$

Exercise

Exercise: Consider the functions **f** and **g** defined as follows

$$\begin{aligned}\mathbf{f}(x) &= \frac{x-3}{x-2}, \quad x \in \mathbb{R}, x \neq 2 \\ \mathbf{g}(x) &= 3 - x, \quad x \in \mathbb{R}\end{aligned}$$

Find **all** the functions that are generated by composing **f** and **g**. It will be a finite number. You may to look at $\mathbf{f} \circ \mathbf{g}$, $\mathbf{g} \circ \mathbf{f}$, $\mathbf{f} \circ \mathbf{f}$, $\mathbf{g} \circ \mathbf{g}$, $\mathbf{f} \circ (\mathbf{g} \circ \mathbf{f})$, and so on. Make sure the compositions are valid.

Outline

Theorem

$\mathbb{R}$ is not countable.

Proof. Suppose we could list the real numbers

$$\mathbb{R} = \{x_1, x_2, x_3, \ldots\}$$

We are going to build a sequence I_n of nested intervals and derive a contradiction to (NIP).
Let $I_1 = [a_1, b_1]$ be an (non-empty) interval not containing x_1. Now, choose a subinterval I_2 of I_1 not containg x_2. This is clearly possible. Proceed in this fashion, for each $n > 1$ pick a subinterval I_n of I_{n-1} not containing x_n.

This produces a nested sequence $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ of non-empty intervals. By (NIP), $\bigcap I_n \neq \emptyset$. Let y be an element in this intersection. It must be one of the x_n, say $y = x_m$. But the intersection is contained in I_m, which does not contain x_m, by construction. $\square$

The cardinality of $\mathbb{R}$ is denoted $\mathrm{card}\,(\mathbb{R}) = c$, c for **continuum.** There are many unresolved questions about c.

Outline

Let us visit, if briefly, the garden universe that Cantor created for us. It was the first great theory of **infinities**, and has had a profound influence on Mathematics.

It helped that his constructions and proofs [sometimes the same thing] were often beautiful, if not even great fun.

We will touch on two of them.

Theorem (Cantor's Proof)

The interval $(0,1)$ *is not countable.*

Proof. It will suffice to show that the open interval $(0,1)$ is not countable. We are going to represent its elements as infinite decimals $x = 0.a_1a_2a_3\cdots a_n\cdots$. We are going to assume, by way of contradiction, that we can list them:

$$\begin{aligned}
x_1 &= 0.\mathbf{a_{11}}a_{12}a_{13}a_{14}\cdots \\
x_2 &= 0.a_{21}\mathbf{a_{22}}a_{23}a_{24}\cdots \\
x_3 &= 0.a_{31}a_{32}\mathbf{a_{33}}a_{34}\cdots \\
x_4 &= 0.a_{41}a_{42}a_{43}\mathbf{a_{44}}\cdots \\
\vdots &\quad\ \vdots
\end{aligned}$$

We are going, by focusing on the diagonal entries a_{nn}, give an an element $x \in (0,1)$ that is not listed.

Define the integer

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2 \end{cases}$$

Set $x = 0.b_1 b_2 b_3 b_4 \cdots b_n \cdots$. Note that x differs from x_n at the n decimal position. So x is not listed. $\square$

Definition

A set S has **cardinality** c iff S is equivalent to the open interval $(0,1)$; we write $\mathrm{card}\,(S) = \mathbf{c}$.

Theorem

The set $\mathbb{R}$ *is uncountable and has cardinality* $\mathbf{c}$.

Proof.

Define $\mathbf{f} : (0,1) \to \mathbb{R}$ by $\mathbf{f}(x) = \tan(\pi x - \pi/2)$. Look at its graph.

□

Exercise

Claim: $(0,1) \times (0,1) \approx (0,1)$, that is the interior of the unit square is equivalent to $(0,1)$. Another form; $\mathbb{R} \times \mathbb{R} \approx \mathbb{R}$.
An element $(a,b) \in (0,1) \times (0,1)$ can be described as

$$\begin{aligned} a &= 0.a_1 a_2 a_3 \ldots a_n \ldots \\ b &= 0.b_1 b_2 b_3 \ldots b_n \ldots \end{aligned}$$

Define the function $\mathbf{f}(a,b) = c \in (0,1)$ where

$$c = 0.a_1 b_1 a_2 b_2 \ldots a_n b_n \ldots$$

f is one-to-one and onto.

Power Sets

If $\mathbf{X}$ is a set, the collection of its subsets is called the **power set** of $\mathbf{X}$: notation $P(\mathbf{A})$.
If $\mathbf{X} = \{0, 1\}$, its subsets are

$$P(\mathbf{X}) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}.$$

One way to represent a subset $A \subset \mathbf{X}$ is as a function

$$\mathbf{f}_A : \mathbf{X} \to \{0, 1\}$$

$$\mathbf{f}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

This leads to the notation $P(\mathbf{X}) = 2^{\mathbf{X}}$.

If $\mathbf{X} = \{x_1, \ldots, x_n\}$, we can also represent its subsets by ordered strings of 0's and 1's as follows:

$$A \leftrightarrow (a_1, a_2, \ldots, a_n)$$

$$a_i = \begin{cases} 1 & \text{if } x_i \in A \\ 0 & \text{if } x_i \notin A \end{cases}$$

This shows that

$$\text{card}\,(P(\mathbf{X})) = 2^{\text{card}\,(\mathbf{X})} = 2^n$$

Cantor's Theorem

The following shows how to build larger infinities from given ones.

Theorem

Given a set $\mathbf{X}$ *there is no function* $\mathbf{f} : \mathbf{X} \to P(\mathbf{X})$ *that is onto.*

Proof. Suppose $\mathbf{f}$ is such a function: For each $a \in \mathbf{X}$, $\mathbf{f}(a)$ is a subset of $\mathbf{X}$ and any subset is a target. Let us build a subset that is not a target.

For each $a \in \mathbf{X}$, $a \in \mathbf{f}(a)$ or $a \notin \mathbf{f}(a)$. Define the subset

$$B = \{a \in \mathbf{X} \mid a \notin \mathbf{f}(a)\}$$

By assumption, $B = \mathbf{f}(x)$ for some $x \in \mathbf{X}$.

Now look how cool:
$x \in \mathbf{f}(x) = B$, contradicts the definition of B, while
$x \notin \mathbf{f}(x) = B$, would make $x \in B$, by the definition of B. $\square$

Outline

Concepts Needed for Workshop

- What are Countable Sets? View some examples
- Counting Techniques: Review the Beautiful zig-zag [diagonal] proof
- Visit Example: $\mathbb{Q}$
- Supremum and Infimum: Least Upper Bound and Greatest Lower Bound of sets of real numbers

Workshop # 2

1. Prove that the set of all polynomials $\mathbb{Q}[x]$ is countable.
2. Prove that the set of all polynomials $\mathbb{Q}[x, y]$ is countable.
3. Consider the following statements about subsets of real numbers and decide whether they are true or false. In the latter case, provide a counterexample:

   (a) A finite, nonempty set always contains its supremum.

   (b) If A and B are sets with the property that $a < b$ for every $a \in A$ and every $b \in B$, then it follows that $\sup A < \inf B$.

   (c) If $\sup A = s$ and $\sup B = t$, then $\sup(A + B) = s + t$. [If A and B are sets of real numbers, their sum is defined as follows: $A + B = \{a + b \mid a \in A, b \in B\}$.]

Some Exercises

Establish a 1–1 correspondence with a set of known cardinality:

1. Is the set of all functions from $\{0, 1\}$ to $\mathbb{N}$ countable or uncountable?
2. Is the set of all functions from $\mathbb{N}$ to $\{0, 1\}$ countable or uncountable?