Math 300–03

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Set 5

Fall 2008

Outline

Equivalence of Sets: Foundation to Counting

Let us begin by introducing a method to size sets. If A and B are two sets we will use functions

$$\mathbf{f} : A \longrightarrow B$$

to compare their sizes.

Definition

For pair of sets (A, B) we write $A \approx B$ if there is a function $\mathbf{f} : A \rightarrow B$ that is both **one-to-one** and **onto**.

Recall

- **f one-to-one**: If $x \neq y \Rightarrow \mathbf{f}(x) \neq \mathbf{f}(y)$
- In particular if $\mathbf{f} : A \to B$ and $\mathbf{g} : B \to C$ are one-to-one

$$x \neq y \Rightarrow \mathbf{f}(x) \neq \mathbf{f}(y) \Rightarrow \mathbf{g}(\mathbf{f}(x)) \neq \mathbf{g}(\mathbf{f}(y)),$$

  so $\mathbf{g} \circ \mathbf{f}$ is one-to-one.

- **f onto**: $\forall b \in B \quad \exists x \in A : \mathbf{f}(x) = b$
- In particular if $\mathbf{f} : A \to B$ and $\mathbf{g} : B \to C$ are onto

$$\forall c \in C \quad \exists b \in B : \mathbf{g}(b) = c \quad \exists a \in A : \mathbf{f}(a) = b.$$

  Thus $\mathbf{g}(\mathbf{f}(a)) = \mathbf{g}(b) = c$ and so $\mathbf{g} \circ \mathbf{f}$ is onto.

Proposition

$\approx$ is an equivalence relation.

Proof. Let us verify the requirements:

1. (reflexivity) $A \approx A$: because $I_A : A \to A$ is one-to-one onto.
2. (symmetry) $A \approx B \Rightarrow B \approx A$: because if $\mathbf{f} : A \to B$ is one-to-one onto then $\mathbf{f}^{-1} : B \to A$ is one-to-one onto.
3. (transitivity) If $A \approx B$ and $B \approx C$ then $A \approx C$: because if $\mathbf{f} : A \to B$ is one-to-one onto and $\mathbf{g} : B \to C$ is one-to-one onto then $\mathbf{g} \circ \mathbf{f} : A \to C$ is one-to-one onto.

Definition

The **equivalence class** of A is called the **cardinality** of A, $\mathrm{card}\,(A)$.

Let E be the set of even numbers,

$$E = \{2, 4, \ldots, 2n, \ldots\}$$

The function $\mathbf{f} : \mathbb{N} \to E$, given by $\mathbf{f}(n) = 2n$, gives a one-to-one & onto correspondence between the sets $\mathbb{N}$ and E.

We write this as $\mathrm{card}\,(E) = \mathrm{card}\,(\mathbb{N})$: There are as many even numbers as natural numbers...

Equivalence of Sets

Definition

Two sets A and B are **equivalent** iff there exists a one-to-one function from A onto B, and denote $A \approx B$.

Example: The set E of even numbers is equivalent to the set O of odd numbers:

$$\mathbf{f} : E \to O, \quad \mathbf{f}(2n) = 2n - 1, \quad n \in \mathbb{N}.$$

Example

Theorem

For $a, b, c, d \in \mathbb{N}$, with $a < b$ and $c < d$, the open intervals (a, b) and (c, d) are equivalent.

Proof. Let **f** be the linear function

$$\mathbf{f}(x) = \frac{d-b}{c-a}(x-a)+c.$$

We must show that $\mathbf{f} : (a, b) \to (c, d)$ is one-to-one and onto.

In some cases, [the case above included], it is possible to build $\mathbf{f}^{-1}$ by solving the equation for x

$$\mathbf{f}(x) = y, \quad x = \mathbf{f}^{-1}(y).$$

$$y = \frac{d-b}{c-a}(x-a) + c,$$

gives

$$x - a = \frac{c-a}{d-b}(y-c)$$

$$x = \mathbf{f}^{-1}(y) = \frac{c-a}{d-b}(y-c) + a$$

$(0, \infty) \approx [0, \infty)$

Split $(0, \infty)$ and $[0, \infty)$ as follows

$$
\begin{aligned}
(0, \infty) &= (0,1) \cup \{1\} \cup (1,2) \cup \{2\} \cup (2,3) \cup \{3\} \cup \cdots \\
\{0\} \cup (0, \infty) &= \{0\} \cup (0,1) \cup \{1\} \cup (1,2) \cup \{2\} \cup (2,3) \cup \cdots
\end{aligned}
$$

Define the function

$$
\begin{aligned}
\mathbf{f}(n) &= n - 1 \\
\mathbf{f}(x) &= x
\end{aligned}
$$

for all other x.

Exercise: Will be in next Homework

As a challenge, prove

Theorem

For $a, b \in \mathbb{R}$, with $a < b$, the intervals (a, b) and $[a, b]$ are equivalent.

Marvelous Example

Claim: Let $\mathcal{F}$ be the set of functions from $\mathbb{N}$ to the set of two elements $\{0, 1\}$. Then $\mathcal{F} \approx \mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$.
Define the correspondence

$$\mathbf{F} : \mathcal{F} \to \mathcal{P}(\mathbb{N}), \quad \mathbf{F}(\mathbf{g}) = \{x \in \mathbb{N} : \mathbf{g}(x) = 1\}.$$

1. One-to-one: If $\mathbf{f}$ and $\mathbf{g}$ are different functions, then there is $x \in \mathbb{N}$ so that $\mathbf{f}(x) \neq \mathbf{g}(x)$. This means one of these values is 1, the other is 0. Thus the sets $\mathbf{F}(\mathbf{f})$ and $\mathbf{F}(\mathbf{g})$ different.
2. Onto: Let A be a subset of $\mathbb{N}$. Let χ_A be the characteristic function of A (someone recalls?) $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise. Note that $\mathbf{F}(\chi_A) = A$.

Product/Sums Rules

Theorem

Suppose A, B, C and D are sets and $A \approx C$ and $B \approx D$. Then

1. $A \times B \approx C \times D$.
2. *If A and B are disjoint and C and D are disjoint, then* $A \cup B \approx C \cup D$.

Proof. Let $\mathbf{f} : A \to C$ and $\mathbf{g} : B \to D$ be one-to-one and onto functions.

1. Let $\mathbf{h} : A \times B \to C \times D$ be given by $\mathbf{h}(a, b) = (\mathbf{f}(a), \mathbf{g}(b))$. It is easy to verify that $\mathbf{h}$ is one-to-one and onto.
2. We can glue the functions $\mathbf{f}$ and $\mathbf{g}$: $\mathbf{f} \cup \mathbf{g} : A \cup B \to C \cup D$, so that if $a \in A$, $(\mathbf{f} \cup \mathbf{g})(a) = \mathbf{f}(a)$, while if $b \in B$, $(\mathbf{f} \cup \mathbf{g})(b) = \mathbf{g}(b)$. Again, it is clear that $\mathbf{f} \cup \mathbf{g}$ is one-to-one and onto.

Those rules extend to other products and sums:

Theorem

Suppose $A_1, A_2, \ldots, A_n$ and $C_1, C_2, \ldots, C_n$ are two families of sets and for all i, $A_i \approx C_i$. Then

1. $A_1 \times A_2 \times \cdots \times A_n \approx C_1 \times C_2 \times \cdots \times C_n$.
2. *If the A_i are disjoint and the C_i are disjoint, then* $A_1 \cup A_2 \cup \cdots \cup A_n \approx C_1 \cup C_2 \cup \cdots \cup C_n$.

The proof earlier will work. Even works for arbitrary collections of sets.

Finite and Countable Sets

We use the following notation:

$$\mathbb{N}_n = \{1, 2, \ldots, n\} \subset \mathbb{N}$$

and the following terminology

Definition

A set S is **finite** if $S = \emptyset$ or $S \approx \mathbb{N}_k$ for some natural number k. A set S is **infinite** if S is not finite.

The attending class today is finite, since we can set a correspondence between it and some $\mathbb{N}_k$ ($k \leq 18$).

Definition

Let S be a finite set. If $S \approx \mathbb{N}_k$, $k \in \mathbb{N}$, we say that S has **cardinal number** k (or **cardinality** k), and write $\overline{\overline{S}} = k$. If $S = \emptyset$ we say that S has **cardinal number** 0 (or **cardinality** 0) and write $\overline{\overline{\emptyset}} = 0$.

Definition

A set A is said to be **countable**, or **denumerable**, if $A \approx \mathbb{N}$:

$$\mathbf{f} : \mathbb{N} \to A$$

$$A = \{\mathbf{f}(1), \mathbf{f}(2), \ldots, \}.$$

We write that $\mathrm{card}\,(A) = \mathrm{card}\,(\mathbb{N}) = \aleph_0$, and say that A has **cardinal number** $\aleph_0$ and write $\overline{\overline{A}} = \aleph_0$.

Warning about Terminology: The correct usage is to call a set **countable** if it is equivalent to $\mathbb{N}$ or finite. We abuse this often by the definition above.

Exercise: If $\mathrm{card}\,(A)$ is countable and $B \subset A$, then B is countable or finite.

It is obviously a tricky thing to determine the cardinality of sets, particularly of infinite sets. Let us get our hands busy!

Pre-Exercise, i.e. a Warm-up

Question: Why/How can we list a subset A of the natural numbers $\mathbb{N}$?

1. If $A = \emptyset$, there is nothing to do.
2. If A is not empty, let a_1 be the smallest element of A. (someone: why can we do this?)
3. Let $A_1 = A \setminus \{a_1\}$. If $A_1 = \emptyset$ we are done; otherwise let a_2 be its smallest element.
4. Let $A_2 = A \setminus \{a_1, a_2\}$. If $A_2 = \emptyset$ we are done; otherwise let a_3 be its smallest element.
5. In this manner we list the elements of A:

$$a_1, a_2, a_3, \cdots$$

6. If the list stops at a_n, we have a one-to-one correspondence $\mathbf{f} : \{1, 2, \ldots, n\} \to A$, $\mathbf{f}(i) = a_i, i \leq n$.
7. Otherwise we have a one-to-one correspondence $\mathbf{f} : \mathbb{N} \to A$, $\mathbf{f}(i) = a_i, i \in \mathbb{N}$.

Exercise

The set $\mathbb{N} \times \mathbb{N}$ is countable: Let define a one-one function

$$\mathbf{f} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

Define

$$\mathbf{f}(m, n) = 2^m 3^n$$

By the unique factorization on integers,

$$2^m 3^n = 2^p 3^q \Rightarrow m = p \quad n = q,$$

which proves the claim that **f** is injective.

Exercise

Exercise: Use the infinity of prime numbers to show that the set $\mathbf{X}$ of all infinite tuples $(x_1, x_2, x_3, \ldots)$, $x_i \in \mathbb{N}$, such that all $x_i = 0$ except for finitely many exceptions is countable.

Let P be the set of primes, $P = \{p_1, p_2, p_3, \ldots, p_n, \ldots\}$.

Now define the function $\mathbf{f} : \mathbf{X} \to \mathbb{N}$ by the rule

$$\mathbf{f}(x_1, x_2, \ldots, x_n, \ldots) = p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n} \cdots .$$

$\mathbf{f}$ is well-defined because almost all x_i are 0. $\mathbf{f}$ is one-to-one by the unique factorization of integers by primes.

A variant of this Example

In one of our examples weeks back, we considered the function

$$\mathbf{f} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N},$$

given by

$$\mathbf{f}(m, n) = 2^{m-1}(2n - 1).$$

We proved that **f** is one-to-one & onto.

Outline

Homework #12

1. Prove that $[0, 1] \approx (0, 1)$
2. 5.1: 3(a, i, n), 6(b), 17(a,b), 20(b)

Pigeonhole Principle

Theorem

Let $n, r \in \mathbb{N}$. *If* $\mathbf{f} : \mathbb{N}_n \to \mathbb{N}_r$ *and* $n > r$ *then* $\mathbf{f}$ *is not one-to-one.*

Proof of the Pigeonhole Principle

We prove this by induction on n.

1. If $n = 2$, since $r < n$, $r = 1$. In this case $\mathbf{f}$ is a constant function, $\mathbf{f}(1) = \mathbf{f}(2) = 1$, so $\mathbf{f}$ is not one-to-one.
2. Suppose the Pigeonhole Principle holds for all $r < n$. We argue by contradiction. Suppose $r < n+1$ and $\mathbf{h} : \mathbb{N}_{n+1} \to \mathbb{N}_r$ is one-to-one. The restriction $\mathbf{h}_0$ of $\mathbf{h}$ to $\mathbb{N}_n$ is one-to-one. Furthermore the range of this function does not contain $\mathbf{h}(n+1)$ **Why Someone?**
3. There is a one-to-one function $\mathbf{g} : \mathbb{N}_r \setminus \{\mathbf{h}(n+1)\} \to \mathbb{N}_{r-1}$. Let $\mathbf{f} = \mathbf{g} \circ \mathbf{h}_0$. Thus $\mathbf{f} : \mathbb{N}_n \to \mathbb{N}_{r-1}$ is one-to-one since it is the composite of one-to-one functions. Thus is a contradiction of the induction hypothesis.
4. By the **PMI**, for every $n \in \mathbb{N}$ if $r < n$ there is no one-to-one function from $\mathbb{N}_n$ to $\mathbb{N}_r$.

Exercise

5.1, 20(a): Prove that if five points are in or on a square with sides of length 1, then at least two points are no farther apart than $\sqrt{2}/2$.

For instance, if 4 points are chosen at the vertices then the fifth point must be chosen in one of the 4 triangles determined by the center. The distance of that point to one of the corner points is at most $sqrt2/2$.

Solution: To use the Pigeonhole Principle, split the square into 4 squares of sides of length $1/2$. According to the Pigeonhole Principle, we would have to put at least two points in the same little square: they could not be further apart than $\sqrt{2}/2$.

Corollaries of the Pigeonhole Principle

- Let A be a finite set, that is $A \approx \mathbb{N}_n$ for some n. If $A \approx \mathbb{N}_m$ then $m = n$.

  Proof: The first hypothesis means: There is $\mathbf{f} : \mathbb{N}_n \to A$ that is one-to-one. The second hypothesis means: There is $\mathbf{h} : A \to \mathbb{N}_m$ that is one-to-one. It follows that

  $$\mathbf{h} \circ \mathbf{f} : \mathbb{N}_n \to \mathbb{N}_m$$

  is one-to-one. Therefore $n \geq m$. Reverse the roles of m and n to get $m \geq n$. Thus $m = n$.

Corollary to Pigeonhole Principle

Corollary

A finite set is not equivalent to any of its proper subsets.

Proof: We first show that the set $\mathbb{N}_k$ is not equivalent to any of its proper subsets.
If $k = 1$, the only proper subset of $\mathbb{N}_k$ is $\emptyset$ and $\{1\}$ is not equivalent to $\emptyset$. Assume $k > 1$ and A is a proper subset of $\mathbb{N}_k$ and $\mathbf{f} : \mathbb{N}_k \approx A$ is one-to-one onto.
There are two cases to consider:

- If $k \notin A$, then $A \subset \mathbb{N}_{k-1}$, and the inclusion function $i : A \to \mathbb{N}_{k-1}$ is one-to-one. But then the composite $i \circ \mathbf{f} : \mathbb{N}_k \to \mathbb{N}_{k-1}$ would be one-to-one, violating the Pigeonhole Principle.

- Suppose $k \in A$. Choose $y \in \mathbb{N}_k \setminus A$. Let $A' = (A \setminus \{k\}) \cup \{y\}$. Then $A \approx A'$ as we simply exchanged k by y in A. Thus $A' \approx \mathbb{N}_k$. From the previous case we get a contradiction.

Outline

Harder Examples

Theorem

The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countable.

We must establish one-one & onto correspondences between $\mathbb{N}$ and each of these sets. In other words, we must describe $\mathbb{Z}$ and $\mathbb{Q}$ as long lists

$$\{\mathbf{f}(1), \mathbf{f}(2), \ldots, \}.$$

For $\mathbb{Z}$, this is very easy

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots, \pm n, \ldots\}$$

for example, $0 = \mathbf{f}(1)$, $23 = \mathbf{f}(46)$, $-55 = \mathbf{f}(111)$. If we cared, **f** can even be made explicit.

A list description of $\mathbb{Q}$ is not much different. Each $x \in \mathbb{Q}$, can be written uniquely as

$$x = \pm\frac{p}{q} \quad | \quad p \geq 0, q > 0$$

$\gcd(p, q) = 1$ when $q \neq 0$. Define the finite subsets of $\mathbb{Q}$, $A_0 = \{0\}$, for $n \geq 1$

$$A_n = \left\{\pm\frac{p}{q} \quad | \quad p + q = n\right\}.$$

$$A_{10} = \{\pm 1/9, \pm 3/7\}$$

$$\mathbb{Q} = A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots$$

is a disjoint union of finite sets. Listing the elements of each A_n gives a desired listing for $\mathbb{Q}$. □

A more general argument is the following:

Theorem

If the sets A_i, $i \geq 1$, are countable, then $A = \bigcup_{i=1}^{\infty} A_i$ is countable.

Proof. Here is a way to list the elements of A. Since the A_i are countable, each comes with an injective mapping $\mathbf{f}_i : A_i \to \mathbb{N}$. We are going to define an injective mapping from A into the set $\mathbb{N} \times \mathbb{N}$. (By a previous exercise $\mathbb{N} \times \mathbb{N}$ is countable.) If $x \in A$, x belongs to some A_i and thus there exists an integer m such that

$$x \in A_m, \quad x \notin A_i, \quad i < m$$

Define $\mathbf{f} : A \to \mathbb{N} \times \mathbb{N}$ by the rule:

$$\mathbf{f}(x) = (m, \mathbf{f}_m(x)).$$

To verify that **f** is one-one we check:

$$\mathbf{f}(x) = \mathbf{f}(y)$$

means

$$(m, \mathbf{f}_m(x)) = (n, \mathbf{f}_n(y))$$

and thus

$$x \& y \in A_m = A_n$$

and therefore

$$\mathbf{f}_m(x) = \mathbf{f}_m(y)$$

implies that

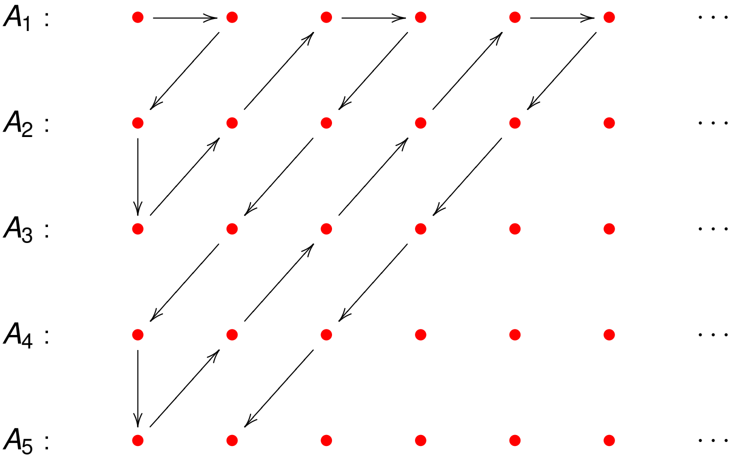
$$x = y$$

since $\mathbf{f}_m$ is one-one. □

Theorem

If the sets A_i, $i \geq 1$, are countable, then $A = \bigcup_{i=1}^{\infty} A_i$ is countable.

Proof. Here is a beautiful way to list the elements of A:



Exercise: Prove that the set $\mathbf{A}$ of finite subsets of $\mathbb{N}$ is countable.

Solution: Let $\mathbf{A}_n$ be the subset of $\mathbf{A}$ made up of subsets of $\mathbb{N}$ with n elements. Note that $\mathbf{A}_0 = \{\emptyset\}$ is not the empty set! and that

$$\mathbf{A} = \bigcup_{n \geq 0} \mathbf{A}_n.$$

To apply the theorem above, we prove that each $\mathbf{A}_n$ is countable. There are various ways to do it.

- The set of n-tuples of natural numbers

$$\mathbb{N}^n = \{(a_1, \ldots, a_n) \mid a_i \in \mathbb{N}\}$$

is countable, by the theorem.

- The set $\mathbf{A}_n$ is on a 1-1 correspondence with the n-tuples

$$\{(a_1, \ldots, a_n) \mid a_1 < a_2 < \cdots < a_n\}$$

so $\mathbf{A}_n$ is countable.

Uncountable Sets

Definition

A set S is **uncountable** if it is neither finite nor denumerable.

Question: Are there such sets?

Outline

Let us visit, if briefly, the garden universe that Cantor created for us. It was the first great theory of **infinities**, and has had a profound influence on Mathematics.
It helped that his constructions and proofs [sometimes the same thing] were often beautiful, if not even great fun.
We will touch on two of them.

Theorem (Cantor's Proof)

The interval $(0,1)$ *is not countable.*

Proof. It will suffice to show that the open interval $(0,1)$ is not countable. We are going to represent its elements as infinite decimals $x = 0.a_1a_2a_3\cdots a_n\cdots$. We are going to assume, by way of contradiction, that we can list them:

$$\begin{aligned}
x_1 &= 0.\mathbf{a_{11}}a_{12}a_{13}a_{14}\cdots \\
x_2 &= 0.a_{21}\mathbf{a_{22}}a_{23}a_{24}\cdots \\
x_3 &= 0.a_{31}a_{32}\mathbf{a_{33}}a_{34}\cdots \\
x_4 &= 0.a_{41}a_{42}a_{43}\mathbf{a_{44}}\cdots \\
\vdots & \qquad \vdots
\end{aligned}$$

We are going, by focusing on the diagonal entries a_{nn}, give an element $x \in (0,1)$ that is not listed.

Define the integer

$$b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2 \end{cases}$$

Set $x = 0.b_1 b_2 b_3 b_4 \cdots b_n \cdots$. Note that x differs from x_n at the n decimal position. So x is not listed. □

Definition

A set S has **cardinality** c iff S is equivalent to the open interval $(0, 1)$; we write $\mathrm{card}\,(S) = \mathbf{c}$.

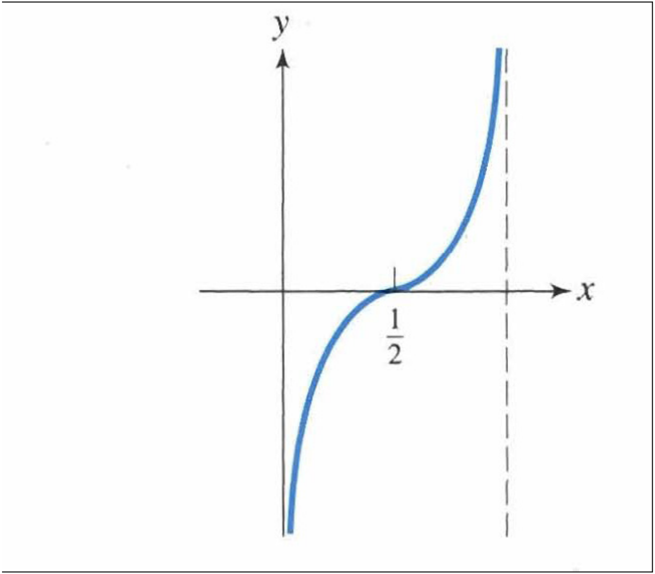
Theorem

The set $\mathbb{R}$ is uncountable and has cardinality $\mathbf{c}$.

Proof.

Define $\mathbf{f} : (0, 1) \to \mathbb{R}$ by $\mathbf{f}(x) = \tan(\pi x - \pi/2)$. Look at the graph: $\square$

$\tan(\pi x - \pi/2) : (0, 1) \approx \mathbb{R}$



Exercise

Claim: $(0,1) \times (0,1) \approx (0,1)$, that is the interior of the unit square is equivalent to $(0,1)$. Another form; $\mathbb{R} \times \mathbb{R} \approx \mathbb{R}$.
An element $(a,b) \in (0,1) \times (0,1)$ can be described as

$$\begin{aligned} a &= 0.a_1 a_2 a_3 \ldots a_n \ldots \\ b &= 0.b_1 b_2 b_3 \ldots b_n \ldots \end{aligned}$$

Define the function $\mathbf{f}(a,b) = c \in (0,1)$ by

$$c = 0.a_1 b_1 a_2 b_2 \ldots a_n b_n \ldots$$

f is one-to-one and onto.

Power Sets

If **X** is a set, the collection of its subsets is called the **power set** of **X**: notation $P(\mathbf{A})$.

If $\mathbf{X} = \{0, 1\}$, its subsets are

$$P(\mathbf{X}) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}.$$

One way to represent a subset $A \subset \mathbf{X}$ is as a function

$$\mathbf{f}_A : \mathbf{X} \rightarrow \{0, 1\}$$

$$\mathbf{f}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

This leads to the notation $P(\mathbf{X}) = 2^{\mathbf{X}}$.

If $\mathbf{X} = \{x_1, \ldots, x_n\}$, we can also represent its subsets by ordered strings of 0's and 1's as follows:

$$A \leftrightarrow (a_1, a_2, \ldots, a_n)$$

$$a_i = \begin{cases} 1 & \text{if } x_i \in A \\ 0 & \text{if } x_i \notin A \end{cases}$$

This shows that

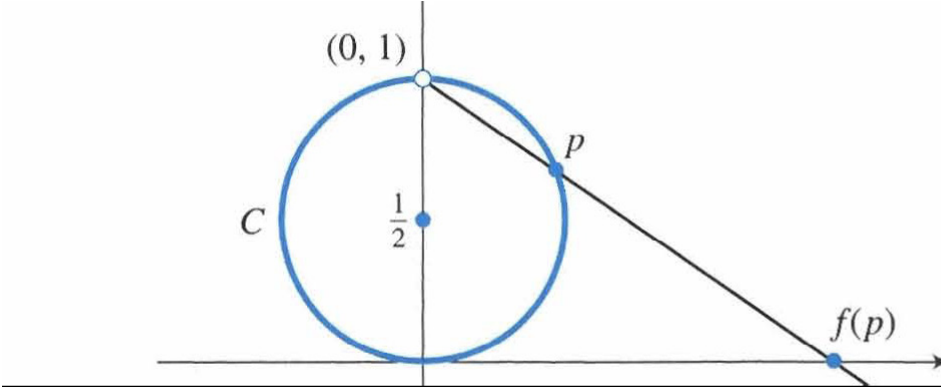
$$\text{card}\,(P(\mathbf{X})) = 2^{\text{card}\,(\mathbf{X})} = 2^n$$

Exercise

Prove the following statements:

- All circles of positive radius are equivalent.
- The circle $(x^2 + (y - 1/2)^2 = 1/4$ is equivalent to $\mathbb{R}$.

$\mathbb{R} \approx$ Circles



Outline

Homework #13

1. 5.1: 3(a, i, n), 6(b), 17(a,b), 20(a)
2. 5.2: 1(g), 5(a, d, e), 10

Outline

The Ordering of Cardinal Numbers

Cantor's Theorem

The following shows how to build larger infinities from given ones.

Theorem

Given a set $\mathbf{X}$ *there is no function* $\mathbf{f} : \mathbf{X} \to P(\mathbf{X})$ *that is onto.*

Proof. Suppose $\mathbf{f}$ is such a function: For each $a \in \mathbf{X}$, $\mathbf{f}(a)$ is a subset of $\mathbf{X}$ and any subset is a target. Let us build a subset that is not a target.

For each $a \in \mathbf{X}$, $a \in \mathbf{f}(a)$ or $a \notin \mathbf{f}(a)$. Define the subset

$$B = \{a \in \mathbf{X} \mid a \notin \mathbf{f}(a)\}$$

By assumption, $B = \mathbf{f}(x)$ for some $x \in \mathbf{X}$.

Now look how cool:
$x \in \mathbf{f}(x) = B$, contradicts the definition of B, while
$x \notin \mathbf{f}(x) = B$, would make $x \in B$, by the definition of B. $\square$

Ever larger cardinals

A consequence of Cantor's Theorem is to provide chains of increasing cardinals:

$$\aleph_0 = \overline{\overline{\mathbb{N}}} < \overline{\overline{\mathcal{P}(\mathbb{N})}} < \overline{\overline{\mathcal{P}(\mathcal{P}(\mathbb{N}))}} < \overline{\overline{\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))}} < \cdots$$

Are they equal?

The cardinality of $\mathbb{N}$ is $\aleph_0$, while we have just proved that

$$\aleph_1 = \mathrm{card}\,(\mathcal{P}(\mathbb{N})) \neq \mathrm{card}\,(\mathbb{N})$$

We have two infinite sets with well-understood cardinalities larger that $\aleph_0$: $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$ which has cardinality **c**. One of the most famous unsolved problems of Mathematics is: True or False

Continuum Hypothesis: $\mathcal{P}(\mathbb{N}) \approx \mathbb{R}$

Cantor-Schröder-Bernstein Theorem

Theorem

If $\overline{\overline{A}} \leq \overline{\overline{B}}$ and $\overline{\overline{B}} \leq \overline{\overline{A}}$, then $\overline{\overline{A}} = \overline{\overline{B}}$.

Fig 511

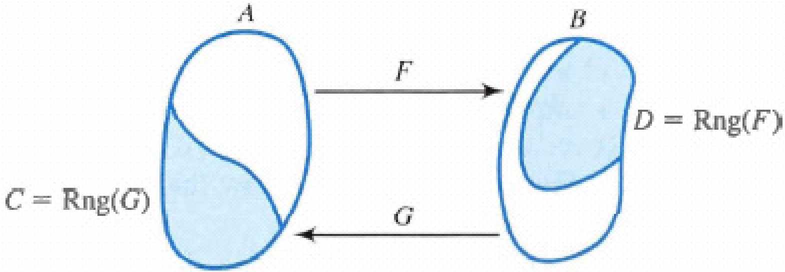


Figure 5.11

Fig 512

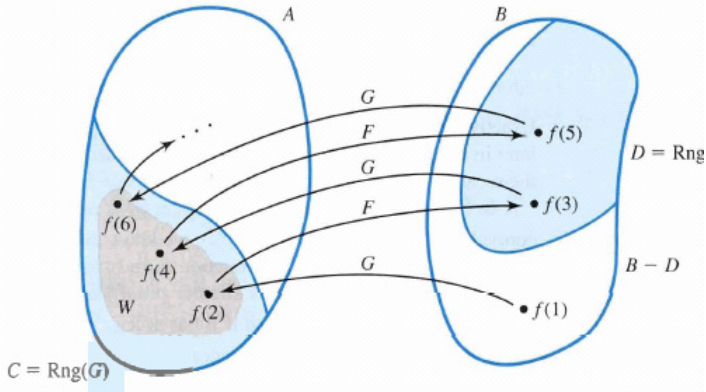


Figure 5.12 String f: $f(1), f(2), f(3), f(4), f(5), f(6), \ldots$

Fig 513

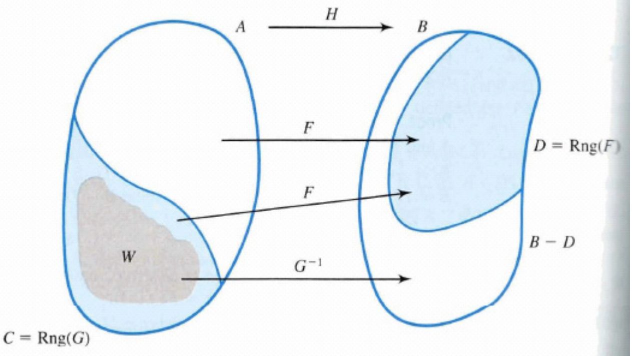


Figure 5.13

Outline

Final Orientation

Final will be comprehensive but topics will be emphasized according to the folliowing classification:

- **VITs: Very Important Topics**
- **BITs: Basic Important Topics**
- **LITs: Basic but Less Important Topics**

VITs

- Propositions, Truth tables
- Basic Methods of Proof
- Mathematical Induction (PMI, PCI, Well-Ordering)
- Relations, Equivalence Relations, Classes of
- Functions: Ingridients and Important Types (1-1, onto)
- Cardinality
- Finite, Countable and Uncountable Sets

BITs

- Logical connectives, quantifiers
- Set Theory/Operations
- Principles of Counting
- More relations, Partitions
- Constructions of Functions
- Functions from Calculus
- Review homework

LITs

- Graphs
- Names to recall: Venn, Fibonacci, Cantor
- Examples in slides
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