Math 300–03

Wolmer V. Vasconcelos

Set 4

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Outline

1. What is a Function?
2. Building Functions
3. Homework #9
4. Onto and One-to-One Functions
5. Homework #10
6. Last Class...and Today...
7. Images of Sets
8. Homework #11
9. Sequences

What is a Function?

- The beginning: Leibniz, Euler (who invented the notation $\mathbf{f}(x)$)
- Definition of function uses notion of Relation: subset of $\mathbf{A} \times \mathbf{B}$

Cartesian Product

Definition

Let **A** and **B** be sets. The set of all ordered pairs having first coordinate in **A** and second coordinate in **B** is called the **Cartesian product** of **A** and **B** and written $\mathbf{A} \times \mathbf{B}$. Thus

$$\mathbf{A} \times \mathbf{B} = \{(a, b) : a \in \mathbf{A} \quad \text{and} \quad b \in \mathbf{B}.\}$$

Example: Let $\mathbf{A} = \{a, b\}$, $\mathbf{B} = \{1, 2, 3\}$. Then:

$$\mathbf{A} \times \mathbf{B} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}.$$

Function

Definition

A **function** (or **mapping**) **A to B** is a relation **f** from **A** to **B** such that

1. the **domain** of **f** is **A**: $Dom(\mathbf{f}) = \mathbf{A}$
2. if $(x, y) \in \mathbf{f}$ and $(x, z) \in \mathbf{f}$, then $y = z$.
3. Convenient notation is $\mathbf{f} : \mathbf{A} \to \mathbf{B}$, and we read "**f** is a function from **A** to **B**", or "**f** maps **A** to **B**". The set **B** is called the **codomain** of **f**.
4. When $\mathbf{A} = \mathbf{B}$, **f** is called a **function on A.**

Example

Let $\mathbf{A} = \{1, 2, 3\}$ and $\mathbf{B} = \{a, b, c\}$. Here are some relations (subsets of $\mathbf{A} \times \mathbf{B}$):

$$\begin{aligned} R_1 &= \{(1, a), (2, b), (3, c), (2, c)\} \\ R_2 &= \{(1, a), (2, b), (3, b)\} \\ R_3 &= \{(1, b), (2, c), (3, b)\} \\ R_4 &= \{(1, a), (3, c)\} \end{aligned}$$

Which relation(s) are functions? Note the basic requirement:

$$(x, y) \in R \quad \vee \quad (x, z) \in R \Rightarrow y = z.$$

This condition is known as the **vertical line test.**

Example

1. Let $H = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$. Is H a function with domain $[-1, 1]$? Note that elements

$$(\sqrt{2}/2, \sqrt{2}/2) \quad (\sqrt{2}/2, -\sqrt{2}/2)$$

   are in H, so the requirement fails.
2. If we consider the subset $H_0 = [0, 1] \times [0, 1]$, for a given first coordinate x, the second coordinate is uniquely given as $y = \sqrt{1 - x^2}$. So H_0 is a function.

Example

Proposition

The set $H = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x + y = 7\}$ *is a function from the set* $\{1, 2, 3, 4, 5, 6\}$ *to* $\mathbb{N}$.

Proof.

First, note that H is a relation from $\mathbb{N}$ to $\mathbb{N}$.

1. Suppose $x \in \{1, 2, 3, 4, 5, 6\}$. Then $7 - x$ is a natural number and $(x, 7 - x) \in H$. Thus $\{1, 2, 3, 4, 5, 6\} \subset Dom(H)$. Suppose $x \in Dom(H)$ and $(x, y) \in H$ for some $y \in \mathbb{N}$. thus $x \leq 6$, so $x \in \{1, 2, 3, 4, 5, 6\}$. Therefore, $Dom(h) = \{1, 2, 3, 4, 5, 6\}$.
2. Suppose $(x, y) \in H$ and $(x, z) \in H$. Then $y = 7 - x$ and $z = 7 - x$, so $y = z$.

By (1) and (2), $H : \{1, 2, 3, 4, 5, 6\} \to \mathbb{N}$. □

Definition

Let $\mathbf{f} : \mathbf{A} \rightarrow \mathbf{B}$. We write $y = \mathbf{f}(x)$ when $(x, y) \in \mathbf{f}$. We say that y is the the value of $\mathbf{f}$ and x (or the image of $\mathbf{f}$ at x), and that x is a pre-image of y under $\mathbf{f}$.

Pay attention to the articles the and a

Definition

1. Let **A** be a set. The function $I_A : \mathbf{A} \to \mathbf{A}$, for $x \in \mathbf{A}$ given by $I_A(x) = x$ is the **identity function of A.**
2. If $\mathbf{A} \subset \mathbf{B}$, the function $i : \mathbf{A} \to \mathbf{B}$ for $x \in \mathbf{A}$ given by $i(x) = x$ is the **inclusion function** from **A** to **B.**
3. If c is a fixed element of **B**, the function $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ such that for $x \in \mathbf{A}$ gives $\mathbf{f}(x) = c$ is the **constant function** c.

Equality of Two Functions

We need to keep in mind the following observation:

Theorem

Two functions $\mathbf{f}$ *and* $\mathbf{g}$ *are equal iff*

1. $Dom(\mathbf{f}) = Dom(\mathbf{g})$, *and*
2. $\forall x \in Dom(\mathbf{f}), \quad \mathbf{f}(x) = \mathbf{g}(x)$.

Proof.

(Just the implication $\Rightarrow$). Assume $\mathbf{f} = \mathbf{g}$.

1. If $x \in Dom(\mathbf{f})$, then $(x, y) \in \mathbf{f}$, for some y and since $\mathbf{f} = \mathbf{g}$, $(x, y) \in \mathbf{g}$. Therefore $x \in Dom(\mathbf{g})$, which shows $Dom(\mathbf{f}) \subset Dom(\mathbf{g})$. The reverse containment is proved in the same manner, so that together will have $Dom(\mathbf{f}) = Dom(\mathbf{g})$.
2. Suppose $x \in Dom(\mathbf{f})$. Then for some y, $(x, y) \in \mathbf{f}$. Since $\mathbf{f} = \mathbf{g}$, $(x, y) \in \mathbf{g}$. Therefore $\mathbf{f}(x) = \mathbf{g}(x)$.

□

Example

$$\begin{aligned}\mathbf{F} : \{-2, 3\} &\to \{4, 9\}, \quad \mathbf{F}(x) = x^2 \\ \mathbf{G} : \{-2, 3\} &\to \{4, 9\}, \quad \mathbf{G}(x) = x + 6\end{aligned}$$

$\mathbf{F} = \mathbf{G}$: Different rules but define the same functions.

So in defining a function $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ one pays attention to all the sets (rules included) needed to defined **f**. Often **f** may be defined in more than one way.

Exercise

One sacrificial volunteer please:
If $\mathbf{A}$ is a set with 5 elements, how many functions are there of the form

$$\mathbf{f} : \mathbf{A} \to \mathbf{A}$$

Answer according to Kristin: 5^5.
How about

$$\mathbf{f} : \mathbf{A} \to \emptyset$$

Answer: None!

Characteristic Function of a Subset

Definition

Let U be a specified universe and $\mathbf{A} \subset U$. Define $\chi_{\mathbf{A}} : \mathbf{A} \to \{0, 1\}$ by

$$\chi_{\mathbf{A}}(x) = \begin{cases} 1 & \text{if } x \in \mathbf{A} \\ 0 & \text{if } x \in U \setminus \mathbf{A}. \end{cases}$$

$\chi_{\mathbf{A}}$ is the **characteristic function** of **A**.

Exercise

Proposition

Let **A** *and* **B** *be subsets of the universe* U. *Then*

1. $\chi_{\tilde{\mathbf{A}}} = \mathbf{1} - \chi_{\mathbf{A}}$, *where* **1** *is the constant function defined by* 1.
2. $\mathbf{A} = \mathbf{B}$ *iff* $\chi_{\mathbf{A}} = \chi_{\mathbf{B}}$.
3. $\chi_{\mathbf{A} \cap \mathbf{B}} = \chi_{\mathbf{A}} \cdot \chi_{\mathbf{B}}$.
4. $\chi_{\mathbf{A} \cup \mathbf{B}} = \chi_{\mathbf{A}} + \chi_{\mathbf{B}} - \chi_{\mathbf{A}} \cdot \chi_{\mathbf{B}}$.
5. $\chi_{\mathbf{A} \Delta \mathbf{B}} = \chi_{\mathbf{A}} + \chi_{\mathbf{B}} - 2 \cdot \chi_{\mathbf{A} \cap \mathbf{B}}$.

Exercise: Use the proposition to prove easily that $\mathbf{A} \cap (\mathbf{B} \Delta \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \Delta (\mathbf{A} \cap \mathbf{C})$: This shows that $\cap$ works as a **product** and Δ as a **sum** operation.

Dirichlet Function

It might be a good idea to have wonderful functions $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ (or from subsets $\mathbf{A} \subset \mathbb{R}$) at hand:

1. (Dirichlet Function)

$$\mathbf{f}(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

2.

$$\mathbf{f}(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

3. Let $\mathbf{f}(x)$ be your favorite function: polynomials, rational functions, trig functions, $\zeta(x)$? You might want to google the last one: after all, it is the most famous function of Mathematics!

Definition

A sequence is a function **f** whose domain is $\mathbb{N}$.

$$\mathbf{f} : \mathbb{N} \to \mathbf{A}.$$

It can be represented as

$$\{\mathbf{f}(1), \mathbf{f}(2), \mathbf{f}(3), \ldots\}$$

$$\{a_n : n \in \mathbb{N}\}$$

Sequences of real numbers

It allows us to look at real numbers in a concrete manner: If

$$x = A.a_1 a_2 \cdots a_n \cdots ,$$

where a_i are the decimal digits, we form the sequence of rational numbers

$$\begin{aligned} x_0 &= A \\ x_1 &= A.a_1 \\ x_2 &= A.a_1 a_2 \\ x_n &= A.a_1 a_2 \cdots a_n, \quad \text{and so on} \end{aligned}$$

This actually says that a real number is what of the sequence?

More Examples

1. $\mathbf{f}(n) = 1/n$, $n \in \mathbb{N}$.
2. $\mathbf{f}(n)$ is the nth digit in the decimal expression of π. Makes sense?
3. $(c, c, c, c, \ldots)$
4. $(1, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \ldots)$
5. $(\frac{1}{2^n})_{n=1}^{\infty} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$
6. (a_n), $a_1 = 1$, and $a_{n+1} = \frac{a_n+1}{2}$
7. (a_n), $a_n = (1 + 1/n)^n$

Equivalence Relation

Recall: R is an equivalence relation on the set X:

Definition

Let X be a set and R a relation on X.

- R is reflexive iff for all $x \in X$, $x \; R \; x$.
- R is symmetric iff for all $x \in X$ and $y \in A$, if $x \; R \; y$, then $y \; R \; x$.
- R is transitive iff for all x, y and z in X, if $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$.

Definition

A relation R on a set X is an **equivalence relation on** A iff R is reflexive, symmetric, and transitive.

Equivalence Class

Definition

Let R be an equivalence relation on the set X. For $x \in X$, the **equivalence class of** x determined by R is the set

$$x/R = \{y \in X : x\ R\ y\}.$$

This is read "the class of x modulo R." The set of all equivalence classes of R is called X **modulo** R and denoted $X/R = \{x/R : x \in A\}$. Other notation for it: $[x]_R$ or $\overline{x}_R$–may drop the R when well-understood.

Example: Two integers have the same **parity** if they are both even or both odd. Let $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \text{ and } y \text{ have the same parity.}\}$ R is an equivalence relation with two equivalence classes: the even integers E and the odd integers D. $\mathbb{Z}/R = \{E, D\}$.

Big Example

Let m be a fixed, nonzero integer. Let $\equiv_m$ be the relation on $\mathbb{Z}$,

$$x \equiv_m y \text{ iff } m \text{ divides } x - y.$$

This is also written $x \equiv y \pmod{m}$ or even $x = y \pmod{m}$.
It is easy to see that $\mathbb{Z}/\equiv_2 = \{E, D\}$. This set is also denoted by $\mathbb{Z}_2$ and called the set of integers modulo 2. For $m = 3$, $\equiv_3$ is also an equivalence relation and there are three distinct equivalence classes.

Theorem

The relation $\equiv_m$ is an equivalence relation on the integers. The set of equivalence relations is called $\mathbb{Z}_m$ and has m distinct elements $\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{m-1}$.

Canonical Map

1. Let m be a fixed integer. The map $\mathbf{f} : \mathbb{Z} \to \overline{\mathbb{Z}}_m$

$$\mathbf{f}(n) := [n] = \text{congruence class of } n \in \mathbb{Z} \text{ modulo } m$$

is an example of a **canonical mapping.**

2. More generally, if R is an equivalence relation on X, the map

$$\mathbf{f}(x) := \text{equivalence class of } x \in X \text{ relative to } R$$

is the **canonical mapping** relative to R.

Outline

Definition

If $\mathbf{f}$ is a relation from A to B, the **inverse** of $\mathbf{f}$ is

$$\mathbf{f}^{-1} = \{(y, x) : (x, y) \in \mathbf{f}\}.$$

If $\mathbf{f}$ is a function from A to B, $\mathbf{f}^{-1}$ is a relation from B to A. Lets us find out when $\mathbf{f}^{-1}$ is a function by applying the requirements.

Theorem

Let $\mathbf{f}$ *be a relation from A to B.*

1. $\mathbf{f}^{-1}$ *is a relation from B to A.*
2. $Dom(\mathbf{f}^{-1}) = Rng(R)$.
3. $Rng(\mathbf{f}^{-1}) = Dom(R)$.

Theorem

If $\mathbf{f} : A \rightarrow B$ *is a function then* $\mathbf{f}^{-1} : B \rightarrow A$ *is a function iff*

1. $Rng(\mathbf{f}) = B$;
2. *if* $(x, z), (y, z) \in \mathbf{f}$ *then* $y = z$.

Proof. Let us prove that if **f** satisfies (1) and (2) then $\mathbf{f}^{-1}$ is a function.

1. By (1), for each $y \in B$ there is $x \in A$ such that $(x, y) \in \mathbf{f}$, and therefore $(y, x) \in \mathbf{f}^{-1}$.
2. (**Vertical Line Test**) If $(y, x) \in \mathbf{f}^{-1}$ and $(y, z) \in \mathbf{f}^{-1}$, then $x = z$ by condition (2).

This proves that $\mathbf{f}^{-1}$ is a function. The converse has a similar proof. $\square$

Summary: Let $\mathbf{f} : A \to B$ be a function such that $\mathbf{f}^{-1} : B \to A$ is also a function:

1. $\mathbf{f}(x) = y$ iff $\mathbf{f}^{-1}(y) = x$;
2. $B = Range(\mathbf{f}) = Dom(\mathbf{f}^{-1};$
3. $A = Dom(\mathbf{f}) = Range(\mathbf{f}^{-1});$

Definition

Let $\mathbf{f}$ be a function from A to B, and let $\mathbf{g}$ be a function from B to C. The **composite** of $\mathbf{f}$ and $\mathbf{g}$ is

$$\mathbf{g} \circ \mathbf{f} = \{(a, c) : \text{there exists } b \in B \text{ such that } (a, b) \in \mathbf{f} \text{ and } (b, c) \in \mathbf{g}\}.$$

This is more simply written as

$$(\mathbf{g} \circ \mathbf{f})(a) = \mathbf{g}(\mathbf{f}(a)) = \mathbf{g}(b) = c.$$

Confusing Diagram...

Let $A = \{1, 2, 3, 4, 5\}$, and $B = \{p, q, r, s, t\}$, and $C = \{x, y, z, w\}$. Let R be the relation from A to B:

$$R = \{(1, p), (1, q), (2, q), (3, r), (4, s)\}$$

and S the relation from B to C:

$$S = \{(p, x), (q, x), (q, y), (s, z), (t, z)\}.$$

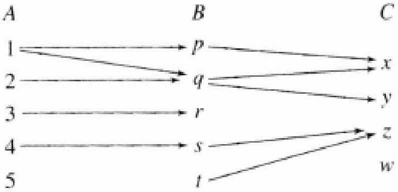


Figure 3.10

Theorem

Suppose A, B, C and D are sets. Let **f** *be a function from A to B,* **g** *a function from B to C, and* **h** *a function from C to D:*

$$A \xrightarrow{\mathbf{f}} B \xrightarrow{\mathbf{g}} C \xrightarrow{\mathbf{h}} D.$$

1. *If $\mathbf{f}^{-1}$ exists, then $(\mathbf{f}^{-1})^{-1} = \mathbf{f}$.*
2. $\mathbf{h} \circ (\mathbf{g} \circ \mathbf{f}) = (\mathbf{h} \circ \mathbf{g}) \circ \mathbf{f}$.
3. $I_B \circ \mathbf{f} = \mathbf{f}$ *and* $\mathbf{f} \circ I_A = \mathbf{f}$.
4. *If $\mathbf{f}^{-1}$ and $\mathbf{g}^{-1}$ exist, then $(\mathbf{g} \circ \mathbf{f})^{-1} = \mathbf{f}^{-1} \circ \mathbf{g}^{-1}$.*

Proof of (2): Note both $\mathbf{h} \circ (\mathbf{g} \circ \mathbf{f})$ and $(\mathbf{h} \circ \mathbf{g}) \circ \mathbf{f}$ are relations from A to D, that is they are subsets of $A \times D$.

To prove $\mathbf{h} \circ (\mathbf{g} \circ \mathbf{f}) = (\mathbf{h} \circ \mathbf{g}) \circ \mathbf{f}$, let $(a, d) \in \mathbf{h} \circ (\mathbf{g} \circ \mathbf{f})$. Note that $\mathbf{g} \circ \mathbf{f}$ is a relation from A to C.

1. Thus there is $c \in C$ such that $(a, c) \in \mathbf{g} \circ \mathbf{f}$ and $(c, d) \in \mathbf{h}$:
   $(a, c) \in \mathbf{g} \circ \mathbf{f}$ and $(c, d) \in \mathbf{h}$
2. Hence there is $b \in B$ such that $(b, c) \in \mathbf{g}$.
3. Therefore $(b, d) \in \mathbf{h} \circ \mathbf{g}$.
4. Since $(a, b) \in \mathbf{f}$ and $(b, d) \in \mathbf{h} \circ \mathbf{g}$, it follows that $(a, d) \in (\mathbf{h} \circ \mathbf{g}) \circ \mathbf{f}$.
5. This shows that $\mathbf{h} \circ (\mathbf{g} \circ \mathbf{f}) \subseteq (\mathbf{h} \circ \mathbf{g}) \circ \mathbf{f}$. The reverse inequality has a similar proof.

Composition Summary

1. $\mathbf{h} \circ (\mathbf{g} \circ \mathbf{f}) = (\mathbf{h} \circ \mathbf{g}) \circ \mathbf{f}$: Because both make sense and in the functional notation

$$(\mathbf{h} \circ (\mathbf{g} \circ \mathbf{f}))(a) = \mathbf{h}(\mathbf{g}(\mathbf{f}(a))) = ((\mathbf{h} \circ \mathbf{g}) \circ \mathbf{f})(a)$$

2. If $\mathbf{f}^{-1}$ and $\mathbf{g}^{-1}$ exist, then $(\mathbf{g} \circ \mathbf{f})^{-1} = \mathbf{f}^{-1} \circ \mathbf{g}^{-1}$.

Restriction and Extension

Let $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ be a function. A simple way to obtain new functions from $\mathbf{f}$ is the following:

Definition

Let $\mathbf{C} \subset \mathbf{A}$. The function

$$\mathbf{g} : \mathbf{C} \to \mathbf{B}, \quad \forall c \in \mathbf{C},\ \mathbf{g}(c) = \mathbf{f}(c)$$

is called the **restriction** of $\mathbf{f}$ to $\mathbf{C}$. In turn, $\mathbf{f}$ is called an **extension or prolongation** of $\mathbf{g}$ to $\mathbf{A}$.

Note that $\mathbf{f}$ has a unique restriction to $\mathbf{C}$, but $\mathbf{g}$ may have several extensions to $\mathbf{A}$.

Example of Restriction/Extension

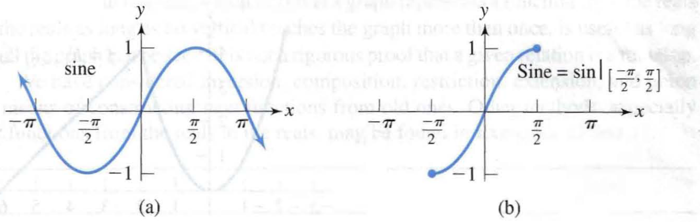


Figure 4.6

Gluing Functions

Let **f** and **g** be two functions with the same target:

$$\mathbf{f} : \mathbf{A} \to \mathbf{C},$$
$$\mathbf{g} : \mathbf{B} \to \mathbf{C}$$

How to define a function

$$\mathbf{h} : \mathbf{A} \cup \mathbf{B} \to \mathbf{C}$$

so that **f** is the restriction of **h** to **A** and **g** is the restriction of **f** to **B**?

Gluing two functions

Proposition

Let $\mathbf{h}$ *be the relation from* $\mathbf{A} \cup \mathbf{B}$ *to* $\mathbf{C}$*,*

$$\mathbf{h} = \{(a, b) \in \mathbf{A} \times \mathbf{C} : b = \mathbf{f}(a)\} \cup \{(c, d) \in \mathbf{B} \times \mathbf{C} : d = \mathbf{g}(c)\}.$$

Then $\mathbf{h}$ *is a function from* $\mathbf{A} \cup \mathbf{B}$ *to* $\mathbf{C}$ *if* $\mathbf{A} \cap \mathbf{B} = \emptyset$*. More generally,* $\mathbf{h}$ *is a function from* $\mathbf{A} \cup \mathbf{B}$ *to* $\mathbf{C}$ *if*

$$\mathbf{f}(x) = \mathbf{g}(x) \forall x \in \mathbf{A} \cap \mathbf{B}.$$

$\mathbf{h}$ is said to be **glued** from **f** and **g along** $\mathbf{A} \cap \mathbf{B}$. Note that if $\mathbf{A} \cap \mathbf{B} = \emptyset$, we have no obstruction.

Gluing of two functions

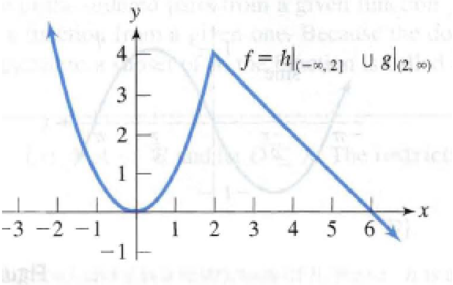


Figure 4.7

Gluing several functions

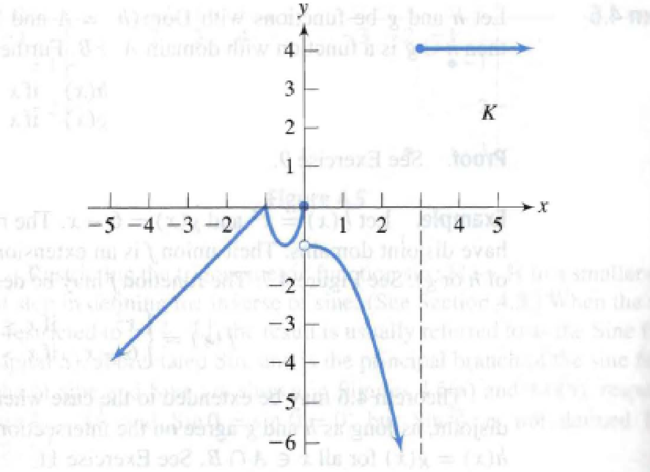


Figure 4.8

Restriction and Extension

Sum and Product of Functions

Functions whose target space is the set of real numbers $\mathbb{R}$, allow many new constructions to obtain new functions from old ones. Here are some

Definition

Let $\mathbf{f}$ and $\mathbf{g}$ be two functions from $\mathbf{A}$ to $\mathbb{R}$. The relations given by

$$\begin{aligned} \mathbf{f}+\mathbf{g} &: \mathbf{A} \to \mathbb{R} \quad (\mathbf{f}+\mathbf{g})(a) = \mathbf{f}(a)+\mathbf{g}(a),\ a \in \mathbf{A}, \\ \mathbf{f}\cdot\mathbf{g} &: \mathbf{A} \to \mathbb{R} \quad (\mathbf{f}\cdot\mathbf{g})(a) = \mathbf{f}(a)\mathbf{g}(a),\ a \in \mathbf{A}, \end{aligned}$$

are called the **Sum** and the **Product** of **f** and **g**, resp.

Moreover, if $\mathbf{g}(a) \neq 0\ \forall a \in \mathbf{A}$, the quotient $\frac{\mathbf{f}}{\mathbf{g}}$ can also be defined.

Outline

Homework #9

1. 4.2: 3(b), 7(c), 14(d), 16(c)

Outline

Onto Functions

Definition

A function $\mathbf{f} : A \to B$ is **onto** B (or is a **surjection**) if $Rng(\mathbf{f}) = B$. That is,

$$\forall b \in B \quad \exists a \in A : \mathbf{f}(a) = b.$$

Note that all functions $\mathbf{f} : A \to B$ gives rise to another function that is onto

$$\mathbf{g} : A \to Rng(\mathbf{f}).$$

Example: The function $\mathbf{f} : \mathbb{N} \to \mathbb{N}$, $\mathbf{f}(n) = 2n$, is not a surjection. Its Range is the set E of nonzero even numbers. The function $\mathbf{g} : \mathbb{N} \to E$, $\mathbf{g}(n) = 2n$, is a surjection.

Solving Equations

Let $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ be the function

$$\mathbf{f}(x) = x^2 + 2x + 1.$$

Is **f surjective**? (another name for a surjection)

To be a surjection means that for any real number b, we must be able to solve the equation

$$\mathbf{f}(x) = x^2 + 2x + 1 = b.$$

This means that

$$(x + 1)^2 = b,$$

so b cannot be < 0, so **f** is not surjective.

Exercise

Let $\mathbf{f}$ be the function $\mathbf{f}: \mathbb{R} \to \mathbb{R}$, given by

$$\mathbf{f}(x) = ax^2 + bx + c, \quad a \neq 0.$$

Prove that $\mathbf{f}$ is NOT surjective.
We must be able to solve $ax^2 + bx + c = e \quad \forall e \in \mathbb{R}$.

$$ax^2 + bx + (c - e) = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4a(c - e)}}{2a}.$$

This requires that

$$b^2 + 4a(e - c) \geq 0 \Rightarrow 4a(e - c) \geq -b^2$$

This means that if $a > 0$, $e \geq c - b^2/4a$, and $e \leq c - b^2/4a$ if $a < 0$.
Both inequalities can be broken by choosing e conveniently.

Cubics

One sacrificial volunteer please: Prove the following

Theorem

Let $\mathbf{f}$ *be the function* $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ *given by*

$$\mathbf{f}(x) = ax^3 + bx^2 + cx + d, a \neq 0.$$

(We may assume $\mathbf{f}(x) = 2x^3 + 4x^2 - 6x + 8$*.) Then* $\mathbf{f}$ *is surjective.*

The proof uses Cal 1. We are going to sketch the graph of $\mathbf{f}$, paying attention to the following points:

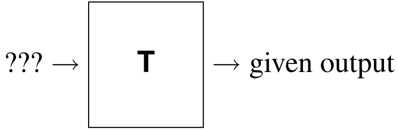
1. Plot a few points, say $x = 0, -1, 1$
2. Find $\lim_{x\to\infty} \mathbf{f}(x)$ and $\lim_{x\to-\infty} \mathbf{f}(x)$
3. Is $\mathbf{f}$ a continuous function?
4. What does this means for the graph of $\mathbf{f}$?
5. Argue that the graph of $\mathbf{f}$ crosses any **horizontal line**!

Bit of Linear Algebra

We can described a linear system of equations in the following manner:
Let **T** (i.e. a matrix) be a linear transformation of source **V** and target **W**,

$$\mathbf{T} : \mathbf{V} \to \mathbf{W}.$$

Problem: Given $w \in \mathbf{W}$ is there $v \in \mathbf{V}$ such $\mathbf{T}(v) = w$? Such v is called **a** solution, or a special solution.



1. Do solutions exist? The answer, in the affirmative case [called CONSISTENT] carries consequences to the next questions.
2. If solutions exist, what is the nature of the set of solutions?
3. Among the solutions, which is the best?
4. How do we find these things anyway?

Example

Let $\mathbf{F} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by $\mathbf{F}(m, n) = 2^{m-1}(2n - 1)$.

Claim: $\mathbf{F}$ is surjective. Let $s \in \mathbb{N}$. We must show that there are $m, n \in \mathbb{N}$ such that $s = 2^{m-1}(2n - 1)$. For example, if $s = 12$, $12 = 4 \times 3 = 2^{3-1}(2 \times 2 - 1) = \mathbf{F}(3, 2)$.

If s is odd, it can be written $s = 2n - 1$, so that $\mathbf{F}(1, n) = 2^0 \times (2n - 1) = s$.

If s is even, it can be written $s = 2^k t$, where t is odd (why Abdel?) Choosing $m = k + 1$ and $t = 2n - 1$, we have $\mathbf{F}(m, n) = s$.

Tools

Theorem

Let $\mathbf{f} : A \to B$ *and* $\mathbf{g} : B \to C$. *Then*

1. *If* $\mathbf{f}$ *and* $\mathbf{g}$ *are surjections, then* $\mathbf{g} \circ \mathbf{f}$ *is a surjection.*
2. *If* $\mathbf{g} \circ \mathbf{f}$ *is a surjection then* $\mathbf{g}$ *is a surjection.*

One-to-One Functions

Definition

A function $\mathbf{f} : A \to B$ is **one-to-one** (or is an **injection**) iff whenever $\mathbf{f}(x) = \mathbf{f}(y)$ then $x = y$.

To prove that a **f** is one-to-one, one often checks it by contradiction:

$$x \neq y \Rightarrow \mathbf{f}(x) \neq \mathbf{f}(y).$$

For example, the function $\mathbf{f} : \mathbb{R} \to \mathbb{R}$, defined by $\mathbf{f}(x) = |x|$ is not injective: $|1| = |-1| = 1$.

Visual Test

The **Horizontal Line Test for One-to-One** :

f : $A \to B$ is one-to-one iff every horizontal line intersects the graph of **f** at most once.

Back to Earlier Example

Claim: The function $\mathbf{F} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $\mathbf{F}(mn) = 2^{m-1}(2n-1)$ is one-to-one.

1. Suppose $\mathbf{F}(m, n) = \mathbf{F}(r, s)$. We must show $(m, n) = (r, s)$.
2. We first prove that $m = r$. We may assume $m \geq r$ (Why Eric?) From $2^{m-1}(2n-1) = 2^{r-1}(2s-1)$ we have
$$2^{m-r}(2n-1) = (2s-1).$$
3. If $m > r$, this gives that $2s - 1$ is an even number, a contradiction. Thus $m = r$.
4. Therefore $2n - 1 = 2s - 1$, which implies $n = s$.
5. We conclude that $(m, n) = (r, s)$, as desired.

Famous Technique from Calculus

Let **f** be a differentiable function on the interval $[a, b]$. Probably the most useful assertion of the differential calculus is the relationship between the value of the slope of the secant to the graph of $\mathbf{f}(x)$,

$$\frac{\mathbf{f}(b) - \mathbf{f}(a)}{b - a},$$

and values of the derivative. Even the so-called Fundamental Theorem of Calculus can be seen as one of its consequences.

MVT of Calculus

Theorem

Let $\mathbf{f} : [a, b] \to \mathbb{R}$ *be a continuous function on* $[a, b]$ *and differentiable on* (a, b)*. Then there exists a point* $c \in (a, b)$

$$\mathbf{f}'(c) = \frac{\mathbf{f}(b) - \mathbf{f}(a)}{b - a}.$$

Corollary

If $\mathbf{f}'(x) \neq 0$ *in* $[a, b]$ *then* $\mathbf{f} : [a, b] \to \mathbb{R}$ *is one-to-one.*

IVT: Intermediate Value Theorem of Calculus

Another of the great theorems of Calculus is

Theorem (IVT)

If $\mathbf{f} : [a, b] \to \mathbb{R}$ *is continuous, and if* L *is any real number satisfying* $\mathbf{f}(a) < L < \mathbf{f}(b)$ *or* $\mathbf{f}(a) > L > \mathbf{f}(b)$*, then there exists a point* $c \in (a, b)$ *where* $\mathbf{f}(c) = L$*. In particular, if* $\mathbf{f}(a) < 0$ *and* $\mathbf{f}(b) > 0$*, there exists a point* $c \in (a, b)$ *such that* $\mathbf{f}(c) = 0$*.*

It is useful to prove that certain functions are surjections.

Example

Theorem

Let $\mathbf{f}(x) = x^n + x$*, where n is an odd natural number. Then the function*

$$\mathbf{f} : \mathbb{R} \to \mathbb{R}$$

is one-to-one and onto.

Proof.

1. (**One-to-One**) Since $\mathbf{f}'(x) = nx^{n-1} + 1$, and $n - 1$ is even, $\mathbf{f}'(x)$ is never 0. Let us argue by contradiction. If for $a < b$, $\mathbf{f}(a) = \mathbf{f}(b)$, then by the **MVT**, for some $c \in (a, b)$, $\mathbf{f}'(c) = 0$, which can't happen. Thus **f** is one-to-one.
2. (**Onto**) Since n is odd, $\lim_{x\to\infty} \mathbf{f}(x) = \infty$ and $\lim_{x\to-\infty} \mathbf{f}(x) = -\infty$. Thus if $L \in \mathbb{R}$, there are a and b such that $\mathbf{f}(a) < L < \mathbf{f}(b)$. By the **IVT**, there is $c \in [a, b]$ such that $L = \mathbf{f}(c)$. So **f** is onto.

□

Bijections

Definition

A function $\mathbf{f} : A \to B$ is a **one-to-one correspondence** (or is a **bijection**) if it is one-to-one and onto.

Example: The function $\mathbf{f} : \mathbb{R} \to \mathbb{R}$, $\mathbf{f}(x) = 2x + 1$, is a bijection:

1. If $\mathbf{f}(x) = 2x + 1 = 2y + 1 = \mathbf{f}(y)$, then $x = y$, so $\mathbf{f}$ is one-to-one.
2. If $b \in \mathbb{R}$, we can find x so that $\mathbf{f}(x) = 2x + 1 = b$: $x = 1/2(b - 1)$, so $\mathbf{f}$ is onto.

Tools

Theorem

If $\mathbf{f} : A \to B$ *and* $\mathbf{g} : B \to C$ *are bijections, then*

1. $\mathbf{g} \circ \mathbf{f} : A \to C$ *is bijective.*
2. *The inverse relation* $\mathbf{f}^{-1} : B \to A$ *is a function and*

$$I_A = \mathbf{f}^{-1} \circ \mathbf{f} : A \to A \quad I_B = \mathbf{f} \circ \mathbf{f}^{-1} : B \to B.$$

Outline

Homework #10

1. If A is a set with 3 elements and B has 4 elements: (a) How many functions are there from A to B, (b) how many of these are surjections, and (c) and how many are injections?
2. 4.3: 1(l), 4, 8(c), 15(b,d), 16(b,c)

Outline

Last Class...and Today...

Some properties of functions that are valuable

1. **Onto/Surjective Function:** $\mathbf{f} : A \to B$, $Rng(\mathbf{f}) = B$, that is

$$\forall b \in A \quad \exists x \in A : \mathbf{f}(x) = b.$$

2. **One-to-One/Injective Function:** If $\mathbf{f}(x) = \mathbf{f}(y)$ then $x = y$, in other words (more properly) if $x \neq y$ then $\mathbf{f}(x) \neq \mathbf{f}(y)$.
3. **Bijection:** **f** is both **onto and one-to-one.** If $\mathbf{f} : A \to B$ is a bijection, $\mathbf{f}^{-1} : B \to A$ is a function. The irony is that **f** may be given by a 'formula' but we may be unable to describe $\mathbf{f}^{-1}$ in a similar manner.

Calculus has wonderful tools to examine these properties.

Outline

Images of Sets

Giiven

$$\mathbf{f} : A \to B$$

we are interested in issues like: if a and b are 'related', what of $\mathbf{f}(a)$ and $\mathbf{f}(b)$?

Definition

Let $\mathbf{f} : A \to B$ and let $X \subset A$ and $Y \subset B$.

- The **image of** X is $\mathbf{f}(X) = \{y \in B : y = \mathbf{f}(x) \quad \text{for some } x \in A\}$.
- The **inverse image of** Y is $\mathbf{f}^{-1}(Y) = \{x \in A : \mathbf{f}(x) \in Y\}$.

Functions of Sets

Observe what this says: If $\mathbf{f} : A \to B$, for every subset $X \subset A$, $\mathbf{f}(X)$ is a subset of B, in other words we have a new function

$$\mathbf{f}_* : \mathcal{P}(A) \to \mathcal{P}(B),$$

from the power set $\mathcal{P}(A)$ to the power set $\mathcal{P}(B)$.

Also, a new function

$$\mathbf{f}_*^{-1} : \mathcal{P}(B) \to \mathcal{P}(A),$$

from the power set $\mathcal{P}(B)$ to the power set $\mathcal{P}(A)$.

Surprisingly, $\mathbf{f}_*^{-1}$ is more well-behaved than $\mathbf{f}_*$. (see later)

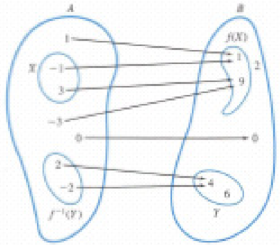


Figure 4.18

Exercise

Let $\mathbf{f} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $\mathbf{f}(m, n) = 2^m 3^n$. Find

1. $\mathbf{f}(A \times B)$ where $A = \{1, 2, 3\}$ and $B = \{3, 4\}$.

   We just collect the images of the 6 elements of $A \times B$:

   $$\mathbf{f}(A \times B) = \{2 \cdot 3^3, 2 \cdot 3^4, 2^2 \cdot 3^3, 2^2 \cdot 3^4, 2^3 \cdot 3^3, 2^4 \cdot 3^4\}$$

2. $\mathbf{f}^{-1}(5, 6, 7, 8, 9, 10)$: We find (m, n) so that $2^m 3^n$ is one of $\{5, 6, 7, 8, 9, 10\}$. Keep in mind that $0 \notin \mathbb{N}$.

   Note that $2^m 3^n$ cannot be $5, 7, 8, 9, 10$, so $\mathbf{f}^{-1}(5, 6, 7, 8, 9, 10) = \{(1, 1)\}$.

Example. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Then $f(\{-2, 2\}) = \{4\}$ since both $f(2) = 4$ and $f(-2) = 4$. From Figure 4.12 we see that $f([1, 2]) = [1, 4]$. Also, $f([-1, 0]) = [0, 1]$.

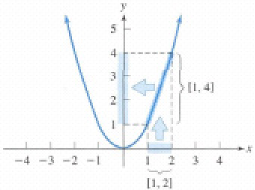


Figure 4.12 $f([1, 2]) = [1, 4]$

In this example it is tempting to believe that $f([-1, 2]) = [(-1)^2, 2^2] = [1, 4]$, but this is incorrect. By definition, $f([-1, 2])$ is the set of all images of elements of $[-1, 2]$. Since $-\frac{1}{2}$, 0, and 0.7 are in $[-1, 2]$, their images $\frac{1}{4}$, 0, and 0.49 must be in $f([-1, 2])$. Figure 4.13 shows that $f([-1, 2]) = [0, 4]$.

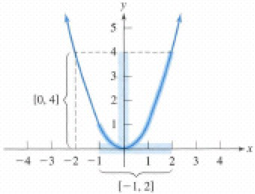


Figure 4.13 $f([-1, 2]) = [0, 4]$

Class Exercise

Let $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ be given by $\mathbf{f}(x) = 10x - x^2$. Find: (a) $\mathbf{f}([1, 6))$, (b) $\mathbf{f}^{-1}((0, 21])$.

Need the sketch of the graph.

Exercise

Let $\mathbf{f} : A \to B$. Prove that if $\mathbf{f}$ is one-to-one, then for all X, Y $\mathbf{f}(X) \cap \mathbf{f}(Y) = \mathbf{f}(X \cap Y)$. Is the converse true?

The R.H.S. is always contained in the in the L.H.S. Let $z \in \mathbf{f}(X) \cap \mathbf{f}(Y)$, that is $z = \mathbf{f}(x) = \mathbf{f}(y)$ with $x \in X$ and $y \in Y$. Since $\mathbf{f}$ is one-to-one, $x = y$. Thus $z \in \mathbf{f}(X \cap Y)$.

Let us prove by contradiction that the converse holds. If $\mathbf{f}(x) = \mathbf{f}(y)$ but $x \neq y$, consider the sets $X = \{x\}$, $Y = \{y\}$. Then $X \cap Y = \emptyset$, and $\mathbf{f}(X) \cap \mathbf{f}(Y) = \{\mathbf{f}(x)\}$ but $\mathbf{f}(X \cap Y) = \mathbf{f}(\emptyset) = \emptyset$.

Properties

Theorem

Let $\mathbf{f} : A \to B$*, C and D subsets of A, and E and F be subsets of B. Then*

1. $\mathbf{f}(C \cap D) \subset \mathbf{f}(C) \cap \mathbf{f}(D)$.
2. $\mathbf{f}(C \cup D) = \mathbf{f}(C) \cup \mathbf{f}(D)$.
3. $\mathbf{f}^{-1}(E \cap F) = \mathbf{f}^{-1}(E) \cap \mathbf{f}^{-1}(F)$.
4. $\mathbf{f}^{-1}(E \cup F) = \mathbf{f}^{-1}(E) \cup \mathbf{f}^{-1}(F)$.

Classroom Proof by non-willing volunteer

Exercise

Let $\mathbf{f} : A \to B$. Let R be the relation on A defined by $x\ R\ y$ iff $\mathbf{f}(x) = \mathbf{f}(y)$.

1. Show that R is an equivalence relation.
2. Describe the partition of A associated with R.

This is like the case in a quiz, when we defined $x\ R\ y$ iff $\sin x = \sin y$. It is easy to prove the relation is reflexive, symmetric and transitive. We have for the equivalence class of x

$$x/R = \{\text{all } y \text{ such that } \mathbf{f}(y) = \mathbf{f}(x)\} = \mathbf{f}^{-1}(\mathbf{f}(x)).$$

The partition is

$$A = \bigcup_{x \in A} \mathbf{f}^{-1}(\mathbf{f}(x)).$$

Note that the subsets $\mathbf{f}^{-1}(\mathbf{f}(x))$ are non-empty, pairwise disjoint and cover A.

Outline

Homework #11

4.4: 4(a,c), 9(a,b), 17, 19(a,b)

Outline

Sequences of real numbers

Definition

A sequence is a function **f** whose domain is $\mathbb{N}$.

It can be represented as

$$\{\mathbf{f}(1), \mathbf{f}(2), \mathbf{f}(3), \ldots\}$$

$$\{\mathbf{f}(0), \mathbf{f}(1), \mathbf{f}(2), \mathbf{f}(3), \ldots\}$$

or

$$\{\mathbf{f}(n), \ldots, \quad n \geq n_0\}$$

We will first examine sequences of real numbers, $\mathbf{f} : \mathbb{N} \to \mathbb{R}$.

Sequences allow us to look at real numbers in a concrete manner: If

$$x = A.a_1 a_2 \cdots a_n \cdots ,$$

where a_i are the decimal digits, we form the sequence of rational numbers

$$\begin{aligned} x_0 &= A \\ x_1 &= A.a_1 \\ x_2 &= A.a_1 a_2 \\ x_n &= A.a_1 a_2 \cdots a_n, \quad \text{and so on} \end{aligned}$$

Examples

We will look for features such as **clustering**

1. $(1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots)$
2. $(c, c, c, c, \ldots)$
3. $(1, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \ldots)$
4. $(\frac{1}{2^n})_{n=1}^{\infty} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$
5. (a_n), $a_1 = 1$, and $a_{n+1} = \frac{a_n+1}{2}$
6. (a_n), a_n is the nth digit in the decimal expansion of π.
7. (a_n), $a_n = (1 + 1/n)^n$

Why Sequences?

We use sequences to make sense of:

- $\sum_{n\geq 1} a_n$: Series

$$1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 + \cdots$$

Question: How to handle

$$(a_0 + a_1 + \cdots + a_n + \cdots)(b_0 + b_1 + \cdots + b_n + \cdots)$$

- $\sum_{m,n\geq 1} a_{m,n}$: Double [multiple] Series

$$\sum_{m,n} \frac{1}{m^2 + n^2}$$

- $\prod_{n\geq 1} a_n$: Infinite Products

$$\prod_p (\frac{1}{1-p}), \quad p \text{ prime number}$$

Convergence of a Sequence

Sequences are wonderful ways to represent data, but we are mostly interested is one of its aspects:

Definition

A sequence (a_n) converges to a real number a if, for every positive real number ϵ, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.

One notation: $\lim a_n = a$, or $(a_n) \to a$. To understand this we introduce the notion of a **neighborhood** of a real number a.

Example

Consider the sequence (a_n), $a_n = \frac{n+1}{n}$. It is natural to expect that $\lim a_n = 1$. Let us follow the template:

- Given $\epsilon > 0$, to determine N we solve

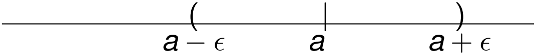
$$\left|\frac{n+1}{n} - 1\right| < \epsilon$$

- That is

$$\left|\frac{1}{n}\right| < \epsilon \quad \Rightarrow \quad n > \frac{1}{\epsilon}$$

- Thus if $\epsilon = 1/100$, $N = 101$ will work.

Neighborhoods



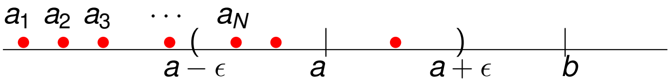
Definition

Given a real number $a \in \mathbb{R}$ and a positive number $\epsilon > 0$, the set

$$V_\epsilon(a) = \{x \in \mathbb{R} \colon |x - a| < \epsilon\}$$

is called the ϵ-**neighborhood** of a.

Limit and Neighborhoods



a is the limit of (a_n) if once a_N **enters** the neighborhood $V_\epsilon(a)$, all a_n that follow will stay in it. That is, the a_n cluster around a in a very specific manner.

Note that this implies that if (a_n) converges, its limit is unique: the a_n cannot be in both $V_\epsilon(a)$ and $V_\epsilon(b)$ if $\epsilon < 1/2|a-b|$.

Exercise

Let $a_n = \frac{2n^2+n+1}{n^2}$. It can be written as

$$a_n = 2 + \frac{1}{n} + \frac{1}{n^2}$$

It is now easy to see that $\lim a_n = 2$: Just notice that

$$|a_n - 2| = \frac{1}{n} + \frac{1}{n^2} \leq 2\frac{1}{n}$$

and we can use the argument of the previous Example to finish.
Exercise: For every real number $x \in \mathbb{R}$, there exists a sequence (a_n) of rational numbers such that $(a_n) \to x$.

Limit Template

Let us summarize the procedure to compute the limit of a sequence:

$(a_n) \to a$ involves all the following steps:

1. Let $\epsilon > 0$ be arbitrary
2. Demonstrate a choice for $N \in \mathbb{N}$: hard work here often
3. Assume $n \geq N$
4. Check that

$$|a - a_n| < \epsilon$$

Example

Define the sequence

$$a_1 = \sqrt{2}, \quad a_2 = \sqrt{2\sqrt{2}}, \quad a_3 = \sqrt{2\sqrt{2\sqrt{2}}}, \cdots$$

Question: $(a_n) \to ?$ Note

$$a_1 = \sqrt{2}, \quad a_2 = a_1 \sqrt[4]{2}, \quad a_3 = a_2 \sqrt[8]{2}, \cdots$$

$$a_n = 2^{1/2+1/4+\cdots+1/2^n} < 2$$

So this sequence is bounded [and increasing]. Show that its least upper bound is 2.

Infinity as the limit of a sequence

If a sequence (a_n) is not **convergent**, we say that it is **divergent.** We also use the following terminology for some divergent sequences:

Definition

The sequence (a_n) converges to ∞, $\lim a_n = \infty$, if given any positive number b, there is an $N \in \mathbb{N}$ such that $a_n \geq b$ for $n \geq N$.

Example: $\{1, 2, 3, \ldots, n, \ldots\}$

Some sequences don't make up their minds:

1. $1, -1, 1, \ldots, \pm 1, \ldots$
2. one gets a very complicated sequence by glueing two unrelated sequences (a_n), (b_n), as in

$$a_0, b_0, a_1, b_1, a_2, b_2, \ldots, a_n, b_n, \ldots,$$