Math 300–03

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Set 3

Fall 2008

Outline

1. Cartesian Products and Relations
2. Equivalence Relations
3. Homework #6
4. Partitions
5. Last Class ... and Today ...
6. Ordering Relations
7. Homework #7
8. Graphs
9. Homework #8

Cartesian Products and Relations

Two of the most important mathematical objects are **relations** and **functions**.

They reflect special relationships between elements of a set A or of pairs of elements of two sets A and B.

Ordered Pair

Definition

Let A and B be sets. For $a \in A$ and $b \in B$, the **ordered pair** (a, b) is the set

$$\{\{a\}, \{a, b\}\}.$$

a is called the first coordinate of the pair, and b the second coordinate.

Note that (a, b) may be different from (b, a):

$$\{\{a\}, \{a, b\}\} \neq \{\{b\}, \{a, b\}\},$$

if $a \neq b$.

Definition

Let A and B be sets. The set of all ordered pairs having first coordinate in A and second coordinate in B is called the **Cartesian product** of A and B and written $A \times B$. Thus

$$A \times B = \{(a, b) : a \in A \quad \text{and} \quad b \in B.\}$$

Example: Let $A = \{a, b\}$, $B = \{1, 2, 3\}$. Then:

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}.$$

Theorem

If A and B are finite sets, then

$$\overline{\overline{A \times B}} = \overline{\overline{A}} \cdot \overline{\overline{B}}.$$

Ordered triples, quad...

Ordered triples, quadruples, *n*-tuples–can also be defined:

Definition

Let A, B and C be sets. For $a \in A, b \in B$ and $c \in c$, the **ordered triple** (a, b, c) is the set

$$\{\{a\}, \{a, b\}, \{a, b, c\}\}.$$

a is called the first coordinate of the pair, b the second coordinate, and c the third coordinate.
The set of all these ordered triples is the cartesian product $A \times B \times C$.

For a finite collection $A_1, A_2, \ldots, A_n$, we may define

$$A_1 \times A_2 \times \cdots \times A_n.$$

Some Tools

Theorem

If A, B, C and D are sets, then

1. $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
2. $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
3. $A \times \emptyset = \emptyset$.
4. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
5. $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.
6. $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$.

Proof. To prove (1), $A \times (B \cup C) = (A \times B) \cup (A \times C)$,

1. The ordered pair $(x, y) \in A \times (B \cup C)$
2. iff $x \in A$ and $y \in B \cup C$
3. iff $x \in A$ and ($y \in B$ or $y \in C$)
4. iff ($x \in A$ and $y \in B$) or ($x \in A$ and $y \in C$)
5. iff $(x, y) \in A \times B$ or $(x, y) \in A \times C$
6. iff $(x, y) = (A \times B) \cup (A \times C)$,
7. Therefore, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Relations

Definition

Let A and B be sets. R is a **relation from A to B** iff R is a subset of $A \times B$,

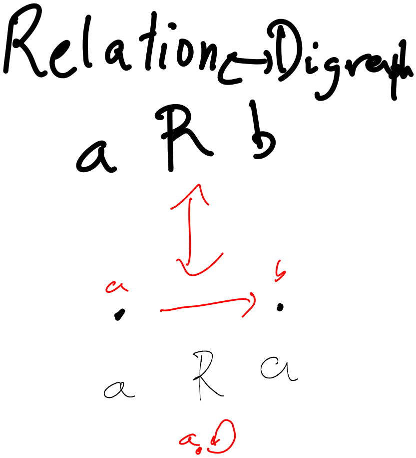
$$R \subset A \times B.$$

If $(a, b) \in R$ we write $a \; R \; b$ and say that a is R-related to b. If $(a, b) \notin R$, we write $a \not R \; b$. A relation R from A to A is called a **relation of A**: $R \subset A \times A$.

There are many notations for relations: familiar ones are $a \simeq b$, $a \geq b$, $a \mid b$, etc.

Example: I_A (**identity** of A is the relation $a \simeq b$ iff $a = b$. Another: $a \; R \; b$ for all $a, b \in A$.

Graphical Representation



Example

Let $A = \{1, 2, 3, 4\}$ and $B = \{-1, 1, 2, 4, 5\}$. We first describe a relation R from A to B with an explicit list of pairs:

$$R = \{(1, 4), (2, 5), (2, -1), (4, 1)\}$$

In table form, we would write R as

1	4
2	5
2	−1
4	1

The same relation R may be written as $R = \{(x, y) \in A \times B : |x - y| = 3\}$. The graph of R is shown in Figure 3.1.

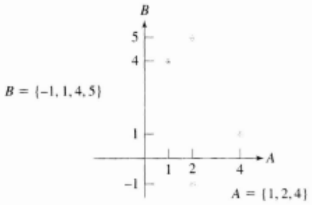


Figure 3.1

Definition

The **domain** of a relation R from A to B is the set

$$\text{Dom(R)}= \{x \in A : \text{there exists } y \in B \text{ such that } x \ R \ y\}.$$

The **range** of the relation R is the set

$$\text{Rng(R)}= \{y \in B : \text{there exists } x \in A \text{ such that } x \ R \ y\}.$$

Example

Let $S = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} : \frac{x^2}{324} + \frac{y^2}{64} \leq 1\right\}$. The graph of S is given in Figure 3.3.

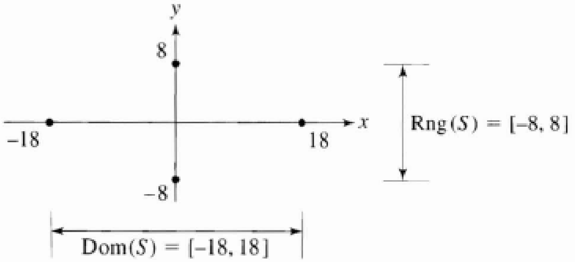
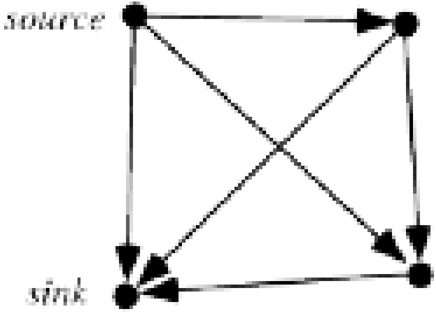


Figure 3.3

From Relations to Graphs



Plenty of Relations

Question: If A is a set with m elements, and B is another set with n elements, how many relations are there from A to B?

Answer: A lot: Since $A \times B$ has mn elements, and relations are subsets of $A \times B$, each relation is an element of the power set $\mathcal{P}(A \times B)$ of $\times B$. Thus there are 2^{mn} relations. Thus, if $A = \{a, b\}$, the number of relations of A (that is, from A to A) is $2^{(2)(2)} = 2^4 = 16$.

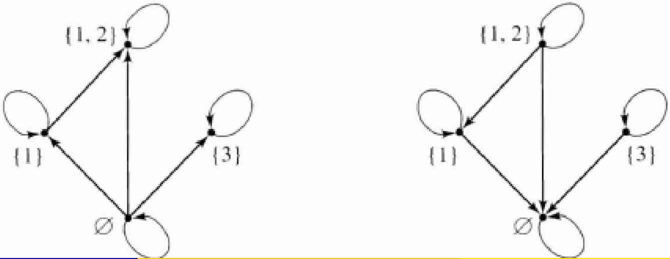
New Relations from Old

Definition

If R is a relation from A to B, the **inverse** of R is

$$R^{-1} = \{(y, x) : (x, y) \in R\}.$$

The digraph of the inverse of a relation on a set differs from the digraph of the relation only in that the directions of the arrows are reversed. Figure 3.8 shows the digraphs of R and R^{-1}, where R is the relation $\subseteq$ on the set $\{\varnothing, \{1\}, \{3\}, \{1, 2\}$



Theorem

Let R be a relation from A to B.

1. R^{-1} *is a relation from B to A.*
2. $Dom(R^{-1}) = Rng(R)$.
3. $Rng(R^{-1}) = Dom(R)$.

Proof.

1. Suppose $(x, y) \in R^{-1}$. Then $(y, x) \in R$. Since R is a relation from A to B, $R \subseteq A \times B$. Thus $y \in A$ and $x \in B$. Therefore $(y, x) \in B \times A$, which proves $R^{-1} \subseteq B \times A$.
2. $y \in$Dom(R^{-1}) iff there exists $y \in A$ such that $(x, y) \in R$ iff $x \in$Rng(R).
3. Same argument as (2).

New Relations from Old

Definition

Let R be a relation from A to B, and let S be a relation from B to C. the **composite** of R and S is

$$S \circ R = \{(a, c) : \text{there exists } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}.$$

Confusing Diagram...

Let $A = \{1, 2, 3, 4, 5\}$, and $B = \{p, q, r, s, t\}$, and $C = \{x, y, z, w\}$.
Let R be the relation from A to B:

$$R = \{(1, p), (1, q), (2, q), (3, r), (4, s)\}$$

and S the relation from B to C:

$$S = \{(p, x), (q, x), (q, y), (s, z), (t, z)\}.$$

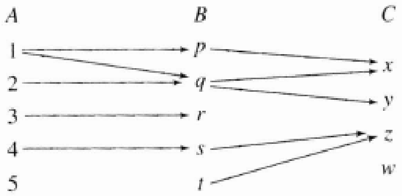


Figure 3.10

Theorem

Suppose A, B, C and D are sets. Let R be a relation from A to B, S a relation from B to C, and T a relation from C to D:

$$A \xrightarrow{R} B \xrightarrow{S} C \xrightarrow{T} D.$$

1. $(R^{-1})^{-1} = R$.
2. $T \circ (S \circ R) = (T \circ S) \circ R$.
3. $I_B \circ R = R$ and $R \circ I_A = R$.
4. $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Proof of (2): Note both $T \circ (S \circ R)$ and $(T \circ S) \circ R$ are relations from A to D, that is they are subsets of $A \times D$.

To prove $T \circ (S \circ R) = (T \circ S) \circ R$, let $a \; T \circ (S \circ R) \; d$. Note that $S \circ R$ is a relation from A to C.

1. Thus there is $c \in C$ such that $a \; S \circ R \; c$ and $c \; T \; d$.
2. Hence there is $b \in B$ such that $b \; S \; c$.
3. Therefore $b \; T \circ S \; d$.
4. Since $a \; R \; b$ and $b \; T \circ S \; d$, it follows that $a \; (T \circ S) \circ R \; d$.
5. This shows that $T \circ (S \circ R) \subseteq (T \circ S) \circ R$. The reverse inequality has a similar proof.

Last Class ... and Today ...

Relation

A **relation** from a set A to a set B is a subset

$$R \subset A \times B.$$

If $(a, b) \in R$, we also write

$$a \; R \; b \quad a \stackrel{R}{\to} b \quad a \to b$$

The two extreme examples are: $R = \{(a, a) : a \in A\}$, $R = \{(a, b) : a, b \in A\}$. The first is the **identity** of A.

4 Examples

Let $\mathbb{N}$ be the set of natural numbers. Define

$$a \rightarrow_1 b \quad a \text{ divides } b$$

$$a \rightarrow_2 b \quad a = b + 1$$

$$a \rightarrow_3 b \quad a \text{ does not divide } b$$

$$a \rightarrow_4 b \quad a \text{ and } b \text{ are prime numbers and } a = b + 2$$

Goldbach conjecture

Outline

Equivalence Relations

Definition

Let A be a set and R a relation on A.

- R is reflexive iff for all $x \in A$, $x \; R \; x$.
- R is symmetric iff for all $x \in A$ and $y \in A$, if $x \; R \; y$, then $y \; R \; x$.
- R is transitive iff for all x, y and z in A, if $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$.

Example

Let R be the set of all $(x, y) \in \mathbb{N} \times \mathbb{N}$ such that $x + y$ is divisible by 3. Is this relation symmetric? reflexive? transitive?

1. If $(x, y) \in R$, $x + y = 3m$, for some m. Then $y + x = 3m$ also, so $(y, x) \in R$: **Symmetric**
2. $(1, 1) \notin R$: **Not Reflexive**
3. $(1, 2), (2, 1) \in R$ but $(1, 1) \notin R$: **Not Transitive**

Example

Let R be the set of all $(x, y) \in \mathbb{N} \times \mathbb{N}$ such that $x + y^2$ is divisible by 2. Is this relation symmetric? reflexive? transitive?

1. If $(x, y) \in R$, $x + y^2 = 2m$, for some m, that is $x + y^2$ is even. If x is even then y^2 must be even so y must be even, while if x is odd then both y^2 and y must be odd. Then $y + x^2 = 2n$ also, so $(y, x) \in R$: **Symmetric**
2. $(x, x) \in R$ since $x + x^2 = x(x + 1)$ is even: **Reflexive**
3. If $(x, y), (y, z) \in R$ we have: If $(x, y) \in R$, if x is even (odd), y is also even (odd), so if $(y, z) \in R$, x is also even (odd).Thus x, z are both even or both odd, so $(x, z) \in R$: **Transitive**

Equivalence Relation

Definition

A relation R on a set A is an **equivalence relation on** A iff R is reflexive, symmetric, and transitive.

Let R be a relation on the set A.

- **R Reflexive**: $a \to a \, \forall a \in A$
- **R Symmetric**: $a \to b \Rightarrow b \to a$
- **R Transitive**: $a \to b$ and $b \to c \Rightarrow a \to c$

Exercise

Problem: If $A = \{a, b\}$, we saw that there are 16 relations of A. How many of these are equivalence relations? List them all. What if $A = \{a, b, c\}$, a set with $2^9 = 512$ relations, how many of these are equivalence relations?

One Volunteer:

Equivalence Class

Definition

Let R be an equivalence relation on the set A. For $x \in A$, the **equivalence class of** x determined by R is the set

$$x/R = \{y \in A : x \; R \; y\}.$$

This is read "the class of x modulo R." The set of all equivalence classes of R is called A **modulo** R and denoted $A/R = \{x/R : x \in A\}$.

Example: Two integers have the same **parity** if they are both even or both odd. Let
$R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x \text{ and } y \text{ have the same parity.}\}$ R is an equivalence relation with two equivalence classes: the even integers E and the odd integers D. $\mathbb{Z}/R = \{E, D\}$.

Big Example

Let m be a fixed, nonzero integer. Let $\equiv_m$ be the relation on $\mathbb{Z}$,

$$x \equiv_m y \text{ iff } m \text{ divides } x - y.$$

This is also written $x \equiv y \pmod{m}$ or even $x = y \pmod{m}$.
It is easy to see that $\mathbb{Z}/\equiv_2 = \{E, D\}$. This set is also denoted by $\mathbb{Z}_2$ and called the set of integers modulo 2. For $m = 3$, $\equiv_3$ is also an equivalence relation and there are three distinct equivalence classes.

Theorem

The relation $\equiv_m$ is an equivalence relation on the integers. The set of equivalence relations is called $\mathbb{Z}_m$ and has m distinct elements $\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{m-1}$.

Proof

We first prove that $\equiv_m$ is an equivalence relation. Observe that $x \equiv_m y$ means that $x - y = am$, for some integer a.

1. reflexive: $x \equiv_m x$: $x - x = 0 \cdot m$.
2. symmetric: $x \equiv_m y$: $x - y = a \cdot m$ for some $a \in \mathbb{Z}$. Thus $y - x = (-a)m$, therefore $y \equiv_m x$.
3. transitive: If $x \equiv_m y$ and $y \equiv_m z$, $x - y = am$ and $y - z = bm$ for a and b in $\mathbb{Z}$. Then

$$x - z = (x - y) + (y - z) = am + bm = (a + b)m.$$

Therefore $x \equiv_m z$.

Now we determine the equivalence classes of $\equiv_m$.

1. The numbers $0, 1, 2, \ldots, m-1$ lie in different equivalence classes: For any pair of them, i and j, we cannot have $i \equiv_m j$ since $i - j$ cannot be divisible by m.
2. If x is an integer, by the Euclidean algorithm,
$$x = qm + r, \quad 0 \leq r < m.$$
Thus, $x \equiv_m r$. Therefore $r \in \mathbb{Z}/\equiv_m$ (usually denoted by $\overline{r}$).
3. Therefore
$$\mathbb{Z}_m = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{m-1}\}.$$

Bonus

The $\mathbb{Z}_2$, the set made up by two elements $\{0, 1\}$ (or (even, odd))with addition defined by the table

$+$	0	1
0	0	1
1	1	0

$1 + 1 = 0!$

and multiplication by

$\times$	0	1
0	0	0
1	0	1

More Bonus

The $\mathbb{Z}_3$, the set made up by three elements $\{\bar{0}, \bar{1}, \bar{2}\}$ (or more simply, $\{0, 1, 2\}$)has similar properties. For instance, with addition defined by the table

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$1 + 2 = 0!$

and multiplication by

$\times$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$2 \times 2 = 1!$

Exercise

Let A be the set of all vectors of the plane. We look at A as the set of all pairs (a, b) of real numbers. The usual notation for A is $\mathbb{R}^2$.
Define the following relation of A:

$$(a, b) \simeq (c, d) \quad (a, b) - (c, d) = (r, r) \text{ for some } r \in \mathbb{R}$$

This means that the two vectors (a, b) and (c, d) differ by a vector along the diagonal of the first quadrant.

1. Prove that $\simeq$ is an equivalence relation of A.
2. Prove that every equivalence class contains exactly a vertical vector (that is, a vector of the form $(0, c)$).

Outline

1. 3.1: 6(g), 8(d), 9(a), 12, 13(c), 16
2. 3.2: 2(f), 4(b), 5(b), 8, 10(a), 13
3. How many equivalence relations are there on the set $\{a, b, c, d, e\}$? Explain [This is a typical exam question]

Outline

Partitions

Definition

Let A be a nonempty set. A **partition** of A is a set $\mathcal{A}$ of subsets of A such that

1. If $X \in \mathcal{A}$, then $X \neq \emptyset$.
2. If $X \in \mathcal{A}$ and $Y \in \mathcal{A}$, $X \neq Y$, then $X \cap Y = \emptyset$.
3. $\bigcup_{X \in \mathcal{A}} X = A$.

How do partitions arise? It will be a pretty straight answer.

Example

Let $A = \{a, b, c\}$–the following are partitions of A:

- $\{\{a, b, c\}\}$
- $\{\{a\}, \{b\}, \{c\}\}$,
- $\{\{a\}, \{b, c\}\}$,
- $\{\{b\}, \{a, c\}\}$
- $\{\{c\}, \{a, b\}\}$

Are they all?

Partitions versus Equivalence Classes

Theorem

Let $\mathcal{B}$ be a partition of the nonempty set A. For x and y in A, define $x\ Q\ y$ iff there exists $C \in \mathcal{B}$ such that $x \in C$ and $y \in C$. Then

1. *Q is an equivalence relation on A.*
2. *$A/Q = \mathcal{B}$.*

Example: Define the following sets of $\mathbb{Z}$:

$$\begin{aligned} A_0 &= \{3k : k \in \mathbb{Z}\} = \{\ldots, -6, -3, 0, 3, 6, \ldots\} \\ A_1 &= \{3k+1 : k \in \mathbb{Z}\} = \{\ldots, -5, -2, 1, 4, 7, \ldots\} \\ A_2 &= \{3k+2 : k \in \mathbb{Z}\} = \{\ldots, -4, -1, 2, 5, 8, \ldots\} \end{aligned}$$

The partition defines the relation we denoted $\equiv_3$.

How Partitions arise

Theorem

Let R be an equivalence relation on a nonempty set A. Then

1. *For all $x \in A$, $x/R \subseteq A$ and $x \in x/R$. (Thus $x/R \neq \emptyset$.*
2. $\bigcup_{x\in A} x/R = A$.
3. *$x\ R\ y$ iff $x/R = y/R$.*
4. *$x \not R\ y$ iff $x/R \cap y/R = \emptyset$.*

Thus, the set $\{x/R : x \in A\}$ of equivalence classes is a partition of A.

In words: An equivalence relation on the set A gives rise to a partition of A and vice-versa.

Outline

Last Class ... and Today ...

- Relations
- Equivalence Relation
- Partitions and Equivalent Classes
- ◦ Order Relations

Outline

Ordering Relations

Definition

A relation R on a set A is antisymmetric if, for all $x, y \in A$, if $x \; R \; y$ and $y \; R \; x$, then $x = y$:

$$x \to y \to x \quad \Rightarrow x = y.$$

Definition

A relation R on a set A is a **partial order** (or **partial ordering**) for A if R is reflexive on A, antisymmetric and transitive. A set A with a partial order R is called a **partially ordered set**, or **poset**.

Typical Digraph

Observe the antisymmetry and transitivity:

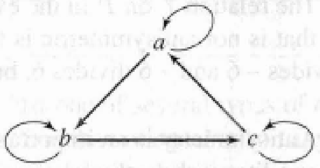


Figure 3.14

Top Example

Let X be a set and let $\mathcal{P}(X)$ be its power set. If A and B are in $\mathcal{P}(X)$, i.e. A and B are subsets of X, define the relation

$$A \quad R \quad B \quad \textit{iff} \quad A \subset B.$$

Let us check for the **reflexive**, **antisymmetric** and **transitive** properties of a relation:

1. $(A, A) \in R$ since $A \subset A$;
2. If $(A, B) \in R$ and $(B, A) \in R$ then $A \subset B$ and $B \subset A$ and therefore $A = B$ by the definition of equality of sets (same elements);
3. If $(A, B) \in R$ and $(B, C) \in R$ then $A \subset C$ and therefore $(A, C) \in R$.

Example: Divisors

Let M be a positive integer and let A be the set

$$A = \{n \in \mathbb{N} : n \mid M.\}$$

Consider the relation D 'divides'
For example, for $M = 12$, $A = \{1, 2, 3, 4, 6, 12\}$. Then D consists of the pairs (a, b) where a divides b. For example, $(2, 6) \in D$ but not $(4, 6)$.

Theorem

If R is a partial order for a set A and $x\ R\ x_1$, $x_1\ R\mathbf{x}_2$, $x_2\ R\ x_3$, $\ldots, x_n\ R\ x$,

$$x_1 \to x_2 \to x_3 \to \cdots \to x_n \to x_1,$$

then $x = x_1 = x_2 = x_3 = \cdots = x_n$.

This means:

$$x \to x_1 \to x_2 \to x \Rightarrow x \to x_2 \to x \Rightarrow x = x_2 \Rightarrow x \to x_1 \to x$$

and

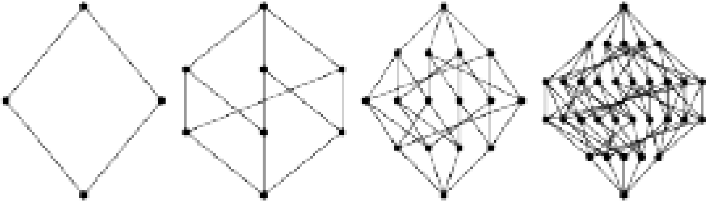
$$x \to x_1 \to x \Rightarrow x = x_1$$

Therefore $x = x_1 = x_2$.

Proof. By induction on n.

1. For $n = 1$: If $x \; R \; x_1$ and $x_1 \; R \; x$, by antisymmetry $x = x_1$.
2. Suppose that whenever $x \; R \; x_1$, $x_1 \; Rx_2$, $x_2 \; R \; x_3, \ldots, x_k \; R \; x$, then $x = x_1 = x_2 = x_3 = \cdots = x_k$ for some natural number k, and suppose $x \; R \; x_1$, $x_1 \; Rx_2$, $x_2 \; R \; x_3, \ldots, x_k \; R \; x_{k+1}, x_{k+1} \; R \; x$. By transitivity applied to $x_k \; R \; x_{k+1}, x_{k+1} \; R \; x$, we have $x_k \; R \; x$. Now we use he induction hypothesis to deduce $x = x_1 = x_2 = \cdots = x_k$. Since $x_k = x$, we have $x \; R \; x_{k+1}$ and $x_{k+1} \; R \; x$, so $x = x_{k+1}$. Therefore $x = x_1 = x_2 = x_3 = \cdots = x_{k+1}$.

Digraphs



Some Terminology

Definition

Let R be a partial ordering on a set A, and let $a, b \in A$ with $a \neq b$. Then a is an **immediate predecessor** of b if $a \; R \; b$ and there does not exist $c \in A$ such that $a \neq c$, $b \neq c$, $a \; R \; c$ and $c \; R \; b$.

Example: For $X = \{1, 2, 3, 4, 5\}$, partially order $\mathcal{P}(X)$ by set inclusion $\subseteq$. For $b = \{2, 3, 5\}$, there are 3 immediate predecessors: $\{2, 3\}$, $\{3, 5\}$ and $\{2, 5\}$.

Example

Let $M = \{1, 2, 3, 5, 6, 10, 15, 30\}$ be the set of divisors of 30, and let D be the relation "divides".

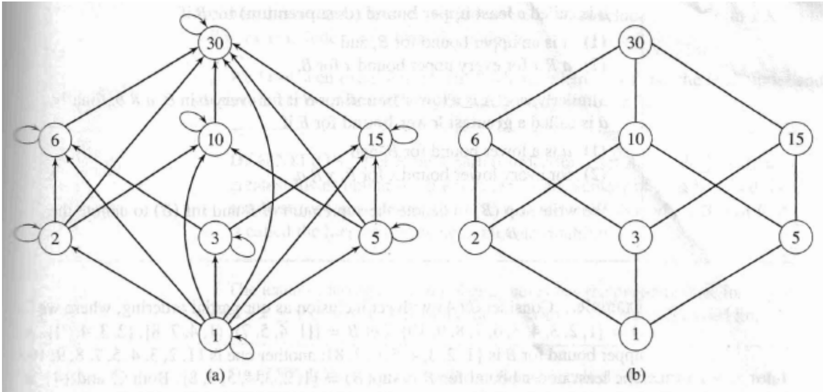


Figure 3.15

Example

Let $A = \mathcal{P}(\{1, 2, 3\})$, ordered by $\subseteq$:

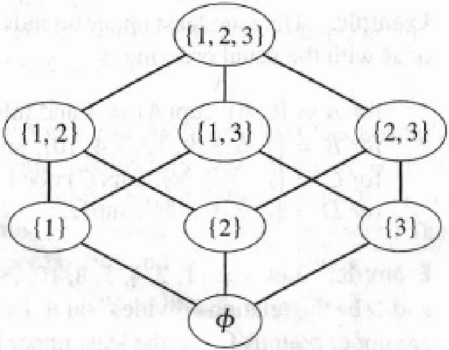


Figure 3.16

More Terminology

Definition

Let R be a partial order for A and let B be a subset of A. Then $a \in A$ is an **upper bound** for B if for every $b \in B$, $b\ R\ a$. Also, a is called a **least upper bound** (or **supremum**) for B if

1. a is an upper bound for B, and
2. $a\ R\ x$ for every upper bound x for B.

Similarly, $a \in A$ is an **lower bound** for B if for every $b \in B$, $a\ R\ b$. Also, is called a **greatest lower bound** (or **infimum**) for B if

1. a is a lower bound for B, and
2. for every lower bound x for B, $x\ R\ a$.

In notation: $\sup(B)$ will denote the supremum of B, and $\inf(B)$ will denote the infimum.

Example

Examples: Subsets of $\mathbb{R}$ with the relation $\leq$:

- $A = [0, 4)$: Sup$(A) = 4$ and Inf$(A) = 0$.
- $B = \{2^k : k \in \mathbb{N}\}$: Sup$(B)$ does not exist and Inf$(B) = 2$.
- $C = \{x : x \in \mathbb{Q},\ x^2 < 2\}$: Sup$(C) = \sqrt{2}$ and Inf(C) does not exist.
- $D =$ the set of roots of $x^2 - 3x + 2 = 0$: ??
- $E =$ numbers in $(0, 1)$ which are not fractions (not $p/q, p \neq 0, q \in \mathbb{N}$): ??

More Terminology

Definition

Let R be a partial order for a set A. Let $B \subseteq A$. If the greatest lower bound for B exists and is an element of B, it is called the **smallest** (or least) **element** of B. If the least upper bound for B is in B, it is called the **largest** (or greatest) **element** of B.

Definition

A partial ordering R on A is called a **linear order**(or **total order**) on A if for any two elements x and y of A, either $x\ R\ y$ or $y\ R\ x$, that is

$$x \rightarrow_R y \quad \text{or} \quad y \rightarrow_R x.$$

Definition

Let L be a linear ordering on a set A. L is a **well ordering** on A if every nonempty subset B of A contains a smallest element.

Examples

1. WOP: The Well-Ordering Principle says that $\mathbb{N}$, with the ordering given by $\leq$, is a linear ordering with property above.
2. Let A be a partial ordered set, called “the alphabet.” Let W be the set of all “words’ of length two–that is, combinations of two letters of the alphabet. Define a relation $\leq$ on W as follows: for $x_1x_2 \in W$ and $y_1y_2 \in W$, $x_1x_2 \leq y_1y_2$ iff (i) $x_1 \leq x_2$ or (ii) $x_1 = x_2$ and $y_1 \leq y_2$. We claim that W is a partial ordering of W (called lexicographic ordering, as in a dictionary).

Example

Let $A = \{a, b, c\}$. Given an example of a relation on A that is

1. antisymmetric and symmetric

$$\{a\ R\ a, b\ R\ b\}$$

2. symmetric and not antisymmetric

$$\{a\ R\ b, b\ R\ a, c\ R\ c\}$$

3. antisymmetric, reflexive on A and not symmetric

$$\{a\ R\ a, b\ R\ b, c\ R, a\ R\ b\}$$

1. antisymmetric, not reflexive on A and not symmetric

$$\{a \; R \; b, b \; R \; c, a \; R \; c\}$$

Exercise

Existence: Let $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = 1 - y\}$. Is S antisymmetric?
Need to show: If $(x, y), (y, x) \in S$ then $(x, x) \in S$

$(1, 0) \in S$ and $(0, 1) \in S$ but $(0, 0) \notin S$.

$x = 1 - y \Rightarrow y = 1 - x$ therefore S is symmetric.

Upper and Lower Bounds

- The number b is said to be an **upper bound** of the set $A \subset \mathbb{R}$ if

$$a \leq b \mid \forall a \in A$$

- A number ℓ is said to be a **lower bound** of the set $A \subset \mathbb{R}$ if

$$a \geq \ell \mid \forall a \in A$$

- Consider the set $A = \{q \in \mathbb{Q} \mid q^2 < 2\}$. -2 is a lower bound of A, while $3/2$ is an upper bound. Clearly there are many other bounds.

Least Upper and Greatest Lower Bounds

- A number b is said to be a **least upper bound** of the set $A \subset \mathbb{R}$ if b is an upper bound of A and $b \leq b'$ for any other upper bound b'. Least upper bounds are also known as the **supremum** of A. If $b \in A$, it is called the **maximum** of A.

$$A = \{x_1 = 1, \forall n \quad x_{n+1} = \frac{x_n}{2} + 1\}$$

has 2 for supremum [needs a proof, as we only proved that 2 is an upper bound]

- Similarly we define **greatest lower bound** [and of **infimum/minimum**].

Example

Define the set $\mathbf{A} = \{a_1, a_2, a_3, \cdots\}$ by the rule

$$a_1 = \sqrt{2}, \quad a_2 = \sqrt{2\sqrt{2}}, \quad a_3 = \sqrt{2\sqrt{2\sqrt{2}}}, \cdots$$

Let us show that $\sup \mathbf{A} = 2$:

$$a_1 = \sqrt{2}, \quad a_2 = a_1\sqrt[4]{2}, \quad a_3 = a_2\sqrt[8]{2}, \cdots$$

$$a_n = 2^{1/2+1/4+\cdots+1/2^n} < 2$$

$$a_n = 2^r, \quad r = \frac{1/2 - 1/2^{n+1}}{1/2} = 1 - 1/2^n$$

Axiom of Completeness

Axiom: Every set *A* of real numbers with an upper bound has a least upper bound.

This is a defining property of $\mathbb{R}$. A lot flows out of it. For example consider the set *A* of all rational numbers x such that $x^2 < 2$. This set has an **upper bound** (in fact many). For instance, $x \leq 3$. The **axiom of completeness** guarantees that there is a **real** number α such that

$$\alpha^2 = 2.$$

Outline

Homework #7

1. 3.3: 3(a,b,c), 7(b), 9, 12
2. 3.4: 2(a,b), 3(all), 9, 12, 13(b), 20
3. Let D be a positive integer and let A be the set

$$A = \{n \in \mathbb{N} : n \mid D.\}$$

Prove that A is linearly ordered iff D is the power of a prime.

Outline

Graphs

Definition

A **graph** G is a pair (V, E), where V is a nonempty set and E is a set of unordered pairs of distinct elements of V.
An element of V is called a **vertex** and an element of E is called an **edge**. An edge between the vertices u and v is written **uv** rather than as the set $\{u, v\}$.

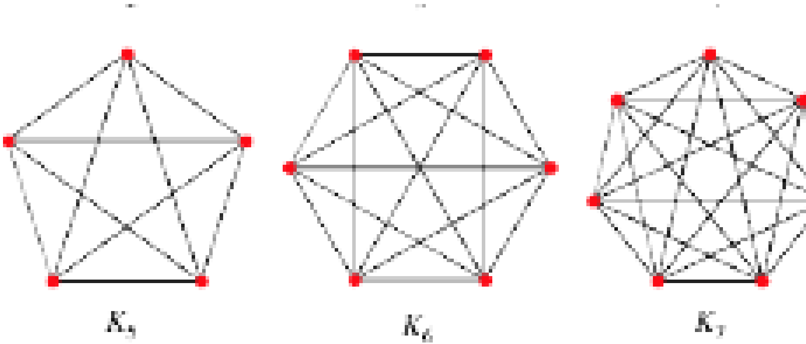
Terminology

Definition

Let $G = (V, E)$ be a graph. The **order** of the graph G is the number of vertices. The **size** of the graph G is the number of edges.
Vertices u and v are **adjacent** if $uv \in E$; the edge uv is said to be **incident** with u and v.
The **degree** of a vertex u is the number of edges incident with u.

A graph G of order n, may have no edges, all the vertices are isolated. Such a graph is called the **null graph**.
A graph such that every pair of vertices are adjacent is called a **complete graph**. If it has n vertices, its size is $\binom{n}{2}$. Its notation; K_n.

K_n



Exercises

If possible, give an example of a graph

1. with order 6 and size 6
2. with order 4 and size 6
3. with order 3 and size 6
4. with order 6 and size 3

Properties of Graphs

Theorem

- *(The Handshaking Lemma) For each graph G, the sum of the degrees of the vertices of G is even.*
- *For every graph G the number of vertices of G having odd degree is even.*

Proof.

- Each edge is incident with two vertices. Thus

$$\sum_{v\in V} \deg(v) = 2 \times \text{(number of edges)}.$$

- Let V_1 be the subset of vertices of odd degree and V_2 the subset of vertices of even degree.

$$\underbrace{\sum_{v\in V} \deg(v)}_{\text{even number}} = \sum_{v\in V_1} \deg(v) + \underbrace{\sum_{v\in V_2} \deg(v)}_{\text{even number}}.$$

Therefore the term

$$\sum_{v \in V_1} \deg(v)$$

being a difference of two even numbers is even. Since each term $\deg(v)$, $v \in V_1$, is odd, we must have an even number of them for the sum to be even. □

Walks

Definition

A **walk** (or **path**) in a graph G is a finite sequence of vertices

$$v_0, v_1, v_2, \ldots, v_m$$

where each $v_i v_{i+1}$ is an edge in G. The vertex v_0 is the **initial vertex** and v_m is the **terminal vertex**. The length of the walk is m, the number of edges. If $v_0 = v_m$ the walk is **closed**. A **path** in G is a walk where all the vertices, except for possibly the initial and terminal vertices, are distinct.

The walk $v_0, v_1, v_2, \ldots, v_m$ is said to **traverse** the vertices of the sequence.

Theorem

Let G be a graph of order n.

1. *If there is a walk originating at v and terminating at u, then there is a path from v to u.*
2. *The length of a path in G that is not closed is at most $n-1$. The length of a closed path is at most n.*

Proof. Let $v, v_1, v_2, \ldots, u$ be a walk with $u \neq v$. If the walk is not a path, some vertex v_i appears twice in the sequence. Let x be the first such vertex. Then the walk contains at least a closed walk of the form $x, v_j, \ldots, v_m, x$. Delete the vertices $v_j, \ldots, v_m, x$. If the result is a path, we are done. Otherwise another repeated vertex occurs and we repeat the process until all duplications are deleted. The process will result in a path.

Proof Cont'd

In case $v = u$ the process produces a closed path [where v is the only repeated vertex].
The walks that arise at the end are:

$$v \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_m \rightarrow u, \quad v \neq u$$

$$v \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_m \rightarrow u, \quad v = u$$

Note that the length is the number of arrows. Counting the $\rightarrow$ gives the assertions: at most $n - 1$ in the first case, and at most n in the second.

Reachability

Definition

Let G be a graph and u a vertex of G. The vertex v is **reachable** (or **accessible**) from u if there is a path from u to v. The number of edges in a path of minimum length from u to v is called the **distance from** u **to** v, written $d(u, v)$. We say that any vertex u is reachable from itself and $d(u, u) = 0$.

Connected Components

Definition

Let G be a graph. If u is a vertex of G, the **component containing** u is $C(u) = \{v \in V : v \text{ is reachable from } u\}$.

Theorem

Let G be a graph with vertex set V and let R be the relation defined by $v\ R\ u$ if u is reachable from v. Then R is an equivalence relation on V and the set of equivalence classes for R is $\{C(u) : u \in V\}$.

Definition

A graph G is **connected** iff every vertex is reachable from every other vertex. G is **disconnected** if it is not connected.

Exercise

Give an example of a graph G with 6 vertices having

1. One component
2. two components
3. three components
4. six components

Outline

Homework #8

1. 3.5: 7, 9, 11
2. 4.1: 2(a,b,c,d), 3(e), 9, 12(a,b), 16(b), 17(a)