ON THE COMPUTATION OF THE JDEG OF BLOWUP ALGEBRAS

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Abstract. For a graded algebra $\mathbf{A}$, its jdeg($\mathbf{A}$) is a global degree that can be used to study issues of complexity of the normalization $\overline{\mathbf{A}}$. Here some techniques grounded on Rees algebra theory are used to estimate jdeg($\mathbf{A}$). A closely related notion, of divisorial generation, is introduced to *count* numbers of generators of $\overline{\mathbf{A}}$.

1. Introduction

Let R be a commutative Noetherian ring, $\mathbf{A}$ a semistandard graded R-algebra, and M a finitely generated graded $\mathbf{A}$-module. The terminology semistandard graded algebra will mean a finite extension of a standard graded algebra. They typically occur in processes of normalization. The jdeg of M is an invariant of M built out of all the local $\mathbf{j}$-multiplicities of M. The local $\mathbf{j}$–multiplicity was introduced and developed by Flenner, O'Carroll and Vogel ([6]), and has evolved into a rich extension of the ordinary multiplicity ([1], [7], [14], [18]). It is defined as follows. If R is a Noetherian local ring with maximal ideal $\mathfrak{m}$, $H^0_{\mathfrak{m}}(M)$ is the largest graded submodule of M which is annihilated by a sufficiently large power of $\mathfrak{m}$. Therefore, $H^0_{\mathfrak{m}}(M)$ can be considered as a graded module over the Artinian ring $R/\mathfrak{m}^k$ for some $k \gg 0$. We will make use of its Hilbert polynomial to define a new multiplicity function of M. If $\dim M = d$, then the $\mathbf{j}$-multiplicity of M with respect to $\mathfrak{m}$ is

$$j_{\mathfrak{m}}(M) = \begin{cases} \deg(H^0_{\mathfrak{m}}(M)) & \text{if } \dim H^0_{\mathfrak{m}}(M) = d, \\ 0 & \text{if } \dim H^0_{\mathfrak{m}}(M) < d. \end{cases}$$

It is noteworthy to observe that the condition on the equality $\dim H^0_{\mathfrak{m}}(M) = d$ can be verified by checking that $\dim M/\mathfrak{m}M = d$, according to [16, Propposition 1].

As the $\mathbf{j}$-multiplicity is defined with respect to a maximal ideal $\mathfrak{m}$, by localization we can extend this definition to all primes. For a ring R (not necessarily local), the $\mathbf{j}$-multiplicity of M at a given prime $\mathfrak{p}$ of R is defined as $j_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$, written simply as $j_{\mathfrak{p}}(M)$. We are now ready to define jdeg(M) ([15]):

Definition 1.1. Let R be a commutative Noetherian ring and $\mathbf{A}$ be a finitely generated semistandard graded R-algebra. For a finitely generated graded $\mathbf{A}$-module M,

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the jdeg of M is the integer

$$\mathrm{jdeg}(M) := \sum_{\mathfrak{p}\in \mathrm{Spec}\ R} j_{\mathfrak{p}}(M).$$

This is a finite sum. Often it does not say much about M, for instance if R is an integral domain and $\mathbf{A} = M = R[x]$, then $\mathrm{jdeg}(M) = 1$. The main use of this degree in [16] was to measure the change between $\mathbf{A}$ and one of its integral extensions $\mathbf{B}$. (Throughout, to ensure the finiteness of integral closure, we assume that R is unmixed.) The original framework for the use of jdeg were two results of [16]:

(1) ([16, Theorem 2]) Let R be a Noetherian domain and $\mathbf{A}$ a semistandard graded R-algebra. Consider a sequence of integral graded extensions
$$\mathbf{A} \subseteq \mathbf{A}_0 \to \mathbf{A}_1 \to \mathbf{A}_2 \to \cdots \to \mathbf{A}_n = \overline{\mathbf{A}},$$
where the $\mathbf{A}_i$ satisfy the S_2 condition of Serre. Then $n \leq \mathrm{jdeg}(\overline{\mathbf{A}}/\mathbf{A})$.

(2) ([16, Theorem 3]) Let R be a reduced, locally analytically unramified ring and I an R-ideal of Briançon-Skoda number $c(I)$. (This is the smallest integer c such that $\overline{I^{n+c}} \subset J^n$ for all n and all reductions J of I.) Then
$$\mathrm{jdeg}(\overline{R[It]}/R[It]) \leq c(I) \cdot \mathrm{jdeg}(\mathrm{gr}_I(R)).$$

Since $\overline{\mathbf{A}}$ may not be accessible, we have sought to estimate $\mathrm{jdeg}(\overline{\mathbf{A}}/\mathbf{A})$ in terms of other invariants of $\mathbf{A}$. Thus in the second of the results above, it devolves on the knowledge of the Briançon-Skoda number $c(I)$ (for which there are estimates in many cases of interest) and of $\mathrm{jdeg}(\mathrm{gr}_I(R))$. For the latter, an important technique is the *length formula* of [1, Theorem 3.8] that asserts that if $(R, \mathfrak{m})$ is a local ring of dimension $d > 0$ and infinite residue field, then for sufficiently general elements $a_1, a_2, \ldots, a_d$ of I, for $n \gg 0$
$$j_{\mathfrak{m}}(\mathrm{gr}_I(R)) = \lambda(R/((a_1, \ldots, a_{d-1}) : I^n + a_d R.$$
There is a detailed discussion of this formula in [14], with numerous applications.

We treat two aspects of these functions, one extending to blowup algebras of modules certain results on Rees algebras of ideals, and give applications on the use of jdeg to estimate the number of generators of a normalization. This paper is a sequel to [16], but while that paper focused on the connection between jdeg and *normalization* processes, here we aim at methods to estimate jdeg. Our approach emphasizes the use of symmetric algebras and approximation complexes. Our main results are Theorem 2.7, Theorem 4.2 and Theorem 5.1. In detail, the first of these asserts:

Theorem 2.7. Let k be a perfect field, and let $(R, \mathfrak{m})$ be a local domain which is a k-algebra essentially of finite type and dimension d. Suppose that R has isolated singularities and L is the Jacobian ideal of R. If $E \subset \mathfrak{m}R^r$ is a module such that $\lambda(R^r/E) < \infty$ and Buchsbaum-Rim multiplicity $\mathrm{br}(E)$, then
$$\mathrm{jdeg}(\overline{S[(E)t]}/S[(E)t]) \leq (d + r + \lambda(R/L) - 2) \cdot (\mathrm{br}(E) + 1).$$

The other results (Theorems 4.2, 5.1 and 5.4) are estimations of the jdeg of certain symmetric algebras of modules and of Rees algebras of prime ideals of dimension one and two.

Most of the notation and terminology are standard. Thus $\nu(E)$ stands for the minimal number of generators of a finitely generated R-module and $\lambda(E)$ for the length if E is also Artinian. For an R-ideal I, a reduction is an ideal $J \subset I$ such that $I^{r+1} = JI^r$ for some integer r, the smallest of which is referred to as the reduction number. In the treatment here, if R is a local ring of infinite residue field the minimal reductions of an ideal I all have the same number of generators, called the analytic spread of I and denoted by $\ell(I)$. For an ideal I, its integral closure is denoted by $\overline{I}$. The extensions of these notions to modules is discussed in the text (see also [20]).

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2. Normalization of modules

Let us recall the notion of the Rees algebra of a module. Let R be a Noetherian ring, let E be a finitely generated torsionfree R–module having a rank, and choose an embedding $\varphi : E \hookrightarrow R^r$. The *Rees algebra* $\mathcal{R}(E)$ of E is the subalgebra of the polynomial ring $R[t_1, \ldots, t_r]$ generated by all linear forms $a_1t_1 + \cdots + a_rt_r$, where $(a_1, \ldots, a_r)$ is the image of an element of E in R^r under the embedding φ. The Rees algebra $\mathcal{R}(E)$ is a standard graded algebra whose nth component is denoted by E^n and is independent of the embedding φ since E is torsionfree and has a rank. Let U be a submodule of E. The module E is *integral* over the module U if the Rees algebra of E is integral over the R-subalgebra generated by U . In this case we say that U is a *reduction* of E. A reduction is said to be *minimal* if it is minimal with respect to containment.

The algebra $\mathcal{R}(E)$ is a subring of the polynomial ring $S = R[t_1, \ldots, t_r]$. Following [10], we consider the ideal (E) of S generated by the forms in E. Denote by G the associated graded ring $\mathrm{gr}_{(E)}(S)$. The $\mathrm{jdeg}(\mathrm{gr}_{(E)}(S))$ encodes information about the Buchsbaum-Rim multiplicity of the module E according to [16, Proposition 7]:

Proposition 2.1. *Let $(R, \mathfrak{m})$ be a Noetherian local domain of dimension d and let E be a torsionfree R–module of rank r with a fixed embedding $E \hookrightarrow R^r$. Then*

(a) *The components of $G = \bigoplus_{n\geq 0} E^nS/E^{n+1}S$ have a natural grading*

$$G_n = E^n + E^nS_1/E^{n+1} + E^nS_2/E^{n+1}S_1 + \cdots .$$

(b) *There is a decomposition*

$$G = \mathcal{R}(E) + H,$$

where $\mathcal{R}(E)$ is the Rees algebra of E and H is the R-torsion submodule of G.

(c) *If* $E \subset \mathfrak{m}R^r$ *and* $\lambda(R^r/E) < \infty$, $H = H^0_{\mathfrak{m}}(G)$ *has dimension* $d + r$ *and multiplicity equal to the Buchsbaum-Rim multiplicity of* E*, that is* $j_{\mathfrak{m}}(G) = \mathrm{br}(E)$.

Corollary 2.2. *Let* E *be a module as above. Then* $\mathrm{jdeg}(G) = \mathrm{br}(E) + 1$.

Proof. It suffices to note that $\mathrm{Ass}_R(G) = \{0, \mathfrak{m}\}$, and use the calculation above for $j_{\mathfrak{m}}(G)$ and take into account that $j_{(0)}(G) = \deg(\mathcal{R}(E)_{(0)}) = \deg(K[t_1, \ldots, t_r]) = 1$. □

We can extract additional degrees information from the decomposition $\mathbf{G} = H + \mathcal{R}(E)$.

Definition 2.3. Suppose $E \hookrightarrow R^r$ is a submodule of rank r. For each $\mathfrak{p} \in \mathrm{Spec}\ (R) \setminus (0)$, the Buchsbaum-Rim multiplicity of E is the j-multiplicity of H,

$$\mathrm{br}_{\mathfrak{p}}(E) = j_{\mathfrak{p}}(H).$$

When $\mathfrak{p}$ is a minimal associated prime of R^r/E, this definition coincides with the Buchsbaum-Rim multiplicity of E (or, of the embedding $E \hookrightarrow R^r$).

Let $(R, \mathfrak{m})$ be a Noetherian local ring and let E be a finitely generated R-module having a rank r. The *analytic spread* of E is defined to be the dimension of the special fiber ring $\mathcal{R}(E)/\mathfrak{m}\mathcal{R}(E)$ and is denoted by $\ell(E)$. For a minimal reduction U of E, one has $\nu(U) = \ell(E)$ where $\nu(\cdot)$ denotes minimal number of generators.

Let R be an integral domain, and let $E \subset R^r$ be an R-module of rank r. Suppose $U \subset E$ is a submodule. Denote $S = R[T_1, \ldots, T_r]$.

Proposition 2.4. *U is a reduction of E if and only if the S-ideal (U) is a reduction of the ideal S-ideal (E).*

Proof. Another way to express that U is a reduction of E is to say that U and E have the same integral closure in R^r, in other words, for every valuation V of R, $VU = VE \subset V^r$. A similar observation applies to the ideals (U) and (E) of S. In this case, instead of using all the valuations of S, it suffices to verify that for every valuation domain V of R, $(U)V[T_1, \ldots, T_r] = (E)V[T_1, \ldots, T_r]$, which is equivalent to $UV = EV$.

Theorem 2.5 ([18, Theorem 1.1])**.** *Let* R *be a universally catenary integral domain and let* E *be a torsionfree* R*-module of rank* r *with an embedding* $E \hookrightarrow R^r$*. Suppose* U *is a submodule of* E *of rank* r*, and fix the induced embedding* $U \hookrightarrow R^r$*. The following conditions are equivalent:*

(1) $\mathrm{br}(U_{\mathfrak{p}}) = \mathrm{br}(E_{\mathfrak{p}})$ *for each* $\mathfrak{p} \in \mathrm{Spec}\ (R)$.
(2) U *is a reduction of* E.

Proof. Under the conditions, it suffices to apply [7, Theorem 3.3] together with Proposition 2.4. (A more general proof is that of [18].) $\square$

One application of these estimations is to provide a module version of [16, Theorem 3]. For that we recall the notion of the Briançon-Skoda number of a module following [10]. If E is a submodule of rank r of R^r, the *Briançon-Skoda number* $c(E)$ of E is the smallest integer c such that $\overline{E^{n+c}} \subset F^n S_c$ for every n and every reduction F of E. In contrast, the Briançon-Skoda number of the ideal (E) of S is the smallest integer $c = c((E))$ such that $\overline{(E)^{n+c}}S \subset F^n S$ for every n and every reduction F of E. In particular this implies that in degree $n+c$, $\overline{E^{n+c}} \subset F^n S_c$, so that $c(E) \leq c((E))$. Note that one has equality of analytic spreads $\ell(E) = \ell((E))$.

We shall refer to both $c(E)$ and $c((E))$ as the Briançon-Skoda numbers of E. When R is a regular local ring, applying the Briançon-Skoda theorem to S, gives $c((E)) \leq \ell(E) - 1$.

Example 2.6. Let $R = K[x,y]$ and E the submodule of the free R–module $Re_1 \oplus Re_2$ generated by x^2e_1, y^2e_2. E is a free R-module so $c(E) = 0$. As for the ideal (E) of $S = R[e_1, e_2]$, an application of Normaliz ([4]) shows that its normalization is $S[Et, xye_1e_2t]$; it follows that $c((E)) = 1$.

Let k be a perfect field, let R be a reduced and equidimensional k-algebra essentially of finite type, and assume that R is affine with $d = \dim R$. Recall that the Jacobian ideal of R is defined as the d-th Fitting ideal of the module of differentials $\Omega_k(R)$, it can be computed explicitly from a presentation of the algebra.

Theorem 2.7. *Let k be a perfect field, and let $(R, \mathfrak{m})$ be a local domain which is a k-algebra essentially of finite type and dimension d. Suppose that R has isolated singularities and L is the Jacobian ideal of R. If $E \subset \mathfrak{m}R^r$ is a module such that $\lambda(R^r/E) < \infty$, then*

$$\operatorname{jdeg}(\overline{S[(E)t]}/S[(E)t]) \leq (d + r + \lambda(R/L) - 2) \cdot (\operatorname{br}(E) + 1).$$

Proof. The argument is nearly identical to that of [16, Theorem 5]. First, note that LS is the Jacobian ideal of $S = R[T_1, \ldots, T_r]$. Let $\mathbf{D}$ be a S_2-ification of $S[(E)t]$. From [16, Theorem 4], setting $\mathbf{C} = \overline{S[(E)t]}$ and $c = \ell(E) - 1 = d + r - 2$, we have $L\overline{(E)^{n+c}} \subset D_n$. Observe that for all large n, $\mathbf{C}_{n+1} = (E)\mathbf{C}_n \subset \mathfrak{m}\mathbf{C}_n$. Consider the diagram

$$\begin{array}{ccccccccc} & & D_n/D_{n+c} & & & & C_{n+c}/LC_{n+c} & & \\ & & \uparrow & & & & \downarrow & & \\ 0 & \longrightarrow & (LC_{n+c} + D_{n+c})/D_{n+c} & \longrightarrow & C_{n+c}/D_{n+c} & \longrightarrow & C_{n+c}/(LC_{n+c} + D_{n+c}) & \longrightarrow & 0. \end{array}$$

Considering that the modules in the short exact sequence have dimension $d+r-1$ and that the module on the right is supported in $\mathfrak{m}$ alone, it follows that

$$\mathrm{jdeg}(\mathbf{C}/\mathbf{D}) = \mathrm{jdeg}((L\mathbf{C}+\mathbf{D})/\mathbf{D})+\mathrm{jdeg}(\mathbf{C}/L\mathbf{C}+\mathbf{D}) = \mathrm{jdeg}((L\mathbf{C}+\mathbf{D})/\mathbf{D})+\deg(\mathbf{C}/L\mathbf{C}+\mathbf{D}).$$

Given the embedding on the left and the surjection on the right, we have that

$$\mathrm{jdeg}((L\mathbf{C}+\mathbf{D})/\mathbf{D}) \leq \mathrm{jdeg}(\mathbf{D}/\mathbf{D}[-c]) = c\cdot \mathrm{jdeg}(\mathrm{gr}(\mathbf{D})),$$

and

$$\deg(\mathbf{C}/L\mathbf{C}+\mathbf{D}) \leq \deg(\mathbf{C}/L\mathbf{C}) \leq \lambda(R/L)\deg(\mathbf{C}/\mathfrak{m}\mathbf{C}).$$

Since $\mathrm{jdeg}(\mathrm{gr}(\mathbf{D})) = \mathrm{jdeg}(\mathrm{gr}_I(R))$ by [16, Theorem 1], we obtain the estimate of multiplicities

$$\mathrm{jdeg}(\mathbf{C}/\mathbf{D}) \leq c\cdot \mathrm{jdeg}(\mathrm{gr}_I(R)) + \lambda(R/L)\cdot f_0(\mathbf{C}),$$

where $f_0(C) = \deg(\mathbf{C}/\mathfrak{m}\mathbf{C})$.

Finally, we consider the exact sequence

$$0 \to \mathfrak{m}C_n/C_{n+1} \longrightarrow C_n/C_{n+1} \longrightarrow C_n/\mathfrak{m}C_n \to 0.$$

Taking into account that $\mathrm{gr}(\mathbf{C})$ is a ring which has the condition S_1, the ideal $\bigoplus_n \mathfrak{m}C_n/C_{n+1}$ either vanishes or has the same dimension as the ring. Arguing as above, we have

$$\mathrm{jdeg}(\mathrm{gr}(\mathbf{C})) = \mathrm{jdeg}(\bigoplus_n \mathfrak{m}C_n/C_{n+1}) + f_0(\mathbf{C}).$$

A final application of [16, Theorem 1] gives

$$\mathrm{jdeg}(\overline{S[(E)t]}/S(E)t]) \leq (d+r+\lambda(R/L)-2)\cdot(\mathrm{br}(E)+1),$$

since $\mathrm{jdeg}(\mathrm{gr}_{(E)}(S)) = \mathrm{br}(E)+1$. □

3. Divisorial generation

Several of the modules that occur in the construction of normalizations have the property S_2 of Serre. It suggests the following notion of *generation*.

Definition 3.1. Let R be a Noetherian ring and M a finitely generated R-module. M is *divisorially generated* by r elements if there exists a submodule $E\subset M$ generated by r elements such that M is the S_2-closure of E.

Theorem 3.2. *Let R be a Noetherian domain and $\mathbf{A}$ a semistandard graded R-algebra of finite integral closure $\overline{\mathbf{A}}$. Then $\overline{\mathbf{A}}$ is the S_2-closure of a subalgebra generated by homogeneous elements $\mathbf{A}[z_1,\ldots,z_r]$, for $r\leq 1+\mathrm{jdeg}(\overline{\mathbf{A}}/\mathbf{A})$.*

Proof. This is a direct consequence of [16, Theorem 2]. Let $\mathbf{A}_0$ be the S_2-closure of $\mathbf{A}$. If $\overline{\mathbf{A}}\neq\mathbf{A}_0$, choose a homogeneous element $z_1\in\overline{\mathbf{A}}\setminus\mathbf{A}_0$. Let $\mathbf{A}_1$ be the S_2-closure of $\mathbf{A}_0[z_1]$. In the same manner, if $\overline{\mathbf{A}}\neq\mathbf{A}_1$, choose $z_2\in\overline{\mathbf{A}}\setminus\mathbf{A}_1$ and let $\mathbf{A}_2$ be the S_2-closure of $\mathbf{A}_1[z_2]$. Iterate, if needed, and form the chain

$$\mathbf{A}_0\subset\mathbf{A}_1\subset\cdots\subset\mathbf{A}_r\subset\overline{\mathbf{A}}.$$

Set $\mathbf{B} = \mathbf{A}[z_1, \ldots, z_r]$. Its S_2-closure contains all $\mathbf{A}_i$, as desired. □

To give an application, consider the observation on bounding number of generators of modules with Hilbert coefficients. If $(R, \mathfrak{m})$ is a local ring, and M is a finitely generated module, we take for its Hilbert coefficients those of the graded module $\mathrm{gr}_{\mathfrak{m}}(M) = \bigoplus_n \mathfrak{m}^n M/\mathfrak{m}^{n+1} M$ over the associated graded ring of $\mathfrak{m}$.

Proposition 3.3. *Suppose that R is a normal standard graded algebra over a field (or a normal Noetherian local domain) and that $E \subset F$ are homogeneous modules of the same rank. If E is Cohen-Macaulay and F is a reflexive module with* $\operatorname{depth} F \geq \dim R - 1$, *then*

$$\nu(F) \leq e_0(F) + e_1(E) - e_1(F),$$

where the $e_i(\cdot)$ denote the coefficients of the corresponding Hilbert functions.

Proof. The module F/E is Cohen-Macaulay of multiplicity $e_1(E) - e_1(F)$, which will bound $\nu(F/E)$. □

Remark 3.4. Suppose $(R, \mathfrak{m})$ is a two-dimensional local normal domain and I is $\mathfrak{m}$-primary with a minimal reduction $J = (a, b)$, then in the sequence

$$0 \rightarrow \mathbf{A} = R[Jt] \longrightarrow \overline{\mathbf{A}} \longrightarrow C \rightarrow 0,$$

the module C is Cohen-Macaulay, since $\mathbf{A}$ is Cohen-Macaulay and $\overline{\mathbf{A}}$ satisfes S_2. These facts can be used to bound the number of generators of $\overline{A}$.

The module $C = \bigoplus_n \overline{J^n}/J^n$ has multiplicity read off the Hilbert polynomials of the filtrations $\{J^n\}$ and $\{\overline{J^n}\}$: The Hilbert polynomial of C is the difference

$$(e_0(J)\binom{n+2}{2} - e_1(J)(n+1) + e_2(J)) - (\overline{e}_0(J)\binom{n+2}{2} - \overline{e}_1(J)(n+1) + \overline{e}_2(J)),$$

and therefore $\deg(C) = \overline{e}_1(J) - e_1(J) = \overline{e}_1(J)$, because $\overline{e}_0(J) = e_0(J)$, and $e_1(J) = 0$ as R is Cohen-Macaulay. Thus $\overline{R[Jt]}$ will be generated over $R[Jt]$ by $1 + \overline{e}_1(J)$ elements.

If I is an ideal of codimension one, what is the bound? Suppose I is unmixed. Then there is a minimal reduction (we may assume the residue field of R is infinite) $J = (a, b)$. It is easy to see that $R[Jt]$ is Cohen-Macaulay. The module $C = \overline{R[Jt]}/R[Jt]$ is Cohen-Macaulay and since at every prime ideal $\mathfrak{p}$ of R of codimension one $R[Jt]$ is normal, the support of C is $\{\mathfrak{m}\}$. Thus $\mathrm{jdeg}(C) = \deg(C)$, therefore $\overline{R[Jt]}$ is generated by

$$1 + \mathrm{jdeg}(\overline{R[Jt]}/R[Jt])$$

elements over $R[Jt]$.

4. Symmetric algebras

Let R be an integral domain and let M be an R-module of rank $r \geq 1$. Let S be the symmetric algebra of M, $S = \mathrm{Sym}_R(M)$. If M is a free (or projective) R-module, $\mathrm{jdeg}(S) = \mathrm{jdeg}(R) = 1$. For more general modules though the associated primes of S are difficult to determine (see [11]). Let us consider one special case.

Proposition 4.1. *Let $(R, \mathfrak{m})$ be a Noetherian local domain of dimension $d > 0$ and let M be a non-free module of rank r that is free on the punctured spectrum, and let $S = \mathrm{Sym}_R(M)$. Then*

$$\mathrm{jdeg}(S) = \begin{cases} 1 + j_{\mathfrak{m}}(S), & \text{if } \nu(M) \geq \dim R + \mathrm{rank}(M) \\ 1, & \text{otherwise} \end{cases}$$

Proof. If S is not an integral domain, the only possible associated primes of S over R are (0) and $\mathfrak{m}$. According to [16, Proposition 1], we must check whether $\nu(M) = \dim S$, and which by [11, Theorem 2.6] (see also [19, Theorem 1.2.1]) is equivalent to the assertion on the dimension of S. □

To obtain more explicit bounds, one needs other kind of data. Let us consider one of these that will be used later.

Theorem 4.2. *Let $(R, \mathfrak{m})$ be a Noetherian local domain of dimension 1, with a finite integral closure. Let M be a module of rank r that is minimally generated by $n = r+1$ elements. Let*

$$0 \to L \to R^n \longrightarrow M \to 0$$

be a minimal presentation of M. Denote by $c(L)$ the R-ideal generated by the coefficients of the elements of L, that is the Fitting ideal $\mathrm{Fitt}_r(M)$. If $S = \mathrm{Sym}_R(M)$, then

$$\mathrm{jdeg}(S) \leq 1 + e_0(c(L)) + \nu(L) \cdot \lambda(\overline{R}/R).$$

Proof. L is a module of rank 1, whose generators define the symmetric algebra of M,

$$0 \to (L) = LB \longrightarrow B = R[x_1, \ldots, x_n] \longrightarrow S = \mathrm{Sym}_R(M) \to 0.$$

If R is integrally closed, that is, a discrete valuation domain (or more generally a PID), (L) would be generated by a linear form, $(L) = (c\mathbf{f})$, where the coefficients of $\mathbf{f}$ generate the unit ideal. In this case $M \simeq R/(c) \oplus R^r$, and therefore $j_{\mathfrak{m}}(S) = \lambda(R/(c)) = \deg(R/(c))$. To take advantage of this elementary fact, let us make a change of rings, $R \to \overline{R}$.

Consider the exact sequence

$$\mathrm{Tor}_1^R(\overline{R}/R, S) \longrightarrow S \longrightarrow \overline{S} = \overline{R} \otimes_R S \longrightarrow \overline{R}/R \otimes_R S \to 0.$$

It gives the inequality

$$\begin{aligned} j_{\mathfrak{m}}(S) &\leq j_{\mathfrak{m}}(\overline{S}) + \deg(\mathrm{Tor}_1^R(\overline{R}/R, S)) \\ &\leq \lambda(\overline{S}/c(L)\overline{S}) + \deg(\mathrm{Tor}_1^R(\overline{R}/R, S)) \\ &\leq e_0(c(L)) + \nu(L)\lambda(\overline{R}/R), \end{aligned}$$

where we used that L is isomorphic to an ideal of R, and therefore $\nu(L) \leq \deg R$, and for $\lambda(\overline{R}/R)$, we need to recall a bound in terms of $\deg(R)$. It will follow that $\mathrm{Tor}_1^R(R/\mathfrak{m}, S) = R/\mathfrak{m} \otimes_R LB$. As for the equality $\lambda(\overline{S}/c(L)\overline{S}) = e_0(c(L))$, this is a general fact. □

If $\dim R > 1$, but M is still free on the punctured spectrum and $\dim R + \mathrm{rank}(M) = \nu(M)$, it is harder to estimate $\mathrm{jdeg}(S)$. One reason is that the ideal LB now has codimension $\nu(M) - \mathrm{rank}(M) = \dim R > 1$, although it is still generated by 1-forms. One approach, is the following. Let $H = H^0_{\mathfrak{m}}(S) \subset S$. Denote by $\mathcal{L}$ the inverse image of H in $B = R[x_1, \ldots, x_n]$. Since S/H is an integral domain, we actually have the primary decomposition

$$LB = \mathcal{L} \cap Q,$$

where Q is $\mathfrak{m}B$-primary. We can express $\mathcal{L}$ as

$$\mathcal{L} = \bigcup_{r \geq 1} (LB :_B \mathfrak{m}^r).$$

The ideal $\mathcal{L}$ is preferably obtained by elimination ([9, Section 4]). The index r where the operation stabilizes could be called the elimination index of M.

Proposition 4.3. *Let R, M and r be as above. Then*

$$j_{\mathfrak{m}}(S) \leq \lambda(R/\mathfrak{m}^r).$$

Proof. We note that $\mathfrak{m}^r H = 0$. Consider the exact sequence

$$0 \to H \longrightarrow S \longrightarrow S/H \to 0.$$

Tensoring it by $k = R/\mathfrak{m}$, we have the exact sequence

$$\mathrm{Tor}_1^R(k, S/H) \longrightarrow k \otimes_R H \longrightarrow k \otimes_R S \longrightarrow k \otimes_R S/H \to 0. \tag{1}$$

Let us examine the dimensions of the terms at the ends. Since $\dim S = \dim R + \mathrm{rank}(M)$ and $\mathfrak{m}$ is not an associated prime of S/H, $\dim k \otimes_R S/H < \dim S$.

At the other end, $\mathrm{Tor}_1^R(k, S/H)$ is a finitely generated S/H-module that is annihilated by $\mathfrak{m}$. Thus its dimension is also bounded by $\dim S - 1$. Counting the multiplicities in (1) we only have to account for the modules of dimension $\dim S$. This gives $\deg(k \otimes_R H) = \deg(k \otimes_R S) = 1$, and therefore using a composition series for $R/\mathfrak{m}^r$, we get $\deg(H) = \deg(H \otimes_R R/\mathfrak{m}^r) \leq \lambda(R/\mathfrak{m}^r)$, as desired. □

Remark 4.4. A computational approach to this calculation is the following. Let $I = (a_1, \ldots, a_n) \subset R$ be an ideal, let

$$0 \to I \longrightarrow B = R[T_1, \ldots, T_n] \longrightarrow R[It] \to 0$$

be a presentation of its Rees algebra. Setting

$$\begin{aligned} L &= (Q, I), \\ L' &= L : \mathfrak{m}^\infty, \end{aligned}$$

one has $H = L'/L$. Consider the following example computed by R. Villarreal,

$$I = (x, y, z)^2 + w(x, y) \subset R = k[x, y, z, w].$$

Using *Macaulay 2* ([8]), one verifies that

$$\begin{aligned} L : \mathfrak{m}^\infty &= L : \mathfrak{m}^3, \\ \deg(H) &= 12, \\ \lambda(R/\mathfrak{m}^3) &= 15. \end{aligned}$$

5. Approximation complexes

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, and let I be an ideal. If $\mathcal{A} = \{I_n, n \geq 0\}$ is an I-good filtration, for

$$\mathbf{A} = \sum_{n \geq 0} I_n t^n,$$

the invariance theorem [16, Theorem 1] asserts that

$$\text{jdeg}(\text{gr}_I(R)) = \text{jdeg}(\text{gr}(\mathbf{A})).$$

We can further profit by selecting a minimal reduction J of I, and seeking to determine $\text{jdeg}(\text{gr}_J(R))$. We are going to discuss two cases: $\dim R/I = 1, 2$.

Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d and let $\mathfrak{p}$ be a prime ideal of dimension 1, that is codim $\mathfrak{p} = d - 1$. Assume that $\mathfrak{p}$ is generically a complete intersection. In order to approach the calculation of $\text{jdeg}(\text{gr}_{\mathfrak{p}}(R))$, we select a minimal reduction J of $\mathfrak{p}$. If $\ell(\mathfrak{p}) = d - 1$, $J = \mathfrak{p}$ and $\text{jdeg}(\text{gr}_{\mathfrak{p}}(R)) = \deg(R/\mathfrak{p})$.

We are going to assume that $\ell(\mathfrak{p}) = d$, and that J is one of its minimal reductions with $\nu(J) = d$. Under these conditions, from the theory of approximation complexes ([19, Chapter 5]), there is an exact complex ($B = R[T_1, \ldots, T_d]$)

$$0 \to H_1(J) \otimes_R B[-1] \longrightarrow B_0 = H_0(J) \otimes_R B \longrightarrow G = \text{gr}_J(R) \to 0,$$

where $H_i(J)$ are the Koszul homology modules on a minimal set of generators of J.

Theorem 5.1. *Let $\mathfrak{p}$ be a one-dimensional prime ideal as above. Set $S = R/\mathfrak{p}$. If Z_1 is the module of syzygies of J,*

$$\text{jdeg}(\text{gr}_{\mathfrak{p}}(R)) \leq 1 + \lambda(\mathfrak{p}/J) + \lambda(\overline{S}/S) \cdot \nu(L) + \lambda(\overline{S}/c(Z_1)\overline{S}).$$

Proof. $H_1(J)$ is the canonical module of $H_0(J) = R/J$. According to [19, Proposition 5.3.3], $H_1(J)$ is isomorphic to the canonical module of $S = R/\mathfrak{p}$. It is thus a torsionfree S-module, while $\mathfrak{p}/J$ is an R-module of finite length. This implies that the submodule $\mathfrak{p}B_0$ maps one-one onto $\mathfrak{p}G$.

The exact sequence

$$0 \to \mathfrak{p}B_0 \simeq \mathfrak{p}G \longrightarrow G \longrightarrow G_0 = R/\mathfrak{p} \otimes_R G \to 0$$

gives the cohomology exact sequence

$$0 \to \mathfrak{p}B_0 \longrightarrow H^0_{\mathfrak{m}}(G)) \longrightarrow H^0_{\mathfrak{m}}(G_0) \to 0,$$

and therefore

$$j_{\mathfrak{m}}(G) = \lambda(\mathfrak{p}/J) + j_{\mathfrak{m}}(G_0).$$

We are now in the setting of Theorem 4.2: In

$$R/\mathfrak{p} \otimes_R \mathrm{gr}_J(R) = R/\mathfrak{p} \otimes_R \mathrm{Sym}_{R/J}(J/J^2) = \mathrm{Sym}_{R/\mathfrak{p}}(J/\mathfrak{p}J),$$

$J/\mathfrak{p}J$ is an $R/\mathfrak{p}$-module of rank $d-1$, free on the punctured spectrum of $R/\mathfrak{p}$, we can apply the estimation of Theorem 4.2. □

Example 5.2. Let k be a field of characteristic $\neq 2$. In the normal hypersurface ring $R = k[x, y, z]/(x^2 + yz)$ let $J = \mathfrak{p} = (x, y)$. Set $S = R/\mathfrak{p}$; then $\overline{S} = S = k[z]$ and $c(Z_1) = (z)$. It follows that $\mathrm{jdeg}(\mathrm{gr}_{\mathfrak{p}}(R)) = 2$.

Let now $\mathfrak{p}$ be a prime ideal of dimension 2 that is a complete intersection on the punctured spectrum (e.g. R is a regular local ring and $R/\mathfrak{p}$ is normal). We shall assume that $\mathfrak{p}$ is Cohen-Macaulay. Suppose $\ell(I) = d$ (as otherwise the analysis is simple), and let J be a minimal reduction (generated by d elements). The approximation complex associated to J,

$$0 \to H_2(J) \otimes_R B[-2] \longrightarrow H_1(J) \otimes_R B[-1] \to H_0(J) \otimes_R B \longrightarrow \mathrm{gr}_J(R) \to 0,$$

where $H_i(J)$ are the Koszul homology modules of J, and $B = R[T_1, \ldots, T_d]$, can be quickly examined. We set ourselves in the context of [19, Theorem 5.3.4].

Proposition 5.3. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d, and let I be a prime ideal of codimension $d-2$. Suppose I is Cohen-Macaulay, and a complete intersection in codimension $d-1$, $\ell(I) = d$, and let J be a minimal reduction. Then the Koszul homology modules of J satisfy: $H_2(J)$ is the canonical module of R/I and* depth $H_1(J) \geq 1$.

The proof is analogous to that of [19, Theorem 5.3.4]. A consequence is that both $H_2(J)$ and $H_1(J)$ are R/I-modules, since $IH_1(J)$ vanishes at each localization in codimension $d-1$. As was the case in Theorem 4.2, these facts lead to the exact sequences: (set $B_0 = B/JB$)

$$0 \to IG = IB_0 \longrightarrow G \longrightarrow G_0 = R/I \otimes_R G \to 0$$

and the exact sequence of R/I-modules

$$0 \to H_2(J) \otimes_R B[-2] \longrightarrow H_1(J) \otimes_R B[-1] \to R/I \otimes_R B \longrightarrow G_0 \to 0.$$

Theorem 5.4. *Let I be as above and J a minimal reduction of I. Then*

$$\operatorname{jdeg}(G) = \lambda(I/J) + \operatorname{jdeg}(G_0) = \lambda(I/J) + 1 + j_{\mathfrak{m}}(G_0).$$

We note that the second sequence says that G_0 is the symmetric algebra of the R/I-module J/IJ, and by the acyclicity lemma G_0 is Cohen-Macaulay. The issue is how to make use of the approximation complex beyond what is already done in Proposition 4.3.

Remark 5.5. A remaining issue is to extend these estimations to the case of a prime $\mathfrak{p}$ of a regular local ring R, such that $R/\mathfrak{p}$ has isolated singularities.

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