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Cohomological Degrees and the HomAB Conjecture

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Abstract. We consider a question, which we label the *HomAB Conjecture*, asserting that the number of generators of $\mathrm{Hom}_R(A, B)$, where A and B are finitely generated modules over the Noetherian ring R, is bounded by a fixed quadratic polynomial defined on invariants of R, and evaluated at invariants of A and B. The main motivation comes from the fact that several algorithms employed in the computation of the normalization of algebras and in primary decomposition involve rounds of the Hom operation. For several classes of modules, in arbitrary dimensions, we derive effective bounds for $\nu(\mathrm{Hom}_R(A, A))$.

Contents

1. Introduction 1
2. Degree functions and Auslander duals 3
3. Isolated singularities and vector bundles 7
4. Syzygies of near-perfect modules 10
5. Ideal modules 14
6. Open questions 17
References 18

1. Introduction

Let R be a Noetherian ring and let A and B be finitely generated R-modules. The HomAB question asks for uniform estimates for the number of generators of $\mathrm{Hom}_R(A, B)$ in terms of invariants of R, A and B.

In addition to the appeal of the question in basic homological algebra, such modules of endomorphisms appear frequently in several constructions, particularly in the algorithms that seek the integral closure of algebras (see [**12**, Chapter 6]). Another natural application is for primary decomposition of ideals, given the prevalence of computation of ideal quotients.

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K. Dalili ([**3**]) was the first to consider various cases of the HomAB problem and raised a broadly based conjecture (the HomAB Conjecture) about the character of the bounds. The scales he used to measure the invariants were the cohomological degree functions introduced in [**4**]. These are extensions (denoted by Deg) of the ordinary multiplicity function deg but encoding also some of the properties of its cohomology (we will discuss them briefly soon). The bounds treated have the general format

$$\nu(\mathrm{Hom}_R(A,B)) \leq f(\mathrm{Deg}(A), \mathrm{Deg}(B)), \tag{1}$$

where $f(x,y)$ is a quadratic polynomials whose coefficients depend on various invariants of R. Although it can be presented in the form $f(\mathrm{Deg}(A), \mathrm{Deg}(B)) = c(R)\mathrm{Deg}(A)\mathrm{Deg}(B)$, we will emphasize polynomials in various other invariants of the modules, such as number of generators $\nu(A)$ and the classical multiplicity $\deg(A)$.

Our outlook here has a slightly different perspective. To obviate the non-interchangeable roles A and B play in $\mathrm{Hom}_R(A,B)$, we consider only the case $E = A = B$. On one hand, this choice enables the additional structure of algebra in $\mathrm{Hom}_R(E,E)$. Naturally the consideration of $E = A \oplus B$ makes for a rough equivalence. The real distinction from [**3**] comes in the classes of modules chosen for examination. We substitute the modules of low dimension treated in [**3**] by a few classes of modules of arbitrary dimension, but that still affirm the basic conjecture (1).

The setup we employ is derived from the analysis of duality of [**1**]: There is a homomorphism of R-algebras

$$E^* \otimes_R E \longrightarrow \mathrm{Hom}_R(E,E) \longrightarrow \mathrm{Tor}_1^R(D(E),E) \to 0 \tag{2}$$

where $D(E)$ is the Auslander dual of E (see Definition 2.9). Considering that the number of generators of E^* can be estimated from certain invariants of E and R (see Theorem 2.12), the question turns on the understanding of $\mathrm{Tor}_1^R(D(E),E)$, a module that by [**6**, Theorem 1.3] is identified with the module of natural endomorphisms $\underline{\mathrm{Hom}}_R(\mathrm{Ext}_R^1(E,\cdot), \mathrm{Ext}_R^1(E,\cdot))$. In several cases, $\mathrm{Tor}_1^R(D(E),E)$ is actually identified to another ring of endomorphisms $\mathrm{Hom}_R(C,C)$, with C having much smaller support than E and explicitly related to E.

To track estimates of the number of generators of $\mathrm{Tor}_1^R(D(E),E)$ to properties of E, we make use of extended degree multiplicity, particularly those labelled hdeg and bdeg (see Definitions 2.2 and 2.6).

We are now going to describe our results. They will be framed in the different ways the analysis of $\mathrm{Tor}_1^R(D(E),E)$ takes place by examining several classes of modules. Since the problem is a local question, we may assume that $(R,\mathfrak{m})$ is a local Noetherian ring of dimension d. We will see that in good many cases this ends up in the consideration of Cohen-Macaulay modules. A successful resolution of this case would have a major impact on all these problems. We are going to list the classes of modules considered and describe the corresponding estimates obtained.

Section 2 introduces the various generalized multiplicity functions used to estimate $\nu(\mathrm{Hom}_R(E,E))$ and the broad setting of Auslander duals. The following 3 sections treat each specific classes of modules. It is grouped under four labels, but the first two are parts of section 3.

- *Vector bundles*: Those are modules free on certain distinguished open $\mathbf{X}$ sets of Spec R. For instance, if $\mathbf{X}$ is the punctured spectrum of R,

$\mathrm{Tor}_1^R(D(E), E)$ is a module of finite length. When E is a module of finite projective dimension, which is free in codimension $\dim R - 2$, this is treated in [**3**]. This remains unresolved in general, with several undecided examples.

- *Isolated singularities*: When $(R, \mathfrak{m})$ is a Gorenstein local ring essentially of finite type over a field, whose Jacobian ideal J is $\mathfrak{m}$-primary, nearly all known results that required the modules to have finite projective resolution are extended. For that we introduce a new extended degree function that makes use of Samuel's multiplicities relative to J. Thus Theorem 3.4 shows that if E is a maximal Cohen-Macaulay module,

$$\nu(\mathrm{Hom}_R(E, E)) \leq \deg(E) \cdot \nu(E) + e(J) \cdot \nu(E)^2.$$

It is then used to show that for any R-module, free in codimension $\leq \dim R - 2$, bounds similar to those of [**3**] (that use proj dim $E < \infty$) are proved.

- *Modules of syzygies of near-perfect modules*: If M is a perfect module (or has this property on the punctured spectrum of R) and E is one of its modules of syzygies, we will exhibit a natural identification $\mathrm{Tor}_1^R(D(E), E) = \mathrm{Hom}_R(M, M)$. The difference of dimensions between E and M leads to several very explicit formulas for $\nu(\mathrm{Hom}_R(E, E))$, e.g. (Theorem 4.2): If M is a perfect module and E is its module of k-syzygies ($k \geq 2$), there exists an exact sequence

$$0 \to E^* \otimes E \longrightarrow \mathrm{Hom}_R(E, E) \longrightarrow \mathrm{Hom}_R(M, M) \to 0.$$

- *Ideal modules*: An ideal module is a torsionfree R-module E such that E^{**} is R-free. This is a property of ideals of grade at least two. The quotient module $C = E^{**}/E$ has the property that $\mathrm{Tor}_1^R(D(E), E) = \mathrm{Hom}_R(C, C)$. One application is (Corollary 5.5): Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d and let E be a torsionfree ideal module free in codimension $d - 3$. Then

$$\nu(\mathrm{Hom}_R(E, E)) \leq (\mathrm{rank}(E) + \nu(E))(\mathrm{hdeg}(E) - \deg(E)) + \mathrm{rank}(E) \cdot \nu(E).$$

These bounds are useful in the estimation of the number of generators of ideal quotients $I : J$ in rings of polynomials.

The last section list some open problems.

2. Degree functions and Auslander duals

In this preliminary section we will recall the scales we employ to measure the various complexities and also the framework were our calculations take place. Our setting is the category $\mathcal{M}(R)$ of finitely generated modules over a Noetherian local ring $(R, \mathfrak{m})$, of residue field k. Basic facts and terminology are be found in [**2**].

Big degs. Clearly, to estimate the number of generators of $\mathrm{Hom}_R(A, B)$, the knowledge $\nu(A)$ and $\nu(B)$ are far from enough. Even more strongly, data provided from the multiplicities $\deg(A)$ and $\deg(B)$ will also fall far short of the goal. We will focus instead on the so-called *extended* and *cohomological* degree functions of [**4**]. (See [**12**, Section 2.4] for a discussion.)

An ingredient to the definition of these functions is a method to select appropriate hyperplane sections, a requirement that may require that the residue fields of the rings to be infinite.

Definition 2.1. A *cohomological degree*, or *extended multiplicity function*, is a function

$$\mathrm{Deg}(\cdot) : \mathcal{M}(R) \mapsto \mathbb{N},$$

that satisfies the following conditions.

(i) If $L = \Gamma_{\mathfrak{m}}(M)$ is the submodule of elements of M that are annihilated by a power of the maximal ideal and $\overline{M} = M/L$, then

$$\mathrm{Deg}(M) = \mathrm{Deg}(\overline{M}) + \lambda(L), \tag{3}$$

where $\lambda(\cdot)$ is the ordinary length function.

(ii) (Bertini's rule) If M has positive depth , there is $h \in \mathfrak{m} \setminus \mathfrak{m}^2$, such that

$$\mathrm{Deg}(M) \geq \mathrm{Deg}(M/hM). \tag{4}$$

(iii) (The calibration rule) If M is a Cohen-Macaulay module, then

$$\mathrm{Deg}(M) = \deg(M), \tag{5}$$

where $\deg(M)$ is the ordinary multiplicity of M.

These functions will be referred to as big Degs. If $\dim R = 0$, $\lambda(\cdot)$ is the unique Deg function. For $\dim R = 1$, $\mathrm{Deg}(M) = \lambda(L) + \deg(M/L)$. When $d \geq 2$, there are several big Degs. An explicit Deg, for all dimensions, was introduced in [**11**].

Definition 2.2. Let M be a finitely generated graded module over the graded algebra A and S a Gorenstein graded algebra mapping onto A, with maximal graded ideal $\mathfrak{m}$. Set $\dim S = r$, $\dim M = d$. The *homological degree* of M is the integer

$$\begin{aligned} \mathrm{hdeg}(M) &= \deg(M) + \\ & \sum_{i=r-d+1}^{r} \binom{d-1}{i-r+d-1} \cdot \mathrm{hdeg}(\mathrm{Ext}^i_S(M,S)). \end{aligned} \tag{6}$$

This expression becomes more compact when $\dim M = \dim S = d > 0$:

$$\begin{aligned} \mathrm{hdeg}(M) &= \deg(M) + \\ & \sum_{i=1}^{d} \binom{d-1}{i-1} \cdot \mathrm{hdeg}(\mathrm{Ext}^i_S(M,S)). \end{aligned} \tag{7}$$

Remark 2.3. A family of such functions arise when we use Samuel multiplicities. That is, if $(R, \mathfrak{m})$ is a local ring and I is an $\mathfrak{m}$-primary ideal, instead of the ordinary multiplicity function $\deg M = e(\mathfrak{m}; M)$ we employ $e(I; M)$. The generic hyperplane section are now to be chosen in $I \setminus \mathfrak{m}I$.

In the case of hdeg, for a local ring with isolated singularities, we shall have the opportunity to employ the construction when I is the Jacobian ideal. The corresponding function will be denoted hdeg_I (see also [**7**]).

We are going to recall some of the bounds afforded by a Deg function, particularly as they pertain to the HomAB issue.

THEOREM 2.4 ([**12**, Theorem 2.94]). *For any* Deg *function and any finitely generated R-module M,*

$$\beta_i(M) \leq \operatorname{Deg}(M)\beta_i(k),$$

where $\beta_i(\cdot)$ is the ith Betti number function.

THEOREM 2.5 ([**8**]). *Let A be an standard graded algebra over an infinite field and let M be a nonzero finitely generated A-module. Then for any* $\operatorname{Deg}(\cdot)$ *function, we have*

$$\operatorname{reg}(M) < \operatorname{Deg}(M) + \alpha(M),$$

where $\alpha(M)$ is the maximal degree in a minimum graded generating set of M.

Another cohomological degree, called bdeg, was introduced in T. Gunston thesis ([**5**]):

DEFINITION 2.6. Let $(R, \mathfrak{m})$ be a Noetherian local ring. For any finitely generated R-module M,

$$\operatorname{bdeg}(M) := \min\{\operatorname{Deg}(M) \mid \operatorname{Deg} \text{ is a cohomological degree}\}.$$

It has the property that if M has positive depth there are generic hyperplane sections such that

$$\operatorname{bdeg}(M) = \operatorname{bdeg}(M/hM).$$

This property has a very useful behavior with respect to certain exact sequences:

PROPOSITION 2.7 ([**3**, Propositions 3.2 and 3.3]). *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let*

$$0 \to A \longrightarrow B \longrightarrow C \to 0$$

be an exact sequence of finitely generated R-modules. Then

$$\begin{aligned} \operatorname{bdeg}(B) &\leq \operatorname{bdeg}(A) + \operatorname{bdeg}(C), \quad \textit{and} \\ \operatorname{bdeg}(A) &\leq \operatorname{bdeg}(B) + (\dim A - 1)\operatorname{bdeg}(C). \end{aligned}$$

QUESTION 2.8. Is there a polynomial $f(x)$ such that for any Deg,

$$\operatorname{hdeg}(E) \leq f(\operatorname{Deg}(E)),$$

for all R-modules E? It would suffice to prove

$$\operatorname{hdeg}(E) \leq f(\operatorname{bdeg}(E)).$$

Auslander dual. The setting of our calculations is the following construction of Auslander ([**1**]).

DEFINITION 2.9. Let E be a finitely generated R-module with a projective presentation

$$F_1 \xrightarrow{\varphi} F_0 \longrightarrow E \to 0.$$

The *Auslander dual* of E is the module $D(E) = \operatorname{coker}(\varphi^t)$,

$$0 \to E^* \longrightarrow F_0^* \xrightarrow{\varphi^t} F_1^* \longrightarrow D(E) \to 0. \tag{8}$$

The module $D(E)$ depends on the chosen presentation but it is unique up to projective summands. In particular the values of the functors $\operatorname{Ext}^i_R(D(E), \cdot)$ and $\operatorname{Tor}^R_i(D(E), \cdot)$, for $i \geq 1$, are independent of the presentation. Its use here lies in the following result (see [**1**, Chapter 2]):

Proposition 2.10. *Let R be a Noetherian ring and E a finitely generated R-module. There are two exact sequences of functors:*

$$0 \to \mathrm{Ext}^1_R(D(E),\cdot) \longrightarrow E\otimes_R \cdot \longrightarrow \mathrm{Hom}_R(E^*,\cdot) \longrightarrow \mathrm{Ext}^2_R(D(E),\cdot) \to 0 \tag{9}$$

$$0 \to \mathrm{Tor}_2^R(D(E),\cdot) \longrightarrow E^*\otimes_R \cdot \longrightarrow \mathrm{Hom}_R(E,\cdot) \longrightarrow \mathrm{Tor}_1^R(D(E),\cdot) \to 0. \tag{10}$$

Corollary 2.11. *Let R be a Noetherian ring and E a finitely generated R-module and denote by $D(E)$ its Auslander dual. Then*

$$\nu(\mathrm{Hom}_R(E,E)) \leq \nu(E^*)\nu(E) + \nu(\mathrm{Tor}_1^R(D(E),E)).$$

In one important case, one can zoom in further ([**3**, Theorem 5.3]):

Theorem 2.12. *Let R be a Gorenstein local ring of dimension d and let E be a finitely generated R-module. Then*

$$\nu(E^*) \leq (\deg(E) + d(d-1)/2)\mathrm{hdeg}(E).$$

As a consequence, the focus is placed on $\mathrm{Tor}_1^R(D(E),E)$, a module with many interesting properties:

Corollary 2.13. *The image of $E^*\otimes_R E$ is a two-sided ideal of $\mathrm{Hom}_R(E,E)$, so $\mathrm{Tor}_1^R(D(E),E)$ has a ring structure. Moreover, the module E is projective if and only if $\mathrm{Tor}_1^R(D(E),E)=0$. In particular, the support of $\mathrm{Tor}_1^R(D(E),E)$ determines the free locus of E.*

Proof. The first assertion is routine, considering the actions of $\mathrm{Hom}_R(E,E)$ on E and on E^*. The surjection of the natural mapping $E^*\otimes_R E \to \mathrm{Hom}_R(E,E)$ gives a representation of the identity of $\mathrm{Hom}_R(E,E)$,

$$I = \sum_i f_i\otimes e_i, \quad f_i\in E^*, e_i\in E,$$

that is

$$e = \sum_i f_i(e)e_i, \quad \forall e\in E,$$

one of the standard descriptions of finitely generated projective modules. □

We single out from the examination of (8) the following effective description of $\mathrm{Hom}_R(E,E)$.

Proposition 2.14. *Let E be a finitely generated R-modules with a presentation*

$$R^m \xrightarrow{\varphi} R^n \longrightarrow E \to 0.$$

Choose bases $\{e_1,\ldots,e_n\}$ and $\{f_1,\ldots,f_m\}$ in R^n and R^m. Then $\mathrm{Hom}_R(E,E)$ is isomorphic to the kernel of the induced mapping $\Phi : E\otimes_R (R^n)^t \to E\otimes (R^m)^t$.

Proof. We let e_i^* be corresponding dual basis of $(R^n)^t$. An element $z=\sum x_i\otimes e_i^*$ lies in the kernel of Φ if $\sum x_i\otimes e_i^*\circ\varphi^t = 0$, a condition that means $[x_1,\ldots,x_n]\cdot\varphi^t=0$. This will imply that if we let a_i be the image of e_i in E, the assignment $a_i\to x_i$ will define an endomorphism $\alpha : E\to E$.

Conversely, given such α, $\zeta=\sum_i \alpha(a_i)\otimes e_i^*\in\ker(\Phi)$. The mappings are clearly inverses of one another. □

3. Isolated singularities and vector bundles

Let $(R, \mathfrak{m})$ be a Gorenstein local ring which is essentially of finite type over the field k. Denote by J be Jacobian ideal of R. We will assume that J is an $\mathfrak{m}$-primary ideal. For any maximal Cohen-Macaulay (MCM for short) module E, we are going to give a bound for $\nu(\mathrm{Hom}_R(E,E))$ in terms of the multiplicity of E and the Samuel multiplicity of J.

We are going to gather information on the number of generators of $\mathrm{Tor}_1^R(D(E),E)$ when E is a MCM module. First, since R is Gorenstein, $E^* = \mathrm{Hom}_R(E,R)$ is also a MCM module of the same multiplicity. In particular we have

$$\nu(\mathrm{Hom}_R(E,E)) \quad \leq \quad \nu(E)\deg(E) + \nu(\mathrm{Tor}_1^R(D(E),E)). \tag{11}$$

Let

$$0 \to L \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E \to 0$$

be a minimal projective presentation of E. Since all these modules are maximal Cohen-Macaulay and R is Gorenstein, dualizing we get another exact sequence of MCM modules

$$0 \to E^* \longrightarrow F_0^* \longrightarrow F_1^* \longrightarrow D(E) \to 0.$$

We now quote in full two results [**13**] that we require.

Theorem 3.1 ([**13**, Theorem 5.3]). *Let R be a Cohen-Macaulay local ring of dimension d, essentially of finite type over a field, and let J be its Jacobian ideal. Then $J \cdot \mathrm{Ext}_R^{d+1}(M,\cdot) = 0$ for any finitely generated R-module M.*

Proposition 3.2 ([**13**, Proposition 1.5]). *Let R be a commutative ring, M and R-module, and $x \in R$. If $x\mathrm{Ext}_R^1(M,\cdot) = 0$ then $x\mathrm{Tor}_1^R(M,\cdot) = 0$.*

Since $D(E)$ is MCM and R is Gorenstein, it is a $d+1$ syzygy for an appropriate module L, and thus $\mathrm{Ext}_R^{d+1}(L,\cdot) = \mathrm{Ext}_R^1(D(E),\cdot)$, a functor that according to Theorem 3.1 is annihilated by the Jacobian ideal J of R. On the other hand, appealing to Proposition 3.2, $J \cdot \mathrm{Tor}_1^R(D(E),\cdot) = 0$.

We now integrate these strands. Let $\mathbf{z} = z_1, \ldots, z_r$ be a regular sequence in J. To provide a modicum of generality, if J has dimension 1, we take $r = d-1$, while if J is $\mathfrak{m}$-primary, we pick $\mathbf{z}$ to be a minimal reduction of J.

Consider the exact sequence induced by multiplication by z_1

$$0 \to E \xrightarrow{z_1} E \longrightarrow E/z_1E \to 0.$$

Since $J \cdot \mathrm{Tor}_1^R(D(E),\cdot) = 0$, $\mathrm{Tor}_1^R(D(E),E)$ embeds in $\mathrm{Tor}_1^R(D(E),E/z_1E)$. Iterating we will get

$$\mathrm{Tor}_1^R(D(E),E) \hookrightarrow \mathrm{Tor}_1^R(D(E),E/\mathbf{z}E).$$

Now we derive a bound for the number of generators of $\mathrm{Tor}_1^R(D(E),E)$ in terms of the properties of $\mathrm{Tor}_1^R(D(E),E/\mathbf{z}E)$.

Let

$$\cdots \longrightarrow G_2 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow D(E) \to 0$$

be a minimal projective resolution of $D(E)$. To tensor with $E/\mathbf{z}E$, we first do with $R/(\mathbf{z})$, which gives a minimal free resolution of $D(E)/\mathbf{z}D(E)$ over the ring $\overline{R} = R/(\mathbf{z})$. We now tensor with $\overline{E} = E/(\mathbf{z})E$ over $\overline{R}$. Denote by K the kernel of

$$\overline{G_1} \otimes \overline{E} \longrightarrow \overline{G_0} \otimes \overline{E},$$

and by B the corresponding module of boundaries

$$0 \to B \longrightarrow K \longrightarrow \mathrm{Tor}_1^R(D(E), \overline{E}) \to 0.$$

We apply the next lemma to obtain

$$\nu(\mathrm{Tor}_1^R(D(E), E)) \leq \mathrm{Deg} K \leq \mathrm{Deg}(\overline{G_1} \otimes \overline{E}),$$

since $\dim \overline{R} \leq 1$.

Lemma 3.3. *Let R be a local Noetherian ring of dimension 1 and consider the diagram of finitely generated R-modules*

$$\begin{tikzcd} A \arrow[r, "\varphi"] & B \arrow[r] & O, \\ & C \arrow[u, hook] & \end{tikzcd}$$

that is, C is a subquotient of A. Then $\nu(C) \leq \mathrm{Deg}(A)$.

Proof. Let $D = \varphi^{-1}(C)$. Since in dimension 1 $\mathrm{Deg}(D) \leq \mathrm{Deg}(A)$, and as $\nu(D) \leq \mathrm{Deg}(D)$ always, the assertion follows. □

We note that $G_1 = F_0^*$ and that $\overline{G_1} \otimes \overline{E}$ is a Cohen-Macaulay that is a homomorphic image of $\overline{G_1} \otimes \overline{F_0}$. If $\dim \overline{R} = 0$, it is an Artin ring of length $e(J)$, the Samuel multiplicity of the ideal J. We have the bound

$$\lambda(\overline{G_1} \otimes \overline{E}) \leq e(J)\nu(E)^2.$$

We can put together these considerations in the following.

Theorem 3.4. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring as above, let E be a MCM module and let J be the Jacobian ideal of R.*

(i) *If J of R is $\mathfrak{m}$-primary,*

$$\nu(\mathrm{Hom}_R(E, E)) \leq \deg(E) \cdot \nu(E) + e(J) \cdot \nu(E)^2.$$

(ii) *If J has dimension 1, then for any regular sequence $\mathbf{z} \subset J$ of length $\dim R - 1$,*

$$\nu(\mathrm{Hom}_R(E, E)) \leq \deg(E)\nu(E) + \deg(R/(\mathbf{z}))\nu(E)^2.$$

We are now going to formulate two results of [**3**] on modules of finite projective dimension. They are [**3**, Theorem 6.5], asserting that the conjectural bound (1) holds for modules which are free on the punctured spectrum of R, and [**3**, Theorem 7.3], where the same bound is established for torsionfree modules if $\dim R = 4$.

Proposition 3.5. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring essentially of finite type over the field k and denote by J its Jacobian ideal. If J is $\mathfrak{m}$-primary and i is a fixed positive integer, then for every maximal Cohen-Macaulay module M and every finitely generated R-module B,*

$$\lambda(\mathrm{Tor}_i^R(M, B)) \leq \beta_i(M)\mathrm{Deg}(B).$$

Proof. We will argue by induction on $\dim B$, the case $\dim B = 0$ being clear. Let $H^0_{\mathfrak{m}}(B) = B_0 \neq 0$; the exact sequence

$$0 \to B_0 \longrightarrow B \longrightarrow B' \to 0$$

yields $\mathrm{Deg}(B) = \lambda(B_0) + \mathrm{Deg}(B')$. On the other hand, the long exact homology sequence

$$\mathrm{Tor}_i^R(M, B_0) \longrightarrow \mathrm{Tor}_i^R(M, B) \longrightarrow \mathrm{Tor}_i^R(M, B')$$

gives

$$\lambda(\mathrm{Tor}_i^R(M, B)) \leq \lambda(\mathrm{Tor}_i^R(M, B_0)) + \lambda(\mathrm{Tor}_i^R(M, B')).$$

It suffices to consider the case of modules of positive depth. Let $z \in J$ be a generic hyperplane section for the module B:

$$0 \to B \xrightarrow{\cdot z} B \longrightarrow \overline{B} \to 0.$$

By Proposition 3.2, $z\mathrm{Tor}_i^R(M, B) = 0$, and therefore

$$\mathrm{Tor}_i^R(M, B) \hookrightarrow \mathrm{Tor}_i^R(M, \overline{B}).$$

Since $\dim \overline{B} = \dim B - 1$ and $\mathrm{Deg}(\overline{B}) \leq \mathrm{Deg}(B)$, the induction is complete. □

We now treat the mentioned results of [**3**], with the assumption of finite projective dimension removed. Let E be a finitely generated R-module with a minimal presentation

$$0 \to F_d \to F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E \to 0,$$

$d = \dim R$, where F_i is a free R-module for $i \leq d - 1$. F_d is a MCM module of multiplicity controlled by $\mathrm{Deg}(E)$. Applying $\mathrm{Hom}_R(\cdot, R)$ we get a complex which is broken up into short exact sequences:

$$\begin{gathered} 0 \to Z_0 \longrightarrow F_0^* \longrightarrow B_1 \to 0 \\ \vdots \\ 0 \to Z_i \longrightarrow F_i^* \longrightarrow B_{i+1} \to 0 \\ \vdots \\ 0 \to B_d \longrightarrow F_d^* \longrightarrow \mathrm{Ext}_R^d(E, R) \to 0 \end{gathered}$$

along with the exact sequences, $1 \leq i \leq d - 1$,

$$0 \to B_i \longrightarrow Z_i \longrightarrow \mathrm{Ext}_R^i(E, R) \to 0$$

and

$$0 \longrightarrow \mathrm{Ext}_R^1(E, R) \longrightarrow D(E) \longrightarrow B_2 \to 0.$$

It is the last exact sequence that will become the focus of interest, as we need to estimate $\nu(\mathrm{Tor}_1^R(D(E), E))$.

Suppose E is free in dimension 2. This implies that the modules $\mathrm{Ext}_R^i(E, R)$, $i \geq 1$, have dimension at most 1. Tensoring the last sequence by E, we obtain a complex

$$\mathrm{Tor}_1^R(\mathrm{Ext}_R^1(E, R), E) \longrightarrow \mathrm{Tor}_1^R(D(E), E) \longrightarrow \mathrm{Tor}_1^R(B_2, E) \tag{12}$$

for which we gather information from the homology of the other exact sequences

$$\begin{gathered} \mathrm{Tor}_2^R(\mathrm{Ext}_R^2(E, R), E) \longrightarrow \mathrm{Tor}_1^R(B_2, E) \longrightarrow \mathrm{Tor}_1^R(Z_2, E) \\ \mathrm{Tor}_1(Z_2, E) = \mathrm{Tor}_2^R(B_3, E) \\ \mathrm{Tor}_3^R(\mathrm{Ext}_R^3(E, R), E) \longrightarrow \mathrm{Tor}_2^R(B_3, E) \longrightarrow \mathrm{Tor}_2^R(Z_3, E) \\ \mathrm{Tor}_2(Z_3, E) = \mathrm{Tor}_3^R(B_4, E) \\ \vdots \end{gathered}$$

$$\mathrm{Tor}_d^R(\mathrm{Ext}_R^d(E,R),E) \longrightarrow \mathrm{Tor}_{d-1}^R(B_d,E) \longrightarrow \mathrm{Tor}_{d-1}^R(F_d^*,E).$$

For each $i \geq 1$, $\mathrm{Tor}_i^R(\mathrm{Ext}_R^i(E,R),E)$ is a subquotient of $R^{\beta_i(E)} \otimes \mathrm{Ext}_R^i(E,R)$, a module whose Deg value is $\beta_i(E)\mathrm{Deg}(\mathrm{Ext}_R^i(E,R))$. By Lemma 3.3, $\dim \mathrm{Ext}_R^i(E,R) \leq 1$, and thus every submodule of $\mathrm{Tor}_i^R(\mathrm{Ext}_R^i(E,R),E)$ is generated by at most $\beta_i(E)\mathrm{Deg}(\mathrm{Ext}_R^i(E,R))$ elements.

Finally, note that $\mathrm{Tor}_{d-1}^R(B_d,E)$ must be handled differently. To estimate a bound for the number of generators for its submodules: to $\beta_d(E)\mathrm{Deg}(\mathrm{Ext}_R^d(E,R))$ we must add the bound for $\nu(\mathrm{Tor}_{d-1}^R(F_d^*,E))$, derived in Proposition 3.5, since F_d^* is a MCM module.

THEOREM 3.6. *Let $(R,\mathfrak{m})$ be a Gorenstein local ring essentially of finite type over the field k and denote by J its Jacobian ideal. If J is $\mathfrak{m}$-primary, then for any R-module that is free in dimension $d-2$,*

$$\begin{aligned} \nu(\mathrm{Hom}_R(E,E)) \quad &\leq \quad \nu(E)(\deg E + d(d-1)/2)\mathrm{hdeg}(E) + \mathrm{hdeg}(E)\sum_{i=1}^{d}\beta_i(E) \\ &+ \quad \beta_{d-1}(k)\beta_d(E)\deg(E)\mathrm{hdeg}_J(E). \end{aligned}$$

Proof. The first summand corresponds to the number of generators for $E^* \otimes E$, according to Theorem 2.12. The second arises from the consideration of short exact sequences of modules of dimension at most 1,

$$A \longrightarrow B \longrightarrow C,$$

where the bounds a and c for the numbers of generators of A and C, respectively, we get that $a+c$ bounds the number of generators for any submodule of B.

To obtain the last term, we apply Proposition 3.5 to the module F_d^*, a Cohen-Macaulay module with $\deg(F_d) \leq \beta_d(E)\deg(R)$. $\square$

REMARK 3.7. Now we outline quickly the proof of the conjecture for rings of dimension 4 and torsionfree modules. (More generally, for rings of arbitrary dimension and torsionfree modules with the condition S_{d-3}.) These modules have a presentation

$$0 \to F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E \to 0,$$

where F_i are free, for $i \leq 2$, and F_3 is a MCM module. The only significant deviation from the proof above, in the exact sequence

$$\mathrm{Tor}_1^R(\mathrm{Ext}_R^1(E,R),E) \longrightarrow \mathrm{Tor}_1^R(B_2,E) \longrightarrow \mathrm{Tor}_1^R(Z_2,E),$$

we must deal with the fact that $\mathrm{Ext}_1^R(E,R)$ has dimension at most two, the inequality $\mathrm{hdeg}(\mathrm{Ext}_R^1(E,R)) < \mathrm{hdeg}(E)$, and all the other homology modules having dimension at most 1. This is handled in [**3**] to give $\nu(\mathrm{Tor}_1^R(\mathrm{Ext}_R^1(E,R),E)) \leq \mathrm{hdeg}(\mathrm{Ext}_R^1(E,R))\nu(E)$.

4. Syzygies of near-perfect modules

Let R be a Gorenstein local ring of dimension d. Let us consider some modules with a very rich structure–the modules of syzygies of Cohen-Macaulay modules, or of mild generalizations thereof.

Let us begin this discussion with an example, the modules of cycles of a Koszul complex $\mathbb{K}(\mathbf{x})$ associated to a regular sequence $\mathbf{x} = \{x_1,\ldots,x_n\}$, $n \geq 4$:

$$\mathbb{K}(\mathbf{x}): \quad 0 \to K_n \to K_{n-1} \to K_{n-2} \to \cdots \to K_2 \to K_1 \to K_0 \to 0.$$

First order syzygies have a more general treatment in the next section. For simplicity we take for module E the 2-cycles Z_2 of $\mathbb{K}$. There is a pairing in the subalgebra $\mathbf{Z}$ of cycles leading to

$$Z_2 \times Z_{n-3} \to Z_{n-1} = R,$$

that identifies Z_{n-3} with the dual E^* of E.

We are now ready to put this data into the framework of the Auslander dual. Dualizing the projective presentation of E,

$$0 \to K_n \longrightarrow \cdots \longrightarrow K_3 \longrightarrow K_2 \longrightarrow E \to 0,$$

gives us the exact complex

$$0 \to E^* \longrightarrow K_2^* \longrightarrow K_3^* \longrightarrow D(E) \to 0.$$

In other words, to the identification $D(E) = Z_{n-4}$:

$$0 \to Z_{n-3} \longrightarrow K_{n-2} \longrightarrow K_{n-3} \longrightarrow Z_{n-4} \to 0.$$

Now for the computation of $\operatorname{Tor}_1^R(D(E), E)$:

$$\begin{aligned} \operatorname{Tor}_1^R(D(E), E) &= \operatorname{Tor}_1^R(Z_{n-4}, E) = \operatorname{Tor}_2^R(Z_{n-5}, E) = \cdots \\ &= \operatorname{Tor}_{n-3}^R(Z_0, E) = \operatorname{Tor}_{n-2}^R(R/I, E) = K_{n-2} \otimes R/I. \end{aligned}$$

REMARK 4.1. The number of generators of $\operatorname{Hom}_R(E, E)$ is bounded by

$$\nu(E)\nu(E^*) + \nu(K_{n-2}) = \binom{n}{2}\left(\binom{n}{3} + 1\right).$$

For the purpose of a comparison, let us evaluate $\operatorname{hdeg}(E)$. The multiplicity of E is $(n-1)\deg(R)$. Applying $\operatorname{Hom}_R(\cdot, R)$ to the projective resolution of E, we get $\operatorname{Ext}_R^{n-2}(E, R) = \operatorname{Ext}_R^n(R/I, R) = R/I$ is Cohen-Macaulay, and its contribution in the formula for $\operatorname{hdeg}(E)$ becomes

$$\operatorname{hdeg}(E) = \deg(E) + \binom{d-1}{n-3}\operatorname{hdeg}(\operatorname{Ext}_R^{n-2}(E, R)) = (n-1) + \binom{d-1}{n-3}\deg(R/I).$$

It is clear that in appealing to [**3**, Theorem 5.2], to get information about $\nu(E^*)$, a similar calculation can be carried out for any module of cycles of a projective resolution of broad classes of Cohen-Macaulay modules.

Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d. Let M be a perfect R-module with a minimal free resolution

$$\mathbb{K}: \quad 0 \to K_n \to K_{n-1} \to K_{n-2} \to \cdots \to K_2 \to K_1 \to K_0 \to M \to 0.$$

We observe that dualizing $\mathbb{K}$ gives a minimal projective resolution $\mathbb{L}$ of $\operatorname{Ext}_R^n(M, R)$. Let E be the module $Z_k = Z_k(\mathbb{K})$ of k-cycles of $\mathbb{K}$,

$$K_{k+1} \longrightarrow K_k \longrightarrow E \to 0.$$

Dualizing, to define the Auslander dual $D(E)$, gives the complex

$$0 \to E^* \longrightarrow L_{n-k} \longrightarrow L_{n-k-1} \longrightarrow D(E) \to 0.$$

It identifies E^* with the $(n-k+1)$–cycles of $\mathbb{L}$, and $D(E)$ with its $(n-k-2)$–cycles. In particular this gives $\nu(E) = \beta_k(M)$ and $\nu(E^*) = \beta_{k-1}(M)$.

Let us now determine $\mathrm{hdeg}(E)$. From the complex

$$0 \to E \longrightarrow K_{k-1} \longrightarrow \cdots \longrightarrow K_1 \longrightarrow K_0 \longrightarrow M \to 0,$$

we obtain that proj. dim. $E = n-k$ and that the Cohen-Macaulay property of M gives that $\mathrm{Ext}_R^n(M,R) = \mathrm{Ext}_R^{n-k}(E,R)$ is also Cohen-Macaulay. This is all that is required:

$$\begin{aligned}\mathrm{hdeg}(E) &= \deg(E) + \binom{d-1}{n-k-1}\deg(\mathrm{Ext}_R^n(M,R))\\ &= \mathrm{rank}(E)\deg(R) + \binom{d-1}{n-k-1}\deg(M).\end{aligned}$$

Note that $\mathrm{rank}(E) = \beta_0(M) - \beta_1(M) + \cdots (-1)^{k-1}\beta_{k-1}(M)$.

Now to make use of the Auslander dual setup, we seek some control over $\mathrm{Tor}_1^R(D(E),E)$. We make use first of the complex

$$0 \to D(E) \longrightarrow L_{n-k-2} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow \mathrm{Ext}_R^n(M,R) \to 0,$$

to get

$$\mathrm{Tor}_1^R(D(E),E) \simeq \mathrm{Tor}_{n-k}^R(\mathrm{Ext}^n(M,R),E),$$

and then of the minimal resolution of E to obtain

$$\mathrm{Tor}_1^R(D(E),E) \simeq \mathrm{Tor}_n^R(\mathrm{Ext}^n(M,R),M).$$

In particular, $\mathrm{Tor}_1^R(D(E),E)$ is independent of which module of syzygies was taken. Furthermore, the calculation shows that $\mathrm{Tor}_2^R(D(E),E) = 0$.

Placing these elements together, we have the exact sequence

$$0 \to E^* \otimes E \longrightarrow \mathrm{Hom}_R(E,E) \longrightarrow \mathrm{Tor}_n^R(\mathrm{Ext}_R^n(M,R),M) \to 0.$$

Note that $E^* \otimes E$ is a torsionfree R-module.

Finally, it follows from Proposition 2.14 that $\mathrm{Tor}_n^R(\mathrm{Ext}_R^n(M,R),M)$ can be identified to $\mathrm{Hom}_R(M,M)$.

Let us sum up these observations in the following:

THEOREM 4.2. *Let R be a Gorenstein local ring and let M be a perfect module with a minimal resolution $\mathbb{K}$. For the module E of k-syzygies of M, there exists an exact sequence*

$$0 \to E^* \otimes E \longrightarrow \mathrm{Hom}_R(E,E) \longrightarrow \mathrm{Hom}_R(M,M) \to 0.$$

This gives the bound

$$\nu(\mathrm{Hom}_R(E,E)) \leq \beta_k(M)\beta_{k-1}(M) + \nu(\mathrm{Hom}_R(M,M)).$$

If M is cyclic, or $\dim M = 2$, the estimation is easy. The formula also shows up the case of MCM modules to be a corner case for the general HomAB problem.

We are going to relax the conditions on M: Assume that it has finite projective dimension, has codimension n and is Cohen-Macaulay on the punctured spectrum. Consider its minimal free resolution:

$$\mathbb{K}: \quad 0 \to K_m \to \cdots \to K_n \to K_{n-1} \to K_{n-2} \to \cdots \to K_2 \to K_1 \to K_0 \to M \to 0.$$

The conditions on M imply that $\mathrm{Ext}_R^i(M,R) = 0$ for $i < n$, and $\mathrm{Ext}_R^i(M,R)$ has finite support for $i > n$. The latter means that the syzygies M of order n, or higher, are vector bundles on the punctured spectrum.

Let E be the module of k-cycles of $\mathbb{K}$:

$$0 \to K_m \to \cdots \to K_{k+1} \xrightarrow{\varphi} K_k \to E \to 0,$$

$$0 \to E \to K_{k-1} \to \cdots \to K_1 \to K_0 \to M \to 0.$$

If $k \geq n$, E being a vector bundle of finite projective dimension, the module $\mathrm{Hom}_R(E,E)$ was dealt with in [**3**]. For this reason we will focus on the case $k < n$. Denote the dual of $\mathbb{K}$ by $\mathbb{L}$. We will set $\mathbb{L}_i = \mathbb{K}^*_{m-i}$. We have the complexes

$$0 \to K_0^* \to K_1^* \to \cdots \to K_{k-1}^* \to E^* \to 0,$$

$$0 \to E^* \to K_k^* \xrightarrow{\varphi^t} K_{k+1}^* \to \cdots \to K_m^* \to 0.$$

The Auslander dual $D(E)$ is expressed in the two exact sequences

$$0 \to E^* \to K_k^* \xrightarrow{\varphi^t} K_{k+1}^* \to D(E) \to 0,$$

$$0 \to \mathrm{Ext}^k_R(M,R) \to D(E) \to Z_{m-k-2}(\mathbb{L}) \to \mathrm{Ext}^{k+2}_R(M,R) \to 0,$$

which we will make use of to calculate $\mathrm{Tor}_1^R(D(E),E)$.

Let us examine the different cases. If $k+2 < n$, $\mathrm{Ext}^k_R(M,R) = \mathrm{Ext}^{k+2}_R(M,R) = 0$, and from the exact complex

$$0 \to Z_{m-k-2} \to L_{m-k-2} \to \cdots \to L_{m-n} \to B_{m-n} \to 0$$

we get

$$\mathrm{Tor}_1^R(D(E),E) = \mathrm{Tor}_1^R(Z_{m-k-2},E) = \mathrm{Tor}^R_{n-k}(B,E) = \mathrm{Tor}^R_n(B,M),$$

the last isomorphism since E is the module of k-cycles of the resolution of M.

If M is Cohen-Macaulay, $B = \mathrm{Ext}^n_R(M,R)$, and we would get the formula of Theorem 4.2. In the general case, we have a series of short exact sequences derived from the boundaries B_i and cycles Z_i of the complex $\mathbb{L}$: For $i \geq 1$

$$0 \to \mathrm{Ext}^n_R(M,R) \to B \to B_{m-n-1} \to 0,$$

$$0 \to B_i \to Z_i \to \mathrm{Ext}^{n+i}_R(M,R) \to 0,$$

$$0 \to Z_{i+1} \to L_{i+1} \to B_i \to 0.$$

From these, we would get the exact complexes whose significant parts we highlight:

$$\mathrm{Tor}^R_n(\mathrm{Ext}^n_R(M,R),M) = \mathrm{Tor}^R_{n-k}(\mathrm{Ext}^n_R(M,R),E) \to \mathrm{Tor}_{n-k}(B,E) \to \mathrm{Tor}^R_{n-k}(B_{m-n-1},E),$$

$$\mathrm{Tor}^R_{n-k}(\mathrm{Ext}^{n+i}_R(M,R),E) \to \mathrm{Tor}^R_{n-k}(B_iE) \to \mathrm{Tor}^R_{n-k}(Z_i,E) \to \mathrm{Tor}^R_{n-k-1}(\mathrm{Ext}^{n+i}_R(M,R),E),$$

$$\mathrm{Tor}^R_{n-k}(B_i,E) = \mathrm{Tor}^R_{n-k-1}(Z_{i+1},E).$$

All the terms that involve $\mathrm{Ext}^{n+i}_R(M,R)$ have finite length that can be assembled in a way bounded by a quadratic polynomial on $\mathrm{Deg}(E)$.

5. Ideal modules

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and let E be a finitely generated torsionfree R-module. E is said be an *ideal module* if E^* is a free R-module (see [**9**]). Such modules are basically the syzygies of modules of codimension at least 2. This property endows E with a natural embedding into a free R-module

$$0 \to E \longrightarrow (E^*)^* \longrightarrow C \to 0. \tag{13}$$

Another property of these modules that is relevant in the HomAB problem is that the Auslander dual $D(E)$ has projective dimension at most two: Dualizing (8) gives

$$0 \to E^* \longrightarrow F_0^* \xrightarrow{\varphi^t} F_1^* \longrightarrow D(E) \to 0.$$

PROPOSITION 5.1. *Let E be an ideal module of rank ℓ as above. There exists an exact sequence*

$$0 \to R^\ell \otimes_R E \longrightarrow \mathrm{Hom}_R(E, E) \longrightarrow \mathrm{Hom}_R(C, C) \to 0.$$

Proof. Apply $\mathrm{Hom}_R(\cdot, E)$ to the exact sequence (13) (note $(E^*)^* \simeq R^\ell$) to get the exact sequence

$$0 \to \mathrm{Hom}_R(R^\ell, E) \longrightarrow \mathrm{Hom}_R(E, E) \longrightarrow \mathrm{Ext}^1_R(C, E) \longrightarrow \mathrm{Ext}^1_R(R^\ell, E) = 0.$$

Since E is free in codimension 1, the annihilator I of C has codimension at least 2. Let f be a regular element in I; reducing (13) modulo f, we get the exact complex

$$0 \to C \longrightarrow E/fE \longrightarrow R^\ell/fR^\ell \longrightarrow C \to 0,$$

that identifies C to the submodule of E/fE supported in dimension at most $d-2$. If we take this fact into the isomorphism $\mathrm{Ext}^1_R(C, E) \simeq \mathrm{Hom}_R(C, E/fE)$, we obtain $\mathrm{Ext}^1_R(C, E) = \mathrm{Hom}_R(C, C)$, as desired. □

We can relate $\mathrm{Deg}(C)$ to $\mathrm{Deg}(E)$, at least in the case when $\mathrm{Deg} = \mathrm{hdeg}$. Considering that a bound for $\nu(\mathrm{Hom}_R(C, C))$ is established in [**3**] for all modules of dimension at most 2, we can apply it to ideal modules that are free in dimension at most 3. The formula obtained is similar to the one we are going to derive now by treating directly the support $\mathrm{Tor}_1^R(D(E), E)$.

PROPOSITION 5.2. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and let E be a torsionfree ideal module that is free in dimension 2. Then*

$$\mathrm{hdeg}(\mathrm{Tor}_1^R(D(E), E)) \leq \mathrm{rank}(E) \cdot (\mathrm{hdeg}(E) - \mathrm{deg}(E)).$$

Proof. We begin by pointing out that the support of $\mathrm{Tor}_1^R(D(E), E)$ has dimension at most 1 since E is free in codimension $d-2$. By the same token, in the natural embedding

$$0 \to E \longrightarrow E^{**} \longrightarrow C \to 0$$

the module C has dimension at most 1. Since E^{**} is free, one has $\mathrm{hdeg}(C) \leq \mathrm{hdeg}(E) - \mathrm{deg}(E)$. We also have that $\mathrm{Tor}_2^R(D(E), C) = \mathrm{Tor}_1^R(D(E), E)$. The first of these modules is the homology of the mapping of modules of dimension at most 1,

$$0 \to \mathrm{Tor}_2^R(D(E), C) \longrightarrow E^* \otimes C \longrightarrow F_0^* \otimes C,$$

and therefore

$$\mathrm{hdeg}(\mathrm{Tor}_2^R(D(E), C)) \leq \mathrm{rank}(E) \cdot \mathrm{hdeg}(C) = \mathrm{rank}(E) \cdot (\mathrm{hdeg}(E) - \mathrm{deg}(E)).$$

Corollary 5.3. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and let E be a torsionfree ideal module that is free in dimension 2. Then*

$$\nu(\mathrm{Hom}_R(E,E)) \leq \mathrm{rank}(E)\cdot(\mathrm{hdeg}(E) + \nu(E) - \deg(E)).$$

One should attempt to cut down the free locus to codimension $d-3$. It would be of interest if we could still have $\dim \mathrm{Tor}_1^R(D(E),E) \leq 1$, for even though C could have dimension 2 its component C_1 of dimension at most 1 would satisfy $\mathrm{hdeg}(C_1) \leq \mathrm{hdeg}(E) - \deg(E)$ (a fact not difficult to verify).

Let us go through the calculations in the case of free locus in codimension $d-3$. Now the module C in

$$0 \to E \longrightarrow E^{**} \longrightarrow C \to 0$$

has dimension at most 2. Let z be a generic element for the following purposes: $\mathrm{hdeg}(R) \geq \mathrm{hdeg}(R/(z))$, $\mathrm{hdeg}(E) \geq \mathrm{hdeg}(E/zE)$, $\dim(C/zC) \leq 1$,

$$\dim \mathrm{Tor}_1^R(D(E),E)/z\mathrm{Tor}_1^R(D(E),E) \leq 1,$$

and the kernels ${}_zC$ and L of multiplication by z on C and $D(E)\otimes E$ have finite length. In addition, since we may assume $\dim R \geq 4$ and $D(E)$ has projective dimension at most 2, z can be assumed to be regular on $D(E)$.

Let us see how reduction modulo (z) affects the degrees data. Tensoring

$$0 \to E \xrightarrow{\cdot z} E \longrightarrow \overline{E} \to 0$$

by $D(E)$, we have the exact sequence

$$0 \to \mathrm{Tor}_1^R(D(E),E)/z\mathrm{Tor}_1^R(D(E),E) \longrightarrow \mathrm{Tor}_1^R(D(E),\overline{E}) \longrightarrow L \to 0.$$

This shows that $\mathrm{Tor}_1^R(D(E),\overline{E})$ has dimension at most one, and by Nakayama Lemma

$$\nu(\mathrm{Tor}_1^R(D(E),E)) \quad \leq \quad \mathrm{hdeg}(\mathrm{Tor}_1^R(D(E),\overline{E})). \tag{14}$$

To get hold of the last degree, we tensor the defining sequence of C by $R/(z)$ to get

$$0 \to {}_zC \longrightarrow \overline{E} \longrightarrow \overline{E^{**}} \longrightarrow \overline{C} \to 0.$$

It shows that ${}_zC$ is the submodule of finite support of $\overline{E}$ and therefore the image E' of $\overline{E}$ satisfies $\mathrm{hdeg}(E') \leq \mathrm{hdeg}(\overline{E}) \leq \mathrm{hdeg}(E)$, and from the exact sequence

$$0 \to E' \longrightarrow \overline{E^{**}} \longrightarrow \overline{C} \to 0$$

we have

$$\mathrm{hdeg}(\overline{C}) \leq \mathrm{hdeg}(E') - \deg(E') \leq \mathrm{hdeg}(E) - \deg(E).$$

Now we get hold of $\mathrm{hdeg}(\mathrm{Tor}_1^R(D(E),\overline{E}))$. From the exact sequence

$$0 \to {}_zC \longrightarrow \overline{E} \longrightarrow E' \to 0$$

we have the exact sequence

$$\mathrm{Tor}_1^R(D(E),{}_zC) \longrightarrow \mathrm{Tor}_1^R(D(E),\overline{E}) \longrightarrow \mathrm{Tor}_1^R(D(E),E').$$

It shows that

$$\begin{aligned}
\mathrm{hdeg}(\mathrm{Tor}_1^R(D(E)),\overline{E}) &\leq \mathrm{hdeg}(\mathrm{Tor}_1^R(D(E),E')) + \mathrm{hdeg}(\mathrm{Tor}_1^R(D(E)),{}_zC)\\
&\leq \mathrm{hdeg}(\mathrm{Tor}_1^R(D(E),E')) + \mathrm{rank}(F_0)\cdot\lambda({}_zC)\\
&\leq \mathrm{hdeg}(\mathrm{Tor}_1^R(D(E),E')) + \mathrm{rank}(F_0)\cdot(\mathrm{hdeg}(E) - \deg(E)). \qquad (15)
\end{aligned}$$

The last equation arises because $\mathrm{hdeg}(\overline{E}) = \mathrm{hdeg}({}_zC) + \mathrm{hdeg}(E')$ and therefore

$$\mathrm{hdeg}({}_zC) \leq \mathrm{hdeg}(\overline{E}) - \mathrm{hdeg}(E') \leq \mathrm{hdeg}(E) - \deg(E).$$

We must now deal with $\mathrm{hdeg}(\mathrm{Tor}_1^R(D(E), E'))$ in the manner of the earlier case. Starting from

$$0 \to E' \longrightarrow \overline{E^{**}} \longrightarrow \overline{C} \to 0,$$

we have the exact sequence

$$\mathrm{Tor}_2^R(D(E), \overline{E^{**}}) \longrightarrow \mathrm{Tor}_2^R(D(E), \overline{C}) \longrightarrow \mathrm{Tor}_1^R(D(E), E') \longrightarrow \mathrm{Tor}_1^R(D(E), \overline{E^{**}}).$$

Since E^{**} is R-free and z is regular on $D(E)$, the two modules at the ends vanish and we have $\mathrm{Tor}_1^R(D(E), E') = \mathrm{Tor}_2^R(D(E), \overline{C})$. As $\overline{C}$ has dimension at most 1 and we have bounds for $\mathrm{hdeg}(\overline{C})$, we make use of the free presentation of $D(E)$ and finally obtain

$$\begin{aligned} \mathrm{hdeg}(\mathrm{Tor}_1^R(D(E), E')) &= \mathrm{hdeg}(\mathrm{Tor}_2^R(D(E), \overline{C})) \leq \mathrm{rank}(E) \cdot \mathrm{hdeg}(\overline{C}) \\ &\leq \mathrm{rank}(E) \cdot (\mathrm{hdeg}(E) - \deg(E)). \end{aligned} \tag{16}$$

We collect the calculation into:

Proposition 5.4. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d and let E be a torsionfree ideal module that is free in dimension 3. Then*

$$\nu(\mathrm{Tor}_1^R(D(E), E)) \leq (\mathrm{rank}(E) + \nu(E))(\mathrm{hdeg}(E) - \deg(E)).$$

Proof. We have from (14)

$$\nu(\mathrm{Tor}_1^R(D(E), E)) \leq \mathrm{hdeg}(\mathrm{Tor}_1^R(D(E), \overline{E})),$$

while from (15) we have

$$\mathrm{hdeg}(\mathrm{Tor}_1^R(D(E), \overline{E})) \leq \mathrm{hdeg}(\mathrm{Tor}_1^R(D(E), E')) + \mathrm{rank}(F_0) \cdot \mathrm{hdeg}(E).$$

Finally, from (16) we obtain

$$\mathrm{hdeg}(\mathrm{Tor}_1^R(D(E), \overline{E})) \leq (\mathrm{rank}(E) + \nu(E))(\mathrm{hdeg}(E) - \deg(E)).$$

Corollary 5.5. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d and let E be a torsionfree ideal module free in codimension $d-3$. Then*

$$\nu(\mathrm{Hom}_R(E, E)) \leq (\mathrm{rank}(E) + \nu(E))(\mathrm{hdeg}(E) - \deg(E)) + \mathrm{rank}(E) \cdot \nu(E).$$

Primary decomposition. Primary decomposition of ideals in a Noetherian ring R, and several other operations as well, involve the computation of ideal quotients: $I \colon J = \{r \in R \mid rI \subset J\}$. It may be relevant to have an estimation for the number of generators of $I \colon J$ given I and J.

The typical ring for us will be a polynomial ring over a field, but given the well-established technique ([**10**]) to convert local bounds for number of generators into global ones, we focus on Gorenstein local rings and on ideals of codimension at least two.

We may consider $I : J = \mathrm{Hom}_R(I, J)$ directly or the variant $\mathrm{Hom}_R(E, E)$ for $E = J \oplus (I, J)$. Then, since by assumption codim $J \geq 2$,

$$\mathrm{Hom}_R(E, E) = (J : J) \oplus J : (I, J) \oplus (I, J) : (I, J) \oplus (I, J) : J) = R^3 \oplus I : J,$$

since $(I, J) : J = I : J$. Note that E is an ideal module of rank 2.

As a direct consequence of Corollary 5.5, we obtain:

THEOREM 5.6. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d and let J be an ideal of codimension at least $d-2$. Then for any ideal I,*

$$\nu(I:J) \leq (2+\nu(J)+\nu(I,J))(\mathrm{hdeg}(J)+\mathrm{hdeg}(I,J)-2\deg(R))+2\cdot(\nu(J)+\nu(I,J))-3.$$

REMARK 5.7. We have to calculate $\mathrm{hdeg}(\mathrm{Ext}^i_R(R/I,R))$ for $i=d-2,d-1,d$. If codim $J=d-2$, $\mathrm{hdeg}(\mathrm{Ext}^{d-2}_R(R/J,R))=\deg(R/J)$, and $\mathrm{hdeg}(\mathrm{Ext}^d_R(R/J,R))=\lambda(H^0_{\mathfrak{m}}(R/J))$.

Let us consider some cases in detail. If J is $\mathfrak{m}$-primary, $\mathrm{Hom}_R(R/I,R/J)=I:J/J$ is a module whose length is at most $\lambda(R/J)$, and thus $\nu(I:J)\leq\lambda(R/J)+\nu(J)$. Meanwhile, given that $\mathrm{hdeg}(J)=\deg(R)+\lambda(R/J)$, the bound from the formula is

$$\nu(I:J)\leq(2+\nu(J)+\nu(I,J))(2+\lambda(R/J)+\lambda(R/(I,J))-7,$$

which is poorer.

The advantage comes in case $\dim R/I=1$, or 2. Let us assume that I and J are Cohen-Macaulay ideals of dimension 2. To make the computation of hdeg simpler, if we take the ideal module: $E=J\oplus I$.

THEOREM 5.8. *Let $(R,\mathfrak{m})$ be a Gorenstein local ring of dimension d and let J and I be Cohen-Macaulay ideals of codimension $d-2$. Then*

$$\nu(I:J)+\nu(J:I)\leq(2+\nu(J)+\nu(I))(\deg(R/J)+\deg(R/I))+2\cdot(\nu(J)+\nu(I))-2.$$

6. Open questions

We will now mention some questions and open problems. We have already mentioned unsolved problems in context. The questions here represent boundary cases.

Test problems. Let R be a Cohen-Macaulay local ring, with a canonical module ω. Let $a_1,\ldots,a_n$ be a minimal set of generators of ω. Consider the complex

$$0\rightarrow R\xrightarrow{\varphi}\omega^{\oplus n}\longrightarrow E=\text{coker }(\varphi)\rightarrow 0,\quad \varphi(1)=(a_1,\ldots,a_n). \tag{17}$$

PROPOSITION 6.1. *E is a Cohen-Macaulay module.*

Proof. Let $\mathbf{z}=z_1,\ldots,z_d$ be a maximal regular sequence of R. It suffices to note that $\mathrm{Tor}^R_1(E,R/(\mathbf{z}))=0$ since $\omega/\mathbf{z}\omega$ is the canonical module of $R/(\mathbf{z})$ and therefore it is a faithful module. □

PROBLEM 6.2. Find a bound for $\nu(\mathrm{Hom}_R(E,E))$. The problem shows the difficulty of dealing with non-Gorenstein rings.

PROBLEM 6.3. A different challenge arises if E is a self-dual module, that is if $E\simeq E^*$. (Examples are rank two reflexive modules over factorial domains.) For these modules $\mathrm{hdeg}(D(E))$ is completely determined by $\mathrm{hdeg}(E)$.

PROBLEM 6.4. Let E be a Cohen-Macaulay module of dimension 3. It would be helpful to understand this corner case.

PROBLEM 6.5. If R is a regular local of dimension 2(!), what is the relationship between bdeg and hdeg?

Vector bundles. Let $(R, \mathfrak{m})$ be a Noetherian local ring and set $\mathbf{X} = \text{Spec } R$. A case of HomAB question is that of a module E that is locally free on a subset $\mathbf{Y} \subset \mathbf{X}$. When $\mathbf{Y}$ is the punctured spectrum Spec $R \setminus \{\mathfrak{m}\}$ of R, it was treated in [**3**], and a solution given for all modules of finite projective dimension that are locally free in codimension $\leq \dim R - 2$. To remove the finite projective dimension requirement reFquires a good understanding of the MCM R-modules.

Here we will consider this issue but also the case of the complement of hyperplanes $\mathbf{Y} = \text{Spec } R \setminus V(z)$, where $R/(z)$ is a regular local ring.

The emphasis on Auslander duals puts a great burden on the algebra

$$\text{Tor}_1^R(D(E), E) = \text{Hom}_R(\text{Ext}_R^1(E, \cdot), \text{Ext}_R^1(E, \cdot)),$$

to determine effective bounds for its number of generators. The cases considered in the previous sections all had a great control over the support of $\text{Tor}_1^R(D(E), E)$. It might be worthwhile to consider the following general question.

Conjecture 6.6. Let E be a torsionfree module. If $\dim \text{Tor}_1^R(D(E), E) \leq 1$,

$$\nu(\text{Hom}_R(E, E)) \leq \text{rank}(E) \cdot (\text{hdeg} E + \nu(E) - \deg E).$$

If $\dim \text{Tor}_1^R(D(E), E) \leq 0$, E is a vector bundle on the punctured spectrum of R. In [**3**], this is dealt even more generally if E is free in codimension $d - 2$, provided proj dim $E < \infty$, while in Theorem 3.6, the case of isolated singularity is dealt with.

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