Complexity of the Normalization of an Algebra

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Outline

Abstract

Let **R** be a normal unmixed integral domain and let **A** be an (usually semistandard graded) **R**-algebra of integral closure $\overline{\mathbf{A}}$. Estimating the number of steps that general algorithms must take to build $\overline{\mathbf{A}}$ can be viewed as an invariant of **A**. We develop a general approach which in two major cases, algebras that allow Noether normalizations (such as affine graded algebras over fields or **Z**) and Rees algebras of ideals or modules, give estimates related to invariants of **A**, in particular they can be said to be known *ab initio*.

This talk uses many sources, but particularly joint work with K. Dalili, J. Hong, T. Pham, C. Polini and B. Ulrich.

Introduction

Let **R** be a geometric/arithmetic integral domain, and let **A** be a graded algebra

$$\mathbf{A} = \mathbf{R}[x_1, \ldots, x_n], \quad x_i \text{ homogeneous.}$$

1. An element u in the total ring of fractions of **A** is integral over **A** if there is an equation

$$u^n + a_{n-1}u^{n-1} + \cdots + a_0 = 0, \quad a_i \in \mathbf{A}.$$

2. The set of all such u is $\overline{\mathbf{A}}$. If **A** is reduced (and the algebra is $\mathbb{Z}^r$-graded)

$$\overline{\mathbf{A}} = \mathbf{R}[y_1, \ldots, y_m], \quad y_i \text{ homogeneous.}$$

3. A main issue is to find such elements.

General Goals

1. Construction of $\overline{\mathbf{A}}$: Find processes $\mathcal{P}$ that create extensions

$$\mathbf{A} \mapsto \mathcal{P}(\mathbf{A}) \subset \overline{\mathbf{A}},$$

$$\mathcal{P}^n(\mathbf{A}) = \overline{\mathbf{A}}, \quad n \gg 0.$$

2. Typically $\mathcal{P}(\mathbf{A}) = \mathrm{Hom}_{\mathbf{A}}(I(\mathbf{A}), I(\mathbf{A}))$, where $I(\mathbf{A})$ is some ideal connected to the conductor/Jacobian of **A.**
3. Predicting properties of $\overline{\mathbf{A}}$ in **A**, possibly by adding up the changes $\mathcal{P}^i(\mathbf{A}) \to \mathcal{P}^{i+1}(\mathbf{A})$.

Specific Goals

1. Numerical Indices for $\overline{\mathbf{A}}$: e.g. Find r such that

$$\overline{\mathbf{A}}_{n+r} = \mathbf{A}_n \cdot \overline{\mathbf{A}}_r, \quad n \geq 0.$$

2. How many “steps” are there between $\mathbf{A}$ and $\overline{\mathbf{A}}$,

$$\mathbf{A} = \mathbf{A}_0 \subset \mathbf{A}_1 \subset \cdots \subset \mathbf{A}_{s-1} \subset \mathbf{A}_s = \overline{\mathbf{A}},$$

   where the $\mathbf{A}_i$ are constructed by an effective process?

3. Express r and s in terms of invariants of $\mathbf{A}$.
4. Generators of $\overline{\mathbf{A}}$: Number and distribution of their degrees

For the first of these goals, bounding the length of the chains

$$\mathbf{A} = \mathbf{A}_0 \subset \mathbf{A}_1 \subset \cdots \subset \mathbf{A}_{s-1} \subset \mathbf{A}_s = \overline{\mathbf{A}},$$

there are 'internal' bounds: If **A** is Gorenstein in codimension one, the descending chain of the conductors, $\mathfrak{c}(\mathbf{A}_i) = \mathrm{Hom}_{\mathbf{A}}(\mathbf{A}_i, \mathbf{A})$

$$\mathbf{A} = \mathfrak{c}(\mathbf{A}_0) \supset \mathfrak{c}(\mathbf{A}_1) \supset \cdots \supset \mathfrak{c}(\mathbf{A}_{s-1}) \supset \mathfrak{c}(\mathbf{A}_s) = \mathfrak{c}(\overline{\mathbf{A}})$$

comes with a descending chain restriction. The point however is to express these bounds by 'external' means.

Affine Domains

Theorem

Let $\mathbf{A}$ *be a semistandard graded domain over a perfect* k*. If* $\mathbf{A}$ *has degree* e *over a (graded) Noether normalization, then for any chain of distinct extensions with the condition* S_2 *of Serre*

$$\mathbf{A} = \mathbf{A}_0 \subset \mathbf{A}_1 \subset \cdots \subset \mathbf{A}_{s-1} \subset \mathbf{A}_s = \overline{\mathbf{A}},$$

1. $s \leq \binom{e}{2}$.
2. *If* $\mathbf{A}$ *is not necessarily homogeneous and char* $k = 0$*, then* $s \leq e(e-1)^2$.

The Tale of Two Degrees

Let **R** be a commutative Noetherian ring, and let **A** be a finitely generated graded algebra (of integral closure $\overline{\mathbf{A}}$). We want to gauge the 'distance' from **A** to $\overline{\mathbf{A}}$. To this end, we introduce two degree functions:

1. $\mathrm{tn} : \mathbf{A} \to \mathbb{N}$, for a certain class of algebras (includes **R** a field). We shall use for distance
$$\mathrm{tn}(\mathbf{A}) - \mathrm{tn}(\overline{\mathbf{A}})$$
2. jdeg : a multiplicity on all finitely generated graded **A**-modules. We shall use for distance
$$\mathrm{jdeg}\,(\overline{\mathbf{A}}/\mathbf{A})$$

The Tracking Number of Graded Algebra

Consider a graded $\mathbf{R} = k[z_1, \ldots, z_r]$–module E: Suppose E has torsionfree rank r. The *determinant* of E is the graded module

$$\det_{\mathbf{R}}(E) = (\wedge^r E)^{**}.$$

If E is a free $\mathbf{R}$–module,

$$E = \mathbf{R}\epsilon_1 \oplus \cdots \oplus \mathbf{R}\epsilon_n$$

with free generator ϵ_i of degree d_i, then

$$\det_{\mathbf{R}}(E) = (\wedge^r E)^{**} = \mathbf{R}(\epsilon_1 \wedge \cdots \wedge \epsilon_n)$$

a free $\mathbf{R}$-module with a generator of degree $\sum d_i$

In general the determinant is a free **R**-module of rank 1 (as **R** is factorial):

$$\det_{\mathbf{R}}(E) \simeq \mathbf{R}[-c(E)], \quad \mathrm{tn}(E) := c(E)$$

The integer $c(E)$ is the **tracking number** of E.
If E has no associated primes of codimension 1, $\mathrm{tn}(E)$ can be read off the Hilbert series $H_E(t)$ of E:

$$H_E(t) = \frac{\mathbf{h}(t)}{(1-t)^d},$$

$$\mathrm{tn}(E) = \mathbf{h}'(1)$$

This shows that $\mathrm{tn}(E)$ is independent of **R**.

Example

Let **f** be a homogeneous polynomial

$$\mathbf{f} = x^e + a_{e-1}x^{e-1} + \cdots + a_0 \in k[x, z_1, \ldots, z_d]$$

and set

$$\mathbf{A} = k[x, z_1, \ldots, z_d]/(\mathbf{f})$$

The Hilbert series of **A** is

$$H(t) = \frac{1 + t + \cdots + t^{e-1}}{(1-t)^d} = \frac{h(t)}{(1-t)^d}$$

Its multiplicity is $e(\mathbf{A}) = h(1) = e$, and its tracking number is

$$\mathrm{tn}(\mathbf{A}) = c(\mathbf{A}) = h'(1) = \binom{e}{2}$$

Tracking number as a coordinate tag

Proposition

Let $\mathbf{R} = k[z_1, \ldots, z_d]$ *and* $E \subset F$ *two graded* $\mathbf{R}$*–modules of the same rank. If* E *and* F *have the condition* S_2 *of Serre,*

$$\mathrm{tn}(E) \geq \mathrm{tn}(F),$$

with equality if and only if $E = F$*.*

Corollary

Let

$$E_0 \subset E_1 \subset \cdots \subset E_n$$

be a sequence of distinct f.g. graded $\mathbf{R}$*-modules of the same rank. Then*

$$n \leq \mathrm{tn}(E_0) - \mathrm{tn}(E_n).$$

Bounding Chains of Algebras with Degrees

Let $\mathbf{R} = k[z_1, \ldots, z_d]$. Let $\mathbf{A}$ (integral domain) be a finitely generated graded $\mathbf{R}$-algebra. If the base field k is perfect, we use the theorem of the primitive element to get

$$\mathbf{A}_0 = k[t, z_1, \ldots, z_d]/(\mathbf{f}) \subset \mathbf{A}$$

with the same field of fractions. Let $\overline{\mathbf{A}}$ be the integral closure of $\mathbf{A}$. Both $\mathbf{A}_0$ and $\overline{\mathbf{A}}$ have the condition S_2 of Serre, so by the Proposition above we have that any chain of distinct subalgebras of $\overline{\mathbf{A}}$

$$\mathbf{A}_0 \subset \mathbf{A}_1 \subset \cdots \subset \mathbf{A}_n$$

has the property

$$n \leq \operatorname{tn}(\mathbf{A}_0) - \operatorname{tn}(\mathbf{A}_n)$$

If in $\mathbf{A}_0 = k[t, z_1, \ldots, z_d]/(\mathbf{f}) \subset \mathbf{A}$, $\deg \mathbf{f} = e$, $\mathrm{tn}(\mathbf{A}_0) = \binom{e}{2}$, to bound n we need:

Theorem

Let $\mathbf{A}$ be a reduced non-negatively graded algebra that is finite over a standard graded Noether normalization $\mathbf{R}$. Then $\mathrm{tn}(\mathbf{A}) \geq 0$. Moreover, if $\mathbf{A}$ is an integral domain and k is algebraically closed, then $\mathrm{tn}(\mathbf{A}) \geq \deg(\mathbf{A}) - 1$.

This gives the bounds on length

1. $n \leq \binom{e}{2}$, in the first case
2. $n \leq \binom{e-1}{2}$, in the other case

jdeg : Generalizing the Classical Multiplicity

Recall the notation: If M is a finitely graded **A**-module and I is a set of elements of **A**,

$$H_I^0(M) = \{m \in M \mid I^n m = 0\}$$

Definition

Let **R** be a Noetherian ring and **A** be a finitely generated graded **R**-algebra where $\mathbf{A} = \mathbf{R}[A_1]$. For a finitely generated graded **A**-module M, and a prime ideal $\mathfrak{p}$ of **R**, let

$$H = H^0_{\mathfrak{p}\mathbf{R}_\mathfrak{p}}(M_\mathfrak{p}).$$

The $j_\mathfrak{p}$-multiplicity of M is the integer

$$j_\mathfrak{p}(M) := \begin{cases} \deg H & \text{if } \dim H = \dim M_\mathfrak{p} \\ 0 & \text{otherwise} \end{cases}$$

jdeg

Definition

Let $\mathbf{R}$ be a Noetherian ring and $\mathbf{A}$ be a finitely generated graded $\mathbf{R}$-algebra where $\mathbf{A} = \mathbf{R}[A_1]$. For a finitely generated graded A-module M,

$$\mathrm{jdeg}\,(M) := \sum_{\mathfrak{p} \in Spec\mathbf{R}} j_{\mathfrak{p}}(M).$$

This is a finite sum, and nonzero if $M \neq O$.

It does not seem to say much: If $\mathbf{R}$ is an integral domain, $\mathrm{jdeg}\,((\mathbf{R}[x]) = 1!$

Properties of jdeg

One the other hand, it has some surprising properties:

Theorem (Invariance)

Let $\mathbf{R}$ be a (somewhat weaker than Cohen-Macaulay) ring and let I be an ideal of $\mathbf{R}$. Let $\mathbf{B}$ be a graded algebra with $\mathbf{A} = \mathbf{R}[It] \subsetneq \mathbf{B} \subset \overline{\mathbf{A}}$ and assume that $\mathbf{B}$ satisfies the condition S_2 of Serre. Then

$$\mathrm{jdeg}\,(\mathrm{gr}\,(\mathbf{A})) = \mathrm{jdeg}\,(\mathrm{gr}\,(\mathbf{B})).$$

jdeg and Normalization

Theorem

Let **R** *be a Noetherian domain and* **A** *a semistandard graded* **R***-algebra with finite integral closure* $\overline{\mathbf{A}}$*. Consider a sequence of distinct integral graded extensions*

$$\mathbf{A} = \mathbf{A}_0 \rightarrow \mathbf{A}_1 \rightarrow \mathbf{A}_2 \rightarrow \cdots \rightarrow \mathbf{A}_s = \overline{\mathbf{A}},$$

where the $\mathbf{A}_i$ *satisfy the* S_2 *condition of Serre. Then*

$$s \leq \mathrm{jdeg}\,(\overline{\mathbf{A}}/\mathbf{A}).$$

This follows from the additivity property of certain exact sequences, such as those of the $\mathbf{A}_i$: From

$$0 \rightarrow \mathbf{A}_2/\mathbf{A}_1 \rightarrow \mathbf{A}_3/\mathbf{A}_1 \rightarrow \mathbf{A}_3/\mathbf{A}_2 \rightarrow 0$$

get

$$\mathrm{jdeg}\,(\mathbf{A}_3/\mathbf{A}_1) = \mathrm{jdeg}\,(\mathbf{A}_2/\mathbf{A}_1) + \mathrm{jdeg}\,(\mathbf{A}_3/\mathbf{A}_2)$$

Divisorial Generation

Question: How many generators must be added to **A** to get $\overline{\mathbf{A}}$?

Corollary

Let **A** *be a finitely generated graded* **R***-algebra as above. If* $\operatorname{jdeg}(\overline{\mathbf{A}}/\mathbf{A}) = m$ *there exist homogeneous elements* $y_1, \ldots, y_m \in \overline{\mathbf{A}}$ *such that* $\overline{\mathbf{A}}$ *is the* S_2*-ification of*

$$\mathbf{A}[y_1, \ldots, y_m].$$

The issue is to express bounds for $\mathrm{jdeg}\,(\overline{\mathbf{A}}/\mathbf{A})$ in terms of invariants of $\mathbf{A}$. There are at least two classes of algebras when this is possible: Rees algebras of ideals/modules and algebras finite over polynomial subrings:

$$\mathbf{R}[x_1, \ldots, x_p] \subset \mathbf{A} \subset \mathbf{R}[t_1, \ldots, t_q]$$

The rings of polynomials serve as referential for various constructions. One knows already quite a lot about these issues. In these situations, one connects $\mathrm{jdeg}\,(\overline{\mathbf{A}}/\mathbf{A})$ to some invariant of $\mathbf{A}$.

Noether Normalizations and Tracking Numbers

Let **R** be a normal domain and let **A** be a semistandard graded **R**-algebra of integral closure $\overline{\mathbf{A}}$. By a Noether normalization of **A** we mean a graded polynomial algebra

$$\mathbf{S} = \mathbf{R}[y_1, \ldots, y_r] \subset \mathbf{A}$$

over which **A** is finite. Unfortunately this does not happen often (e.g. fails for $\mathbb{C}[t]$), although Shimura established it for $\mathbf{R} = \mathbb{Z}$, in one of his first papers.

Tracking number of a pair of algebras

In this setting one can define

$$\det_{\mathbf{S}}(\mathbf{A}) \simeq (\wedge^r \mathbf{A})^{**},$$

where r is the rank of **A** as an **S**-module. Given a pair of algebras $\mathbf{A} \subset \mathbf{B}$ of the same rank, one can attach a degree as follows. Fix $\det(\mathbf{B})$ and the image of $\det(\mathbf{A})$ in it, which we still denote by $\det(\mathbf{A})$. Now set

$$I = \operatorname{ann}(\det(\mathbf{B})/\det(\mathbf{A})).$$

This ideal is independent of the choices made. Let

$$I = (\bigcap \mathfrak{p}_i^{(r_i)}) \cap (\bigcap \mathfrak{q}_j{}^{(s_j)}),$$

is its primary decomposition, where we denote by $\mathfrak{p}_i$ the primes that are extended from R, and $\mathfrak{q}_i$ those that are not.

This means that $\mathfrak{q}_j \cap R = (0)$ and thus $\mathfrak{q}_j KS = (f_j)KS$, $\deg(f_j) > 0$. We associate a *degree* to I by setting

$$\deg(I) = \sum_i r_i + \sum_j s_j \deg(f_j).$$

It depends only on the two algebras and of the embedding $\mathbf{A} \subset \mathbf{B}$. We will denote it by $\mathrm{tn}(\mathbf{A}, \mathbf{B})$. If R is a field, it is possible to define an invariant $\mathrm{tn}(\mathbf{A})$ directly as the degree of $\det(\mathbf{A})$. It has many positivity properties One has $\mathrm{tn}(\mathbf{B}, \mathbf{A}) = \mathrm{tn}(\mathbf{A}) - \mathrm{tn}(\mathbf{B})$.

If $\mathbf{A} = \mathbb{Z}[x, y, z]/(z^3 + xz^2 + x^2y)$, $\mathbf{S} = \mathbb{Z}[x, y]$, then $\mathrm{tn}(\mathbf{A}, \overline{\mathbf{A}}) = 1$.

jdeg and Tracking Number

Theorem

Let **R** *be a normal domain and let* **A** *be a semistandard graded* **R***-algebra that admits a Noether normalization. Then*

$$\operatorname{jdeg}(\overline{\mathbf{A}}/\mathbf{A}) = \operatorname{tn}(\mathbf{A}, \overline{\mathbf{A}}).$$

In case **R** is a field, a **tracking number** $\operatorname{tn}(\mathbf{A})$ can be defined, and

$$\operatorname{tn}(\mathbf{A}, \overline{\mathbf{A}}) = \operatorname{tn}(\mathbf{A}) - \operatorname{tn}(\overline{\mathbf{A}}).$$

$\operatorname{tn}(\mathbf{A})$ is obtained from Hilbert coefficients of the highest dimensional component of **A.**

Hilbert functions and jdeg

Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and let I be an $\mathfrak{m}$–primary ideal. Let $\mathbf{B} = \bigoplus_{n \geq 0} B_n t^n$ be a graded $\mathbf{R}$–subalgebra of $\mathbf{R}[t]$ with $\mathbf{R}[It] \subset \mathbf{B} \subset \mathbf{R}[t]$ and assume that $\mathbf{B}$ is a finite $\mathbf{R}[It]$–module. For any such algebra we consider the Hilbert–Samuel function $\lambda(R/B_n)$.

For $n \gg 0$ this function is given by the Hilbert–Samuel polynomial

$$e_0(\mathbf{B})\binom{n+d-1}{d} - e_1(\mathbf{B})\binom{n+d-2}{d-1} + \text{lower terms}\,.$$

Notice that the $e_i(\mathbf{R}[It])$ coincide with the usual *Hilbert coefficients* $e_i(I)$ of I. Furthermore $e_0(\mathbf{B}) = e_0(I)$. By $\overline{\mathbf{R}[It]}$ we will always denote the integral closure of $\mathbf{R}[It]$ in $\mathbf{R}[t]$.

We write $\overline{e}_i(I)$ for the *normalized Hilbert coefficients* $e_i(\overline{\mathbf{R}[It]})$ of I in case $\overline{\mathbf{R}[It]}$ is a finite $\mathbf{R}[It]$–module.

Proposition

In this case,

$$\operatorname{jdeg}(\overline{\mathbf{R}[It]}/\mathbf{R}[It]) = \overline{e}_1(I) - e_1(I).$$

Briançon-Skoda numbers

Definition

If I is an ideal of a Noetherian ring $\mathbf{R}$, the Briançon-Skoda number $c(I)$ of I is the smallest integer c such that $\overline{I^{n+c}} \subset J^n$ for every n and every reduction J of I.

The motivation for this definition is a result of Briançon-Skoda asserting that for the rings of convergent power series over $\mathbb{C}^n$ (later extended to regular local rings by Lipman- Sathaye) $c(I) < \dim \mathbf{R}$.

Theorem

Let $(\mathbf{R}, \mathfrak{m})$ be an analytically unramified Cohen–Macaulay local ring and let I be an $\mathfrak{m}$–primary ideal of Briançon-Skoda number $c(I)$. Let $\mathbf{A}$ and $\mathbf{B}$ be distinct graded algebras with

$$\mathbf{R}[It] \subset \mathbf{A} \subsetneq \mathbf{B} \subset \overline{\mathbf{R}[It]}$$

and assume that $\mathbf{A}$ satisfies the condition S_2 of Serre. Then

$$c(I)e_0(I) \geq \overline{e}_1(I) \geq e_1(\mathbf{B}) > e_1(\mathbf{A}) \geq e_1(I) \geq 0.$$

In particular, any chain of subalgebras that satisfy the condition S_2 of Serre has length at most $\overline{e}_1(I)$.

Theorem: Let $(\mathbf{R}, \mathfrak{m})$ be a local Cohen-Macaulay algebra of type t essentially of finite type over a perfect field k. Let I be an $\mathfrak{m}$–primary ideal.

- If $\delta \in \mathrm{Jac}_k(\mathbf{R})$ is a non zerodivisor, then

$$\overline{e}_1(I) \leq \frac{t}{t+1}((d-1)e_0(I) + e_0(I + \delta\mathbf{R}/\delta\mathbf{R})).$$

- If the assumptions above hold, then

$$\overline{e}_1(I) \leq (d-1)(e_0(I) - \lambda(\mathbf{R}/\bar{I})) + e_0(I + \delta\mathbf{R}/\delta\mathbf{R}).$$

- If $\mathbf{R}/\mathfrak{m}$ is infinite, then

$$\overline{e}_1(I) \leq c(I) \min\{\frac{t}{t+1}\, e_0(I), e_0(I) - \lambda(\mathbf{R}/\bar{I})\}.$$

Corollary

If **R** *is a regular local ring of dimension* d, $\overline{e}_1(I) \leq \frac{1}{2}(d-1)e_0(I)$.

If I is a monomial ideal, R. Villarreal derived a refined version from Ehrhart's reciprocity theorem.

Theorem

Let **R** *be a Cohen-Macaulay reduced quasi-unmixed ring and let* I *be an ideal of Briançon-Skoda number* $c(I)$ *(or the modified value mentioned above in the case of isolated singularities). Then*

$$\operatorname{jdeg}\,(\overline{\mathbf{R}[It]}/R[It]) \leq c(I) \cdot \operatorname{jdeg}\,(\operatorname{gr}_I(\mathbf{R})).$$

Let $\mathbf{R}$ be a Noetherian ring, let E be a finitely generated torsionfree $\mathbf{R}$–module having a rank, and choose an embedding $\varphi : E \hookrightarrow \mathbf{R}^r$. The *Rees algebra* $\mathcal{R}(E)$ of E is the subalgebra of the polynomial ring $\mathbf{R}[t_1, \ldots, t_r]$ generated by all linear forms $a_1 t_1 + \cdots + a_r t_r$, where $(a_1, \ldots, a_r)$ is the image of an element of E in $\mathbf{R}^r$ under the embedding φ.

The Rees algebra $\mathcal{R}(E)$ is a standard graded algebra whose nth component is denoted by E^n and is independent of the embedding φ since E is torsionfree and has a rank.

We will also consider another Rees algebra. The algebra $\mathcal{R}(E)$ is a subring of the polynomial ring $\mathbf{S} = \mathbf{R}[t_1, \ldots, t_r]$. We consider the ideal (E) of $\mathbf{S}$ generated by the forms in E, and its Rees algebra $\mathcal{R}((E))$. Denote by $\mathbf{G}$ the associated graded ring $\mathrm{gr}_{(E)}(\mathbf{S})$.

Proposition

Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian integral domain of dimension d and let E be a torsionfree $\mathbf{R}$–module of rank r with a fixed embedding $E \hookrightarrow R^r$. Then

1. *The components of $\mathbf{G} = \oplus_{n\geq 0} E^n S/E^{n+1}$ have a natural grading*

$$G_n = E^n + E^n S_1/E^{n+1} + E^n S_2/E^{n+1} S_1 + \cdots .$$

2. *There is a decomposition $\mathbf{G} = \mathcal{R}(E) + H$, where $\mathcal{R}(E)$ is the Rees algebra of E and H is the R-torsion submodule of $\mathbf{G}$.*
3. *If $E \subset \mathfrak{m}\mathbf{R}^r$ and $\lambda(\mathbf{R}^r/E) < \infty$, $H = H^0_{\mathfrak{m}}(\mathbf{G})$ has dimension $d + r$ and multiplicity equal to the Buchsbaum-Rim multiplicity of E.*

Theorem

Let $E \subset \mathbf{R}^r$ *be a module as above and let* $(E) = E\mathbf{R}[t_1, \ldots, t_r] \subset \mathbf{S}$*. If* $\mathbf{G} = \mathrm{gr}_{(E)}(\mathbf{S})$ *then*

$$\mathrm{jdeg}\,(\mathbf{G}) = \mathrm{br}(E) + 1.$$

1. This recovers Buchsbaum-Rim multiplicities.
2. It also permits the definition of a global BR multiplicity.

Theorem

Let **R** *be a reduced quasi-unmixed ring and let E be a module of Briançon-Skoda number $c((E))$ (or as modified in the case of isolated singularities). Then*

$$\operatorname{jdeg}\,(\overline{\mathbf{S}[(E)t]}/\mathbf{S}[(E)t]) \leq c((E)) \cdot \operatorname{jdeg}\,(\operatorname{gr}_{(E)}(\mathbf{S})).$$

Summary

1. Do it all without Cohen-Macaulay baselines.
2. Target specific normalization processes.
3. The length of general normalization processes can be predicted for graded algebras over arbitrary fields.
4. In the arithmetic case the picture is less rosy. Even localization algorithms are lacking.
5. Several complementary notions of degree have promise in the development of a general theory.

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