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Corrections and Complements to

Computational Methods in Commutative Algebra and Algebraic Geometry

With Chapters by David Eisenbud, Daniel Grayson, Jürgen Herzog and Michael Stillman

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Preface

1. Fundamental Algorithms

CORRECTIONS

– p. 25, l. 7: Hilbrt $\to$ Hilbert

– Re Proposition 1.4.3: Francesco Piras has pointed out another algorithm, due to M. Janet, to find these decompositions: J. de Math. **8** (1920), 65–151; *Leçons sur les Systèmes d'Équations aux Dérivées Partielles*. Gauthier-Villars, Paris, 1929.

COMPLEMENTS

2. Toolkit

CORRECTIONS

- p. 25, l. 7: Hilbrt $\to$ Hilbert
- p. 31, l. 14: $I :< J >= \cup_{n\geq 1}(I :_R f^n) \to I :< f >= \cup_{n\geq 1}(I :_R f^n)$
- p. 32, in Exercise 2.1.1: I must be a homogeneous ideal
- p. 54, *Conjecture* 2.6.1: M. Kwieciński has shown ([229]) that if A is an affine normal domain over $\mathbb{C}$ and B is a finitely generated A–algebra, then B is A–flat if and only if *all* tensor powers $B^{\otimes n}$ are torsionfree over A. A stronger assertion, that only requires that $B^{\otimes \dim A}$ be torsionfree over A has been proved by Galligo and Kwieciński ([224]); it still requires that B is pure equidimensional.
- p. 57, l. 16:

$$\frac{h_r + h_{r-1}\mathbf{t} + \cdots + h_0\mathbf{t}^r}{(1-\mathbf{t})^d}. \to \frac{h_s + h_{s-1}\mathbf{t} + \cdots + h_0\mathbf{t}^s}{(1-\mathbf{t})^d}.$$

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2.1 Chevalley's Constructibility Theorem

Let k be a field or the ring of integers $\mathbb{Z}$. A *universal construction* over k is a homomorphism

$$\varphi : A = k[x_1, \ldots, x_n] \longrightarrow B = k[x_1, \ldots, x_n, T_1, \ldots, T_m]/I$$

such that the corresponding morphism

$${}^a\varphi : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$$

is surjective. In other words, φ has the *lying over* property.

In a simple illustration, consider the construction of the roots of an equation

$$x^2 + ax + b = 0$$

over a field k of characteristic $\neq 2$. It is effected by the homomorphism

$$\varphi : k[a, b] \longrightarrow k[a, b, u, v]/(u^2 - a^2 + 4b, 2v + a - u).$$

More typically, k is a field and one is interested in the fact that the restriction of ${}^a\varphi$ to the k–rational points of $\mathrm{Spec}(B)$ is a surjection on the corresponding set of points of $\mathrm{Spec}(A)$. In this case we say that φ is *k–rationally universal.*

Two typical examples of surjective morphisms are: (i) integral, injective maps—where the property is a consequence of the lemma of Nakayama; and (ii) faithfully flat morphisms. A simpler one is when the ideal I of relations is contained in the ideal $(T_1, \ldots, T_m)$.

The issue often arises in modifying an object defined by parameters $x_1, \ldots, x_n$, which must by brought to some canonical form, in a construction involving parameters $T_1, \ldots, T_m$; the relationships that the two sets of values bear is expressed by the ideal I—if possible.

Consider the problem of diagonalizing a symmetric $n \times n$ matrix over the field k of real numbers. Let $A = k[x_{ij}]$ be the ring of polynomials in $\binom{n+1}{2}$ variables parametrizing such matrices and let $B = k[x_{ij}, T_{k\ell}]$ be the ring of polynomials in n^2 additional variables. Let now I be the ideal generated by the entries of

$$\mathbb{T} \cdot \mathbb{T}^t - \mathbb{E},$$

and the off-diagonal entries of

$$\mathbb{T} \cdot \mathbb{X} \cdot \mathbb{T}^t.$$

The spectral theorem is the assertion that the corresponding homomorphism φ is k–rationally universal. (It will be a consequence of a result later in the note that φ is not universal.)

Unlike the first example, the second illustration is a rephrasing of an existence theorem coming to us without explicit formulas to solve for the $T_{k\ell}$ in terms of the x_{ij}.

Elementary Properties

Let k be a field and

$$\varphi : A = k[x_1, \ldots, x_n] \longrightarrow B = A[T_1, \ldots, T_m]/I$$

a ring homomorphism. Before we discuss some elementary properties of morphisms, it is worth noting that the lying over property of the homomorphism φ is can be phrased as

$$\mathfrak{p} = \varphi^{-1}(\varphi(\mathfrak{p})B), \quad \forall \mathfrak{p} \in \mathrm{Spec}(A).$$

In the case of interest here one has:

Proposition 2.1.1. *The homomorphism φ has lying over if and only if for every maximal ideal $\mathfrak{m}$ of A,*

$$(\mathfrak{m}, I) \neq (1).$$

Proof. For simplicity we write the equality above as $\mathfrak{m}B \cap A = \mathfrak{m}$. We must show the equality for all prime ideals. But this follows as (see [103]) every prime ideal of A is the intersection of the maximal ideals that contain it. □

Corollary 2.1.1. *Let $A \subset B \subset C$ be affine algebras over a field k. If the embedding $A \subset C$ has lying over then $A \subset B$ has lying over.*

This applies in case C is a polynomial ring over A and B is a finitely generated subalgebra.

Proposition 2.1.2. *Suppose k is a field and K is one of its algebraic extensions. Then φ has lying over if and only if*

$$\varphi' : A' = A \otimes_k K \longrightarrow B' = B \otimes_k K$$

has lying over.

Proof. We consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "\varphi'"] & B' \end{tikzcd}$$

where the vertical maps are the natural ones; since they are both integral homomorphisms they have lying over. It follows that φ will inherit the lying over property from φ'.

For the converse we use Proposition 2.1.1. Suppose $\mathfrak{m}'$ is a maximal ideal of A' and set $\mathfrak{m} = \mathfrak{m}' \cap A$. We have the corresponding diagram

$$\begin{tikzcd} A/\mathfrak{m} \arrow[r] \arrow[d] & B/\mathfrak{m}B \arrow[d] \\ A'/\mathfrak{m}A' \arrow[r] & B'/\mathfrak{m}B' \end{tikzcd}$$

where by assumption the top horizontal mapping is an embedding and therefore so will be the bottom mapping since it was obtained by a flat base change. This means that we also have an embedding

$$C = (A'/\mathfrak{m}A')_{\mathfrak{m}'} \hookrightarrow D = (B'/\mathfrak{m}B')_{\mathfrak{m}'},$$

where C is an Artinian local ring whose maximal ideal is $\mathfrak{m}'C$. This will imply that we cannot have $D = \mathfrak{m}'D$, as $\mathfrak{m}'C$ is nilpotent. □

Remark 2.1.1. Suppose B is a hypersurface ring, with I generated by the polynomial

$$f = \sum_{\alpha \neq 0} a_\alpha \mathbf{T}^\alpha + b, \quad a_\alpha, b \in A.$$

From (2.1.1) and the Nullstellensatz it follows that one has lying over if and only if

$$b \in \sqrt{(a_\alpha)}.$$

This example illustrates the difficulty one will encounter in verifying condition (2.1.1) in almost any other situation, such as when I is a complete intersection of codimension at least two.

Chevalley Theorem

For most geometric purposes in studying the image of a morphism it suffices to work with its closure. The set theoretic image is harder to describe explicitly. It is the assertion (see [125]):

Theorem 2.1.1. *Let A be a ring with Noetherian spectrum and let B be a finitely generated A–algebra*

$$\varphi : A \longrightarrow B.$$

Then ${}^a\varphi(\mathrm{Spec}(B))$ *is a constructive subset of* $\mathrm{Spec}(A)$*, that is,*

$${}^a\varphi(\mathrm{Spec}(B)) = \bigcup_{1 \leq i \leq r} (D_i \cap V_i),$$

where the D_i and the V_i are, respectively, open and closed subsets of $\mathrm{Spec}(A)$.

Proof. We give a proof to seek elements that are helpful to decide when the morphism is surjective.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal primes of A. The image of ${}^a\varphi$ is the union of the morphisms obtained by reduction mod $\mathfrak{p}_i$ for each minimal prime, so that we may assume that A is an integral domain. If $B = A[T_1, \ldots, T_m]/I$, pick a term order $>$ for the T_i and let

$$in_>(I) = (a_\alpha \mathbf{T}^\alpha, a_\alpha \in A),$$

be a generating set of initial ideal. (We point out that even when A is a ring of polynomials the elements a_α are not necessarily monomials as the orderings of A are mostly ignored for the purpose here.)

If we change base rings by tensoring with the field of fractions K of A, the same elements $a_\alpha \mathbf{T}^\alpha$ will generate an initial ideal of $IK \subset C = K[T_1, \ldots, T_m]$. Pick finitely many $a_{\alpha_j} \mathbf{T}^{\alpha_j}$ of the $a_\alpha \mathbf{T}^\alpha$ that generate $in_>(IK)$ and set

$$b = \prod_{1 \leq j \leq s} a_{\alpha_j}.$$

Note that B_b is a free A_b–module (see Section 2.6 for more details). It follows that

$$ {}^a\varphi(\mathrm{Spec}(B)) = D(b) \cup {}^a\varphi(\mathrm{Spec}(B/bB)). $$

The assertion of the theorem now uses that the corresponding sequence $(b_1 = b, \ldots, b_k)$ of ideals of A will stabilize at the unit ideal. $\square$

Corollary 2.1.2. *Let A be an integral domain and let $b \in A$ be such that B_b is A_b–free. The homomorphism $\varphi : A \to B$ has lying over if and only if $\overline{\varphi} : A/(b) \to B/bB$ has lying over and $bB \cap A \subset \sqrt{(b)}$.*

The second condition assures that B_b does not vanishes. It is also useful in some special cases such as when $A = k[x]$.

The difficulty in using this criterion through Gröbner bases computation is the following. In the first iteration of the generic flatness condition, when A is a ring of polynomials, the coefficients a_α can be found by grouping together the coefficients of $\mathbf{T}^\alpha$ in a Gröbner basis of I with respect to a term order involving all variables. Beyond this point, the minimal primes $\mathfrak{p}_i$ must be found (if $\dim A > 1$) and the computation is done in the rings $A/\mathfrak{p}_i$.

Remark 2.1.2. One approach to involve only computation over rings of polynomials over the same ground field k, that of Noether normalizations, only works partially. Let $\varphi : A \to B$ be a homomorphism of affine rings over a field k, and let C be a Noether normalization of A. If φ has lying over, its restriction to C will have lying over. The converse however does not hold. Consider the mappings

$$ \mathbb{Q}[x] \longrightarrow \mathbb{Q}[x, \sqrt{x}] = \mathbb{Q}[\sqrt{x}] \longrightarrow \mathbb{Q}[\sqrt{x}]_{1+\sqrt{x}}. $$

The only point of concern is the maximal ideal $(1 - x) \subset \mathbb{Q}[x]$ which extends to

$$ (1 - x)\mathbb{Q}[\sqrt{x}] = (1 - \sqrt{x})(1 + \sqrt{x})\mathbb{Q}[\sqrt{x}], $$

and will not blow up in the localization; thus the composite has lying over but not the second mapping.

Case Studies

Symmetric Matrices. Consider the morphism associated to the spectral theorem for real symmetric matrices. First observe that it is not a universal constrction. Indeed the complex matrix

$$ \begin{bmatrix} 2 & i \\ i & 0 \end{bmatrix} $$

has $(t - 1)^2$ for minimal polynomial, so will not be diagonalizable even using similarity by non-orthogonal matrices.

This argues against the possibility of 'automatic' algebraic diagonalization of real symmetric matrices as such a process might not distinguish complex data.

In addition, despite the neat set of defining equations, a calculation will show that despite the likelihood, the ring B, for $n = 2$, is not Cohen–Macaulay.

Product of Two Skew–Symmetric Matrices. The following was brought to our attention by Jon Sjogren, who provided all the references as well. Let A and B be $n \times n$ (n even) be skew-symmetric matrices. One is interested on the invariant factors of $C = A \cdot B$. When A is non–singular one may assume that

$$A = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

This means that

$$C = \begin{bmatrix} \alpha & \beta \\ \gamma & \alpha^t \end{bmatrix},$$

where β and γ are skew–symmetric and α^t is the transpose of α.

It is a result of [219] and [**?**] that such matrices (even when both A and B are singular) are similar to matrices made up of two blocks of a same matrix

$$D = \begin{bmatrix} \phi & 0 \\ 0 & \phi \end{bmatrix},$$

but the argument is not constructive. A constructive but partial 'diagonalization', using orthogonal matrices, given in [219] brings C to the form

$$\begin{bmatrix} \alpha & 0 \\ \gamma & \alpha^t \end{bmatrix}.$$

We could at this point set up a morphism precisely in the fashion used for the case of symmetric matrices and inquire about its universality. A more significant issue was raised by Sjogren: the extent to which the final 'diagonalization' can be realized by symbolic computation. As starters, one should consider the case $n = 4$:

$$C = \begin{bmatrix} \alpha & 0 \\ \begin{bmatrix} 0 & z \\ -z & 0 \end{bmatrix} & \alpha^t \end{bmatrix}$$

and solve for a matrix X such that $\det(X) = 1$ and

$$X \cdot C \cdot \operatorname{adjoint}(X) = D.$$

We believe if the corresponding morphism is not universal then it will not be possible to benefit from Gröbner basis theory to obtain the desirable automatic diagonalization.

2.2 Extra Tools

The torsion submodule of a module. Let $R = k[x_1, \ldots, x_n]$ be a ring of polynomials over the field k and E a finitely generated module defined over the integral domain $A = R/\mathfrak{p}$. We assume that E is given with a presentation over the ring R,

$$R^m \xrightarrow{\varphi} R^n \longrightarrow E \to 0,$$

and set ourselves the task of determining the torsion submodule of E as a module over A.

We use the Fitting ideals of E as an R–module. Let $F_i(E) = I_{n-i}(\varphi)$, $i \geq 0$, be these ideals. First, the rank of E as an A–module is r, if r is the smallest integer such that $F_r(E)$ is not contained in $\mathfrak{p}$. Pick $f \in F_r(E) \setminus \mathfrak{p}$, and let Z be the image of φ in R^n. Set

$$L = Z :_{R^n} f^\infty = \bigcup_{s \geq 1} (Z :_{R^n} f^s).$$

Then $T = L/Z$ is the torsion submodule of E. This is clear since T/Z is just the kernel of the localization mapping

$$R^n/Z = E \longrightarrow E_f,$$

and E_f is a projective module over A_f, and therefore it is a torsionfree module over A.

We note that the operations are standard computation commands in several of the computer algebra systems; their outputs are usually in the form of a presentation of T/Z.

3. Principles of Primary Decomposition

CORRECTIONS

- p. 75, proof of (b) should be
  First observe that the support of a module $\mathrm{Ext}^r_R(M, R)$ has codimension at least r, since R is a Gorenstein ring and therefore for each prime ideal $\mathfrak{p}$ with height $\mathfrak{p} < r$, the injective dimension of $R_{\mathfrak{p}} < r$. We claim that J_r is the intersection of the primary components of codimension r of the annihilator of $\mathrm{Ext}^r_R(R/I, R)$. On the other hand, if the support of $\mathrm{Ext}^r_R(R/I, R)$ has codimension $> r$ then $J_r = R$, because for the same reasons as above, the double Ext will vanish. More precisely, let E_0 be the subset of all elements of $E = R/I$ whose support has codimension $> r$; E_0 is a submodule of E and $\mathrm{Ass}(F = E/E_0)$ consists of primes of codimension $\leq r$. Consider the long exact sequence of the functor Ext:

$$\mathrm{Ext}^{r-1}_R(E_0, R) \longrightarrow \mathrm{Ext}^r_R(F, R) \longrightarrow \mathrm{Ext}^r_R(E, R) \longrightarrow \mathrm{Ext}^r_R(E_0, R).$$

  Since $\mathrm{Ext}^{r-1}_R(E_0, R) = \mathrm{Ext}^r_R(E_0, R) = 0$, $\mathrm{Ext}^r_R(F, R) \simeq \mathrm{Ext}^r_R(R/I, R)$. We already observed that $\mathrm{Ext}^r_R(F, R)$ has no associated primes of codimension $< r$, let us show that it does not have associated primes of codimension $> r$. Let f be an element of R regular on F:

$$0 \to F \xrightarrow{f} F \longrightarrow F/fF \to 0.$$

  In the cohomology sequence

$$\mathrm{Ext}^r_R(F/fF, R) \longrightarrow \mathrm{Ext}^r_R(F, R) \xrightarrow{f} \mathrm{Ext}^r_R(F, R) \longrightarrow \mathrm{Ext}^{r+1}_R(F/fF, R),$$

  $\mathrm{Ext}^r_R(F/fF, R) = 0$ since f is regular on F and therefore the support of F/fF has codimension $r+1$. This implies that f is regular on $\mathrm{Ext}^r(F, R)$, which from the fact that F has no associated prime of codimension $> r$ proves that $\mathrm{Ext}^r_R(F, R)$ is equidimensional (and similarly for the double Ext).
- p. 76, l. -5: b ases $\to$ bases
- p. 95: formula (3.17) should read

$$\mathfrak{P}^{(n)} = \mathfrak{P}^n \colon f^{\infty}.$$

COMPLEMENTS

4. Computing in Artin Algebras

CORRECTIONS

COMPLEMENTS

5. Nullstellensätze

CORRECTIONS

COMPLEMENTS

6. Integral Closure

CORRECTIONS

– p. 169, lines 6–10 should be replaced by

This means that $v = \dfrac{u}{\ell} + \alpha$, where α belongs to the convex hull of $v_1, \ldots, v_m$. The vector v can be written as (set $w = \frac{u}{\ell}$)

$$v = \lfloor w \rfloor + (w - \lfloor w \rfloor) + \alpha,$$

and it is clear that the integral vector

$$v_0 = (w - \lfloor w \rfloor) + \alpha$$

also has the property that $\mathbf{x}^{v_0} \in \overline{I}$.

A first bound results directly from this description of the integral closure:

Proposition 6.0.1. *Let I be a monomial ideal of $k[x_1, \ldots, x_n]$, generated by monomials of degree at most d. Then $\overline{I}$ is generated by monomials of degree at most $d + n - 1$.*

COMPLEMENTS

6.1 Chain Conditions and the Integral Closure

Several constructions in algebra consist of iterations of a basic procedure that modifies the object–an ideal or an algebra–to achieve improvement of one of its aspects. This is typical of smoothing operations such as the construction of the radical of ideals, of the integral closure of affine algebras or even of resolution of singularities. Usually different rounds of the application of the procedure represent different computational difficulties, as the degrees of the generators of the ideals or even the underlying ring of polynomials has changed. It is therefore extremely difficult to give bit count estimates for the algorithms. One might try to achieve other complexity counts by simply estimating the number of cycles the basic procedure will be employed, in other words to define a *unit of basic operations*, UBO for short, or more simply, given the difficulty each such unit represents, an *astronomical unit* (AU).

In this section we employ this perspective to give an analysis of the construction of the integral closure of algebras. Given the nature of the procedures employed–smoothing operations–as expected the counting will be expressed in terms of the multiplicity of the algebra. In fact, one of the motivations for Chapter 9 is the development of families of 'multiplicities' that might be used in the counting.

We describe here the approach to the analysis of the construction of the integral closure $\overline{A}$ of an affine ring A presented in [238]. It has the the following general properties:

- All calculations are carried out in the same ring of polynomials. If the ring A is not Gorenstein, it will require one Noether normalization.
- It uses the Jacobian ideal of A, or of an appropriate hypersurface subring, only theoretically to control the length of the chains of the extensions.
- There is an explicit quadratic bound on the multiplicity for the number of passes the basic operation has to be carried out. In particular, and surprisingly, the bound is independent of the dimension.
- It can make use of known properties of the ring A.

It will thus differ from the algorithms that uses iterated Jacobian ideals. At the same time we use a different counting of complexity, which with tongue-in-cheek, we called AUs (astronomical units). The primitive operation itself is based on elementary facts of the theory of Rees algebras. To enable the calculation we will introduce the notion of a *proper construction* and discuss instances of it.

Key to our discussion here is the elementary, but somehow surprising, observation that the set $\mathcal{S}_2(A)$ of extensions with the condition S_2 of Serre (see the next section) between an equidimensional, reduced affine algebra A and its integral closure $\overline{A}$ satisfy the *ascending* chain condition inherited from the finiteness of $\overline{A}$ over A, but the *descending* condition as well. Actually this is a phenomenon typical of other algebras as well.

The explanation requires the presence of a Gorenstein subalgebra $S \subset A$, over which A is birational and integral. If A is not Gorenstein, S is obtained from a Noether normalization of A and the theorem of the primitive element. One shows (Theorem 6.1.1) that there is a inclusion reversing one–one correspondence between $\mathcal{S}_2(A)$ and a subset of the set $\mathrm{Div}(\mathfrak{c})$ of divisorial ideals of S that contain the conductor $\mathfrak{c}$ of $\overline{A}$ relative to S. Since $\mathfrak{c}$ is not accessible one uses the Jacobian ideal as an approximation in order to determine the maximal length of chains of elements of $\mathcal{S}_2(A)$. In Corollary 6.1.3, it is shown that if A is a standard graded algebra over a field of characteristic zero, and of multiplicity e, then any chain in $\mathcal{S}_2(A)$ has at most $(e-1)^2$ elements (correspondingly, $e(e-1)^2$, in the non-homogeneous case).

In the next section we describe a process to create chains in $\mathcal{S}_2(A)$. Actually the focus is on processes that produce shorter chains in $\mathcal{S}_2(A)$ and its elements are amenable to computation. A standard construction in the cohomology of blowups provides chains of length at most $\left\lceil \frac{(e-1)^2}{2} \right\rceil$ (correspondingly, $\frac{e(e-1)^2}{2}$, in the non-homogeneous case). The possible computations are discussed in the last section. They take place in the ring S and deals entirely with ideals of S.

Divisorial Extensions of Gorenstein Rings

Let $A = k[x_1, \ldots, x_n]/I$ be a reduced equidimensional affine algebra over a field k of characteristic zero, let $R = k[x_1, \ldots, x_d] \subset A$ be a Noether normalization and let $S = k[x_1, \ldots, x_d, x_{d+1}]/(f)$ be a hypersurface ring such that the extension $S \subset A$ is birational. Denote by J the Jacobian ideal of S, that is the image in S of the ideal generated by the partial derivatives of the polynomial f.

From $S \subset A \subset \overline{S} = \overline{A}$ and [232] we have that J is contained in the conductor of $\overline{S}$. To fix the terminology, we denote the annihilator of the S–module A/S by $\mathfrak{c}(A/S)$. Note the identification $\mathfrak{c}(A/S) = \mathrm{Hom}_S(A, S)$.

We want to benefit from the fact that S is a Gorenstein ring, in particular that its divisorial ideals have a rich structure. Let us recall some of those. Denote by K the total ring of fractions of S. A finitely generated submodule L of K is said to be *divisorial* if it is faithful and the canonical mapping

$$L \mapsto \mathrm{Hom}_S(\mathrm{Hom}_S(L, S), S)$$

is an isomorphism. Since S is Gorenstein, for a proper ideal $L \subset S$ this simply means that all the primary components of L have codimension 1. We sum up some of these properties in (see [25] for general properties of Gorenstein rings and Section 6.3 for specific details on the S_2 condition of Serre):

Proposition 6.1.1. *Let S be a hypersurface ring as above (or more generally a Gorenstein ring). Then*

(a) *A finitely generated faithful submodule of K is divisorial if and only if it satisfies the S_2 condition of Serre.*
(b) *Let $A \subset B$ be finite birational extension of S such that A and B have the condition S_2 of Serre. Then $A = B$ if and only if $\mathfrak{c}(A/S) = \mathfrak{c}(B/S)$.*
(c) *If A is an algebra such that $S \subset A \subset \overline{S}$, then*

$$\mathrm{Hom}_S(\mathrm{Hom}_S(A, S), S) = \mathrm{Hom}_S(\mathfrak{c}(A/S), S)$$

is the S_2–closure of A.

Remark 6.1.1. We will also denote $A^{-1} = \mathrm{Hom}_S(A, S)$, and the S_2–closure of an algebra A by $\mathcal{C}(A) = (A^{-1})^{-1}$. Note that one has the equality of conductors $\mathfrak{c}(A/S) = \mathfrak{c}(\mathcal{C}(A)/S)$. Actually these same properties of the duality theory over Gorenstein rings hold if S is S_2 and is Gorenstein in codimension at most one.

We now define our two notions and begin exploring their relationship.

Definition 6.1.1. *Let I be an ideal containing regular elements of a Noetherian ring S. The* degree *of I is the integer*

$$\deg(I) = \sum_{\text{height } \mathfrak{p}=1} \ell((S/I)_{\mathfrak{p}}).$$

Definition 6.1.2. *A* proper operation *for the purpose of computing the integral closure of S is a method such that whenever $S \subset A \subsetneq \overline{S}$ are strict inclusions, the operation produces a divisorial extension B such that $A \subsetneq B \subset \overline{S}$.*

Theorem 6.1.1. *Let A be a Gorenstein ring with a finite integral closure $\overline{A}$. Let $\mathcal{S}_2(A)$ be the set of extensions $A \subset B \subset \overline{A}$ that satisfy the S_2 condition of Serre and denote by $\mathfrak{c}$ the conductor of $\overline{A}$ over A. There is an inclusion reversing one–one correspondence between that elements of $\mathcal{S}_2(A)$ and a subset of divisorial ideals of S containing $\mathfrak{c}$. In particular $\mathcal{S}_2(A)$ satisfies the descending chain condition.*

Proof. Any ascending chain of divisorial extensions

$$A \subsetneq A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_n \subset \overline{A}$$

gives rise to a descending chain of divisorial ideals

$$\mathfrak{c}(A_1/A) \supset \mathfrak{c}(A_2/A) \supset \cdots \supset \mathfrak{c}(A_n/A)$$

of the same length by Proposition 6.1.1(b). But each of these divisorial ideals contain $\mathfrak{c}$, which gives the assertion. $\square$

We sum up these properties in the following diagram. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the associated primes of $\mathfrak{c}$ and let $U = \bigcup_{i=1}^{n} \mathfrak{p}_i$. There is an embedding of partially ordered sets

$$\mathcal{S}_2(A)' \hookrightarrow \text{Ideals}(S_U/\mathfrak{c}_U),$$

where $\mathcal{S}_2(A)'$ is $\mathcal{S}_2(A)$ with the order reversed. Note that $S_U/\mathfrak{c}_U$ is an Artinian ring.

Of course the conductor ideal $\mathfrak{c}$ is usually not known in advance, or the ring A is not always Gorenstein. In case A is a reduced equidimensional affine algebra over a field of large characteristic we may replace it by a hypersurface subring S with integral closure $\overline{A}$. On the other hand, by [232] (see more general results in [113], [121]), the Jacobian ideal J of S is contained in the conductor of S. We get the less tight but more explicit rephrasing of Theorem 6.1.1:

Theorem 6.1.2. *Let S be a reduced hypersurface ring*

$$S = k[x_1, \ldots, x_{d+1}]/(f)$$

over a field of characteristic zero and let J be its Jacobian ideal. Then the integral closure of S can be obtained by carried out in at most $\deg(J)$ *proper operations on S.*

In characteristic zero, the relationship between the Jacobian ideal J of S and the conductor $\mathfrak{c}$ of $\overline{S}$ is difficult to express in detail. In any event the two ideals have the same associated primes of codimension one, a condition we can write as the equality of radical ideals $\sqrt{\mathfrak{c}} = \sqrt{(J^{-1})^{-1}}$.

Main results

Theorem 6.1.3. *Let k be a field of characteristic zero and consider the reduced hypersurface ring*

$$S = k[x_1, \ldots, x_d, x_{d+1}]/(f), \quad \deg f = e.$$

Then

$$\ell(\mathrm{Quot}(S/\gamma(\overline{S}))) \leq \begin{cases} (e-1)^2 & f \text{ is homogeneous} \\ e(e-1)^2 & \text{otherwise} \end{cases}$$

Let us highlight the boundedness of chains of divisorial subalgebras.

Corollary 6.1.1. *Let A be a reduced equidimensional standard graded algebra over a field of characteristic zero, and set $e = \deg(A)$. Then any sequence*

$$A = A_1 \subset A_2 \subset \cdots \subset A_n \subset \overline{A}$$

of finite extensions of A with the property S_2 of Serre has length at most $(e-1)^2$ ($e(e-1)^2$ in the non-homogeneous case).

A Rees Algebra Approach

In this section we introduce and analyze in detail one proper operation which does not use the Jacobian ideals. The reason for this move lies with the estimates of [192] which predict non–elementary embedding dimensions for the integral closure of A.

Proposition 6.1.2. *There are proper operations whose iteration leads to the integral closure of S.*

Proof. We assume that we have $S \subsetneq A \subset \overline{S}$ and now we introduce a construction of a divisorial algebra B that enlarges A whenever $A \neq \overline{S}$. Denote $I = \mathfrak{c}(A/S)$. More strictly we shall first find a finite, proper extension and then take its divisorial closure.

A natural choice is:

$$B = \bigcup_{n \geq 1} \mathrm{Hom}_S(I^n, I^n).$$

In other words, if $R[It]$ is the Rees algebra of the ideal I, $X = \mathrm{Proj}(R[It])$, then $B = H^0(X, O_X)$. To show that B is properly larger than A it suffices that exhibit this at one of the associated prime ideals of I. Let $\mathfrak{p}$ be a minimal prime of I for which $S_\mathfrak{p} \neq A_\mathfrak{p}$. If $I_\mathfrak{p}$ is principal we have that $I_\mathfrak{p} = tS_\mathfrak{p}$ and therefore $S_\mathfrak{p} = A_\mathfrak{p}$, which is a contradiction. This means that the ideal $I_\mathfrak{p}$ has a minimal reduction of reduction number at least one, that is, $I_\mathfrak{p}^{r+1} = uI_\mathfrak{p}^r$ for some $r \geq 1$. This equation shows that

$$I_\mathfrak{p}^r u^{-r} \subset B_\mathfrak{p},$$

which gives the desired contradiction since $I_\mathfrak{p}^r u^{-r}$ contains $A_\mathfrak{p}$ properly as $I_\mathfrak{p}$ is not principal.

We observe that a bound for the integer r can be found as well: Let $g \in J$ be a form of degree $e-1$ which together with f form a regular sequence of $R = k[x_1, \ldots, x_{d+1}]$. According to Corollary 9.5.3, the reduction number of the ideal $I_{\mathfrak{p}}$ is bounded by the Hilbert–Samuel multiplicity $e(I_{\mathfrak{p}})$ of $I_{\mathfrak{p}}$ minus one. In a manner similar as we have argued earlier,

$$e(I_{\mathfrak{p}}) \leq e((g_{\mathfrak{p}})) \leq e(e-1).$$

By taking divisorial closures, we have

$$\mathcal{C}(B) = \mathcal{C}(\mathrm{Hom}_S(I^r, I^r))$$

for $r < e(e-1)$, as desired. □

Another choice is the variation obtained by considering the quotient $\bigcup_{m\geq 1} I^{m+n} :_S I^m$ for choices of m up to $e(e-1)$.

The proper operations above lead to a 'fast' approach (i.e. beats the theoretical limit by a factor of at least 2) to the integral closure according to the following observation:

Proposition 6.1.3. *Let $S \subset A$ be a divisorial extension and let $I = \mathfrak{c}(A/S)$. If B is the divisorial closure of $\bigcup_{n\geq 1} \mathrm{Hom}_S(I^n, I^n)$, then*

$$\deg(\mathfrak{c}(A/S)) \geq \deg(\mathfrak{c}(B/S)) + 2.$$

In particular, for any standard graded algebra of multiplicity e, the number of terms in any chain of algebras obtained in this manner will have at most $\left\lceil \frac{(e-1)^2}{2} \right\rceil$ divisorial extensions.

Proof. Suppose that the degrees of the conductors of A and B differ by 1. This means that the two algebras agree at all localizations of S at height 1 primes, except at $R = S_{\mathfrak{p}}$, and that $\ell((B/A)_{\mathfrak{p}}) = 1$. We are going to show that for the given choice of how B is built that is a contradiction.

We localize S, A, B, I at $\mathfrak{p}$ but keep a simpler notation $S = S_{\mathfrak{p}}$, etc. Let (u) be a minimal reduction of I, $I^{r+1} = uI^r$. We know that $r \geq 1$ since I is not a principal ideal, as $A \neq B$. We note that $A \subset Iu^{-1} \subset B$. Since B/A is a simple S–module, we have that either $A = Iu^{-1}$ or $B = Iu^{-1}$. We can readily rule out the first possibility. The other leads to the equality

$$Iu^{-1} \cdot Iu^{-1} = Iu^{-1},$$

since B is an algebra. But this implies that

$$Iu^{-1} \cdot I \subset I \subset S,$$

in other words, that $Iu^{-1} \subset \mathrm{Hom}_S(I, S) = A$. □

Remark 6.1.2. In case A is a non–homogeneous algebra, its cubic estimate must be considered.

Computational Notes

- To begin, what is a nice way to carry out the calculation of $\mathfrak{c}(A/S)$,

$$S = k[x_1, \ldots, x_d, x_{d+1}]/(f) \hookrightarrow k[x_1, \ldots, x_n]/I = A?$$

One approach is the following. Let g be a nonzero element in the Jacobian ideal of S and set $L = gA \subset S$. Note that

$$\mathfrak{c}(A/S) = g \cdot \mathrm{Hom}_S(gA, S) = Sg :_S L.$$

For this we need gA expressed as an ideal of S. This is an elimination question, for example gA is the image in S of the ideal

$$g, I) \bigcap k[x_1, \ldots, x_{d+1}].$$

If we want to track closely the computation, pick an elimination term order for the x_i such that $x_i > x_{i+1}$ (such ordering as might have been required earlier to achieve the Noether normalization). Let $\mathcal{G}_>(I)$ be a Gröbner basis of I and let $NF(\cdot)$ be the corresponding normal form function. Then $gA \subset S$ is generated by all

$$NF(g\mathbf{x}^\alpha), \quad \mathbf{x}^\alpha = x_{d+2}^{\alpha_{d+2}} \cdots x_n^{\alpha_n}, \quad \text{where } \alpha_i < e.$$

- From the point we have the ideal L, the computations are always with ideals of S, until we reach the integral closure. That is, given an extension $S \subset A$, we assume that A is represented as a fractionary ideal, $A = Lx^{-1}$, where $L \subset S$ and x is a regular element of S. The closure of A is

$$\mathcal{C}(A) = (L^{-1})^{-1}x^{-1}$$

while its conductor is

$$\mathfrak{c} = Sx :_S L.$$

Note that the conductors of A and of $\mathcal{C}(A)$ are the same. This means that the closure operation is being used to control the chain of extension but it is only required at the last step of the computation.
- The hard part is the calculation of $B = H^0(X, O_X)$, as indicated above. We don't know an elegant way of doing it.
- Note that the test of termination is given by the equality $A = B$. If reached, say $\overline{S} = Lx^{-1} = (a_1, \ldots, a_n)x^{-1}$, a set up for a presentation of $\overline{S}$ can go as follows. Consider the homomorphism

$$k[x_1, \ldots, x_{d+1}, T_1, \ldots, T_n] \xrightarrow{\varphi} Lx^{-1} \subset Sx^{-1} \subset S[x^{-1}],$$

where T_i is mapped to $a_i x^{-1}$, $i = 1, \ldots, n$. If we use the presentation

$$S[x^{-1}] = k[x_1, \ldots, x_d, x_{d+1}, u]/(f, u\tilde{x} - 1),$$

where u is a new indeterminate and $\tilde{x}$ is a lift of x, the defining ideal of $\overline{S}$ is then

$$\ker(\varphi) = (f, u\tilde{x} - 1, T_i - \widetilde{a_i}u, i = 1 \ldots n) \bigcap k[x_1, \ldots, x_{d+1}, T_1, \ldots, T_n].$$

Broadly speaking, an algorithm to compute the integral closure of the algebra A consists iterations of an operation $\mathcal{P}$ on the extensions $A \subset B \subset \overline{A}$ with the properties:

(a) $B \subset \mathcal{P}(B) \subset \overline{A}$;
(b) If $B \neq \overline{A}$ then $\mathcal{P}(B) \neq B$.

For point of reference, the operations used in [199] and [**?**] are

(i) $\mathcal{P}(B) = \mathrm{Hom}_B(\mathrm{Hom}_B(J, B), \mathrm{Hom}_B(J, B))$
(ii) $\mathcal{P}(B) = \mathrm{Hom}_B(\sqrt{J}, \sqrt{J})$,

respectively, where J is the Jacobian ideal of the algebra B. Both processes are guaranteed to produce proper extensions of B if $B \neq \overline{A}$.

Unlike the operation defined earlier, each application of $\mathcal{P}$ takes place in a different ring, as the new Jacobian ideal has to be assembled from a presentation of the algebra $\mathcal{P}(B)$.

Remark 6.1.3. It is worth noting that the first of these Jacobian basic operations has the property that if A has the S_2–condition of Serre, then all $\mathcal{P}^i(A)$ also have S_2. This will put the same limits on the order of $\mathcal{P}$ that were developed in this section. What it is not clear is if the general quadratic (or cubic) bounds are still required, or whether a much lower one can be obtained.

In the case of (ii) it would be interesting to find bounds for the 'order' of the basic operation. There a minor modification of (ii) that makes it amenable to the bounds discussed. Let us assume that the algebra A satisfies the condition S_2 and let J be its Jacobian ideal (assume that the characteristic is either zero or does not place any obstruction). The radical of J is made up (possibly) of

$$\sqrt{J} = K \cap L,$$

where K has codimension one and L has codimension at least two (if $\sqrt{J} = L$, A is normal by the Jacobian criterion). Observe that

$$\mathrm{Hom}_A(K \cap L, K \cap L) \subset \mathrm{Hom}_A(K, K).$$

Indeed, for any element $f \in \mathrm{Hom}_A(K \cap L, K \cap L)$, we have

$$L \cdot f \cdot K = f \cdot (KL) \subset f(K \cap L) \subset K \cap L \subset K.$$

Since K is a divisorial ideal and L has grade at least two, $f \cdot K \subset K$, as desired (we have used that A has the condition S_2 of Serre, which it passes to all its divisorial ideals).

There is a payoff: the computation of $\sqrt{J}$ often requires the computation of K as a preliminary step. The observation above states that $\mathrm{Hom}_A(K, K)$ is a possibly larger ring than $\mathrm{Hom}_A(\sqrt{J}, \sqrt{J})$, and requires fewer steps.

Let us indicate the calculation of K in the case of a hypersurface. Suppose $A = k[x_1, \ldots, x_{d+1}]/(f)$, and assume that the characteristic is greater than deg f

(we are assuming that f is a square-free polynomial). Let g be a partial derivative of f for which $\gcd(f, g) = 1$. Let

$$J = (f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{d+1}}),$$

and let J_0 be the Jacobian ideal of (f, g). Then K is given (see Theorem 5.4.3) by the image in A of the ideal of $k[x_1, \ldots, x_{d+1}]$

$$((f, g) : J_0) : (((f, g) : J_0) : J).$$

Example 6.1.1. Let us illustrate with one example how the two methods differ markedly in the presence of the condition R_1 of Serre. Let A be an affine domain over a field of characteristic zero, and let J be its Jacobian ideal. Suppose $\operatorname{height} J \geq 2$ (the R_1 condition). As we have discussed,

$$\overline{A} = J^{-1} \subset \operatorname{Hom}_A(J^{-1}, J^{-1}).$$

Consider now the following example. Let n be a positive integer, and set

$$A = \mathbb{R}[x, y] + (x, y)^n \mathbb{C}[x, y],$$

whose integral closure is the ring of polynomials $\mathbb{C}[x, y]$. Note that $A_x = \mathbb{C}[x, y]_x$ and $A_y = \mathbb{C}[x, y]_y$. This means that $\sqrt{J}$ is the maximal ideal

$$M = (x, y)\mathbb{R}[x, y] + (x, y)^n \mathbb{C}[x, y].$$

It is clear that

$$\operatorname{Hom}_A(M, M) = \mathbb{R}[x, y] + (x, y)^{n-1} \mathbb{C}[x, y].$$

It will take precisely n passes of the operation to produce the integral closure. In particular, neither the dimension ($d = 2$), nor the multiplicity ($e = 1$) play any role.

6.2 NORMALIZ

by **Winfried Bruns and Robert Koch**

The main purpose of NORMALIZ ([220]) is the computation of the integral closure of an affine subsemigroup S in a lattice L. This means that S is a finitely generated subsemigroup S of L, and the integral closure of S in L is

$$\bar{S}_L = \{x \in L \mid mx \in S \text{ for some } m \in \mathbb{Z},\ m > 0\}.$$

This terminology is consistent with its use in commutative algebra where S is usually the semigroup of exponent vectors of the monomials in a monomial subalgebra of a Laurent polynomial ring $K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$.

The input of NORMALIZ is a file containing a set E of generators of S embedded into $\mathbb{Z}^n$, and in the present implementation one has the choices $L = \mathbb{Z}^n$ or $L = \mathrm{gp}(S)$, the subgroup of $\mathbb{Z}^n$ generated by S. In the latter case, $\bar{S}_L$ is just the normalization of S. NORMALIZ requires that S is positive, i. e. contains no vector $x \neq 0$ such that $-x \in S$.

NORMALIZ has input options for the following related tasks:

1. the computation of the normalization of the Rees algebra of a monomial ideal $I \subset K[X_1, \dots, X_n]$ and the integral closure of I,
2. the computation of the Ehrhart ring of a lattice polytope.

The output written to a file (or several files) consists of

1. the uniquely determined minimal set of generators of $\bar{S}_L$ embedded into $\mathbb{Z}^n$,
2. the linear forms defining the support hyperplanes of the cone $C(S)$ generated by S in $\mathbb{R}^n$, provided $\mathrm{rank}(S) = n$,
3. (optionally) the h-vector, Hilbert polynomial, and multiplicity of $\bar{S}_L$ if E is homogeneous with respect to L (see below),
4. (optionally) extra output files holding the output data separately,
5. additional data if the above mentioned input options are used.

We say that E is homogeneous (with respect to L) if there exists an integral linear form ϕ on L with $\phi(x) = 1$ for all $x \in E$.

In future versions it will be possible to define a semigroup by relations or (a necessarily integrally closed semigroup) as the intersection of a lattice and a cone.

Algorithms

NORMALIZ uses simplicial methods. It first embeds S into the smallest direct summand of L containing S. Replacing L by it, we may assume that $L = \mathbb{Z}^r$ where $r = \mathrm{rank}\,\mathrm{gp}(S)$. Then $\bar{S}_L = C(S) \cap \mathbb{Z}^r$ where $C(S) = \mathbb{R}_+ S$ is the polyhedral subcone of $\mathbb{R}^r$ generated by S. It has an apex in the origin since S is positive.

The second step is the computation of the supüport hyperplanes of $C(S)$, combined with the computation of a triangulation of $C(S)$ into simplicial subcones Σ_i, each of which is generated by a linearly independent subset of r elements of E.

In order to determine a system of generators of $\bar{S}_L$, it is enough to find a system of generators for each of the subsemigroups $T_i = \Sigma_i \cap L$, and to take their union. Each generator found is tested for reducibility against the ones already known and vice versa, so that the final result is the set of irreducible elements of $\bar{S}_L$, as desired.

The triangulation therefore reduces the problem to the simplicial case, in which S is generated by r linearly independent vectors $e_1, \dots, e_r$. A system of generators of $\bar{S}_L$ is then given by the union of $E = \{e_1, \dots, e_r\}$ and the set Q consisting of all vectors $q_1 e_1 + \dots + q_r e_r$ with $q_i \in \mathbb{Q}$, $0 \leq q_i < 1$. Note that Q is in 1–1–correspondence with the elements of $L/\mathrm{gp}(S)$. Roughly speaking, one can therefore find Q by the elementary divisor algorithm and division with remainder.

For the Hilbert function one has to consider the decomposition of $C(S)$ into the disjoint union of the relative interiors of all faces of $C(S)$ (and not just the maximal ones). Therefore the computation of the Hilbert function takes more time than the computation of $\bar{S}_L$ itself.

The documentation file `normaliz.tex` contains a detailed description of the algorithms.

Distribution

NORMALIZ is available by anonymous `ftp` from

`ftp.mathematik.Uni-Osnabrueck.DE/pub/osm/kommalg/software/`

as a `zip` file. It contains the documentation `normaliz.tex`, examples, the source code, and executables for DOS/Windows and SUN Sparc.

There are two versions: `normaliz` using 32 bit integer arithmetic and `enormalz` with infinite precision. The source is a `C++` program written for `g++`.

7. Ideal Transforms and Rings of Invariants

CORRECTIONS

- p. 183: both lines 13 and 14 should be replaced simply by

$$\text{height}(\text{ann}(\text{Ext}^i_B(A, B))) \geq r + i, \; i > g.$$

- p. 199, l. -8: for all $\lambda^{(j)}_{nu} \neq 0 \to$ for all $\lambda^{(j)}_{\nu} \neq 0$
- p. 200, l. 13: $in(R[It]) \to in(R[IT])$
- p. 200, l. 14: $x_1, \ldots, x_n, \; f_1T, \ldots, \; f_kT \to x_1, \ldots, x_n, \; f_1T, \ldots, \; f_mT$
- p. 201, l. 10: *rings of invariants* → *ring of invariants*
- Add to p. 203, l. -6: In fact, according to [227], $a(R^G) \leq -\dim R^G = -d$. In particular,

$$e_j \leq \sum_{1 \leq i \leq d} d_i - d, \quad \forall j,$$

- p. 208: Bernd Sturmfels has solved affirmatively Conjecture 7.4.1 for abelian matrix rings (see [236]).

COMPLEMENTS

8. Computation of Cohomology

CORRECTIONS

COMPLEMENTS

9. Degrees of Complexity of a Graded Module

CORRECTIONS

– p. 231, l. 11: let M be a graded algebra $\rightarrow$ let M be a graded module
– p. 262: *Conjecture 9.4.3* has been affirmed independently by A. Conca [221] and N. V. Trung [237].
– p. 262, l. 4: case instance it holds $\rightarrow$ instance

COMPLEMENTS

A. A Primer on Commutative Algebra

CORRECTIONS

- p. 286, l. 11: there exists $h \in R$ satisfying $\rightarrow$ there exists $h \in A$ satisfying
- p. 286, l. -9: $0 : (0 : hA) = R \rightarrow 0 : (0 : hA) = A$
- p. 287, l. 2: hear $\rightarrow$ here
- p. 301, l. -11: should be

$$\operatorname{proj\,dim}_{R[x]} M \leq 1 + \operatorname{proj\,dim}_{R[x]} (R[x] \otimes_R M) \leq 1 + \operatorname{proj\,dim}_R M,$$

- p. 326 In Corollary A.4.8 should be
  With R as above, let M be a finitely generated reflexive R–module. Then M is Cohen–Macaulay if and only if for each prime ideal $\mathfrak{p}$ of codimension ≥ 2

$$\operatorname{depth} M_{\mathfrak{p}} + \operatorname{depth} \operatorname{Hom}_R(M, R)_{\mathfrak{p}} \geq \dim R_{\mathfrak{p}} + 2.$$

COMPLEMENTS

B. Hilbert Functions

CORRECTIONS

– p. 339 l. 10: (i) $S/t \to$ (i) $S/(t)$

COMPLEMENTS

C. Using *Macaulay 2*

CORRECTIONS

COMPLEMENTS

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