Mathematische Zeitschrift manuscript No.
(will be inserted by the editor)

Complexity of the Normalization of Algebras

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Received: date / Revised version: date –

Abstract We introduce and develop new techniques to study the complexity of normalization processes of graded algebras. The construction of a new degree function on graded modules, with a global nature, permits a broad extension of recent bounds for the length of the chains of subalgebras that general algorithms must transverse to build the integral closure, particularly of blowup algebras. It achieves this by relating the values of the new degree with invariants of the algebra known *ab initio*. As a by-product, it reveals new inequalities among Hilbert coefficients.

Mathematics Subject Classification (2000): Primary 13A30, 13B21, 13D40; Secondary 13H10, 13H15

1 Introduction

Let R be a normal, unmixed integral domain and let A be a semistandard graded R-algebra of integral closure $\overline{A}$. The terminology implies that locally R is analytically unramified and that A is a finite integral extension of a graded R-algebra S generated by its component of degree 1. Estimating the number of steps that general algorithms must take to build $\overline{A}$ can be viewed as an invariant of A. This is the perspective of [3], [6], [10], [12] and [14], where for several classes of algebras, noteworthy being affine graded algebras over fields and Rees algebras of ideals and modules, multiplicity dependent bounds were derived in each of these cases. A side effect of these calculations (in this regard, see especially [3] and [10]) is the derivation of various bounds on Hilbert coefficients of ideals and modules, which in no direct way involves normalization but whose intervention was required nevertheless.

* The second author was partially supported by the NSF

Our general goal is to extend the range of these results to very general graded algebras. It will come at a cost in some specificity but achieves two gains beyond naturality: Successful calculations can be moved across domains and new problem areas (to be discussed later) are opened up. Let us indicate overall targets for our investigation:

(i) Numerical indices for $\overline{A} = \sum_{n\geq 0}(\overline{A})_n$: e.g. Find r such that

$$(\overline{A})_{n+r} = (A)_n \cdot (\overline{A})_r, \quad n \geq 0.$$

(ii) How many "steps" are there between A and $\overline{A}$,

$$A = A_0 \subset A_1 \subset \cdots \subset A_{s-1} \subset A_s = \overline{A},$$

where the A_i are constructed by an effective process?
(iii) Express r and s in terms of invariants of A.
(iv) Generators of $\overline{A}$: Number and distribution of their degrees.

One of the tools we required was a mechanism to track the normalization process. We chose to use multiplicity theory for its various opportunities for calculation. Let R be a Noetherian ring, A be a finitely generated graded R-algebra where $A = R[A_1]$ and let M be a graded A-module. There are several ways to attach numerical data to M seeking to account for its behavior at its associated primes relative to A and/or to R. These numbers seek to capture the size of M. They usually go by the name of *degrees*.

Here we will assign to every finitely generated graded A-module M a new multiplicity, namely jdeg(M). It is based on the notion of the **j**-multiplicity $j_{\mathfrak{m}}(M)$ of M at a maximal ideal $\mathfrak{m} \subset R$ of [5], which by localization extends to all prime ideals. This integer coincides with the classical multiplicity $\deg(M)$ when R is an Artinian local ring. The new degree is defined by

$$\text{jdeg}(M) = \sum_{\mathfrak{p}\in\text{Spec } R} j_{\mathfrak{p}}(M).$$

It captures various aspects of M besides its sheer size usually expressed in the ordinary multiplicity $\deg(M)$. In contrast to other extensions of $\deg(M)$, such as the *arithmetic degree* arith-deg(M) or the *geometric degree* gdeg(M), that require that R be a local ring, jdeg(M) places no such restrictions on R, it has a truly global character.

One of the properties jdeg addresses our main issue. Let R be a Noetherian domain and A a semistandard graded R-algebra. Consider a sequence of integral graded extensions

$$A \subseteq A_0 \to A_1 \to A_2 \to \cdots \to A_n = \overline{A},$$

where the A_i satisfy the S_2 condition of Serre. Then $n \leq$ jdeg$(\overline{A}/A)$ (Theorem 2).

The main goal becomes to express jdeg$(\overline{A}/A)$ in terms of *priori* data, that is, one wants to view this number as an invariant of A. Relating it to other, more accessible, invariants of A would bring considerable predictive power to the function jdeg. This is the point of view of [10] and [6], when jdeg is the Hilbert-Samuel multiplicity or the Buchsbaum-Rim multiplicity, respectively.

There are two classes of algebras treated. In the case of Rees algebras of ideals, we achieve just that in case R is a reduced quasi-unmixed ring and I is an ideal of Briançon-Skoda number $c(I)$. Then

$$\text{jdeg}(\overline{R[It]}/R[It]) \leq c(I)\cdot \text{jdeg}(\text{gr}_I(R))$$

(Theorem 3), which extends the equimultiple case of [10, Theorem 4.3] to general ideals. A very effective assertion holds in case R has isolated singularities. Thus in Theorem 5, if $(R,\mathfrak{m})$ is essentially of finite type over a perfect field and its Jacobian ideal L has finite co-length and $\ell(I)=\dim R$, then

$$\text{jdeg}(\overline{R[It]}/R[It]) \leq (\ell(I)+\lambda(R/L)-1)\cdot \text{jdeg}(\text{gr}_I(R)).$$

A module theoretic version (Theorem 6) has several surprising consequences. Simultaneously it provides a new interpretation and an extension of Buchsbaum-Rim multiplicities.

Another application is to graded algebras A admitting a Noether normalization, that is algebras which are finite over a polynomial subalgebra S. This is a restriction for arbitrary base rings R but valid for all $\mathbb{Z}$-algebras. The Noether normalization allows for the construction of a determinantal divisor on a pair $A\subset B$ of graded algebras, to which a degree $\text{tn}(A,B)$ is attached by the formula

$$\text{tn}(A,B)=\deg(\det_S(A) :_S \det_S(B)).$$

When R is a field of arbitrary characteristic, [3] introduced a numerical tag $\text{tn}(A)$ associated to each graded algebra derived from Chern numbers. In this case, $\text{tn}(A,B) = \text{tn}(A)-\text{tn}(B)$. (When $R=\mathbb{Z}$ the existence of a function $\text{tn}(A)$ is unknown.) Whenever the algebras A and B satisfy the condition S_2 of Serre, Theorem 8 gives the equality

$$\text{jdeg}(B/A)=\text{tn}(A,B).$$

This is quite interesting given that the functions $\text{jdeg}(\cdot)$ and $\text{tn}(\cdot,\cdot)$ have very distinctive make ups.

2 Construction of jdeg

To construct and develop our notion of degree, we will use extensively the notion of **j**–multiplicity introduced and developed by Flenner, O'Carroll and Vogel ([5]). It is employed in [9] to develop the notion of jdeg. Here we will only recall the construction jdeg and two of its properties that are required in our examination of normalization.

Let $(R,\mathfrak{m})$ be a Noetherian local ring and A be a finitely generated graded R-algebra where $A=R[A_1]$. For a finitely generated graded A-module M, $H^0_{\mathfrak{m}}(M)$ is a graded submodule of M which is annihilated by a sufficiently large power of $\mathfrak{m}$. Therefore, $H^0_{\mathfrak{m}}(M)$ can be considered as a graded module over the Artinian ring $A/\mathfrak{m}^k A$ for some $k \gg 0$. We will make use of its Hilbert polynomial to define a new multiplicity function of M.

Definition 1. Consider an integer d such that $d \geq \dim M$. We set

$$j_d(M) = \begin{cases} 0 & \text{if } \dim H^0_{\mathfrak{m}}(M) < d, \\ \deg(H^0_{\mathfrak{m}}(M)) & \text{if } \dim H^0_{\mathfrak{m}}(M) = d. \end{cases}$$

Moreover, we set $j(M) = j_{\dim M}(M)$, and call it the j-multiplicity of M.

Remark 1. If R is an Artinian local ring, then $j(M)$ is the ordinary multiplicity $\deg(M)$.

Let $(R, \mathfrak{m})$ be a Noetherian local ring and A be a finitely generated graded R-algebra where $A = R[A_1]$. Let M be a finitely generated graded A-module. Recall that the *analytic spread* $\ell(M)$ of M is the Krull dimension of $M/\mathfrak{m}M$ as an $A/\mathfrak{m}A$-module.

Lemma 1. *With the above notations,* $\ell(M) = \dim(M/\mathfrak{m}^n M)$ *for any* n.

Proof. It is enough to show that $\dim M/\mathfrak{m}M = \dim M/\mathfrak{m}^n M$ for any $n \geq 1$. Suppose that $\mathfrak{m}^n = (a_1, \ldots, a_r)$, the surjection $M^{\oplus r}/\mathfrak{m}M^{\oplus r} \to \mathfrak{m}^n M/\mathfrak{m}^{n+1}M$ given by sending $(m_1, \ldots, m_r) \longmapsto \sum_{k=1}^r a_k r_k$ shows $\dim \mathfrak{m}^n M/\mathfrak{m}^{n+1}M \leq \dim M^{\oplus r}/\mathfrak{m}M^{\oplus r} = \dim M/\mathfrak{m}M$. Observe that the exact sequence

$$0 \to \mathfrak{m}^{n-1}M/\mathfrak{m}^n M \to M/\mathfrak{m}^n M \to M/\mathfrak{m}^{n-1}M \to 0$$

implies $\dim M/\mathfrak{m}^n M = \max\{\mathrm{dim\,\mathfrak{m}^{n-1}M/\mathfrak{m}^nM, dim\,M/\mathfrak{m}^{n-1}M}\}$. The conclusion then follows immediately by induction. □

Proposition 1. *The equality* $\dim H^0_{\mathfrak{m}}(M) = \dim M$ *holds iff* $\ell(M) = \dim(M)$.

Proof. Consider the short exact sequence

$$0 \to H^0_{\mathfrak{m}}(M) = H \to M \to C \to 0.$$

We know that $\mathfrak{m}^n H = 0$ for $n \gg 0$, so we have the following exact sequence:

$$0 \to H \cap \mathfrak{m}^n M \to H \to M/\mathfrak{m}^n M = M' \to C/\mathfrak{m}^n C = C' \to 0.$$

It follows from the Artin-Rees lemma that $H \cap \mathfrak{m}^n M = \mathfrak{m}^{n-r}(H \cap \mathfrak{m}^r M) = 0$ for some $r \gg 0$ and $n > r$. The sequence above becomes

$$0 \to H \to M/\mathfrak{m}^n M = M' \to C/\mathfrak{m}^n C = C' \to 0$$

Note that $\dim M' = \ell(M)$ by the previous lemma and $\dim H \leq \dim M' \leq \dim M$, so if $\dim H = \dim M$ then $\ell(M) = \dim M$.

Conversely, if $\ell(M) = \dim M$ then $\dim M' = \ell(M) = \dim M \geq \dim C > \dim C'$. It then shows $\dim H = \dim M'$. □

This notion has several properties that contrast to $\deg(\cdot)$, while others are similar. First we recall that this **j**-multiplicity behaves well with respect to short exact sequences (see [5, Proposition 6.1.2]).

Proposition 2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and A be a finitely generated graded R-algebra where $A = R[A_1]$. Assume that*

$$0 \to M_0 \to M_1 \to M_2 \to 0$$

is an exact sequence of finitely generated graded A-modules. Then for $d \geq \dim M_1$

$$j_d(M_1) = j_d(M_0) + j_d(M_1).$$

In particular, if $\dim M_0 = \dim M_1 = \dim M_2$ then

$$j(M_1) = j(M_0) + j(M_1).$$

Next we introduce the **j**-multiplicity at a given prime $\mathfrak{p}$ of R.

Definition 2. Let R be a Noetherian ring and A be a finitely generated graded R-algebra where $A = R[A_1]$. For a finitely generated graded A-module M, a prime ideal $\mathfrak{p}$ of R, the j-multiplicity of M at $\mathfrak{p}$ is the integer

$$j_{\mathfrak{p}}(M) = \begin{cases} \deg(H^0_{\mathfrak{p}}(M_{\mathfrak{p}})) & \text{if } \dim H^0_{\mathfrak{p}}(M_{\mathfrak{p}}) = \dim M_{\mathfrak{p}} \\ 0 & \text{otherwise.} \end{cases}$$

We are now ready to define jdeg(M).

Definition 3. Let R be a Noetherian ring and A be a finitely generated graded R-algebra where $A = R[A_1]$. For a finitely generated graded A-module M, the jdeg of M is the integer

$$\text{jdeg}(M) := \sum_{\mathfrak{p} \in \text{Spec } R} j_{\mathfrak{p}}(M).$$

Remark 2. Since the summands depend on R, the strict notation should be $\text{jdeg}_R(M)$. It is easy to see that this is a finite sum. It is also a straightforward verification to show that $\text{jdeg}(M) = 0$ iff $M = 0$.

Associated graded rings and invariance

Let R be a Noetherian ring and let I be an ideal. We are going to examine some properties of jdeg$(\cdot)$ as it applies to modules over the Rees algebra $R[It]$ of the ideal I. We set $A = R[It]$.

Proposition 3. *Let $B = \bigoplus_{i \geq 0} B_i$ be a finitely generated graded A-subalgebra of $\overline{A}$. If B satisfies the condition S_2 of Serre then the filtration of B is decreasing, i.e.* gr(B) *is well-defined.*

Proof. This follows from [15, Proposition 4.6]. □

An interesting property of jdeg is the following invariance.

Theorem 1. *Let R be a reduced universally catenary ring of dimension d and let I be an ideal of R. Let B be a finitely generated graded A-subalgebra with $A = R[It] \subsetneq B \subset \overline{A}$, and assume that B satisfies the condition S_2 of Serre. Then* $\text{jdeg}(\text{gr}(A)) = \text{jdeg}(\text{gr}(B))$.

Proposition 4. *Let R be a Noetherian ring, let A be an R-module and let $\Phi : A \to A$ be a nilpotent endomorphism of A. Then $\dim \ker \Phi = \dim A = \dim \text{coker}\, \Phi$.*

Proof. This follows from [4, Exercise 12.8]. Indeed

$$\text{rad}\,(\text{ann}\,(\ker \Phi)) = \text{rad}\,(\text{ann}\,(A)) = \text{rad}\,(\text{ann}\,(\text{coker}\,\Phi))$$

so the R-modules $\ker \Phi$, A, coker Φ have the same support. Hence, they all have the same dimension. $\square$

Proof of Theorem 1: Let $C = B/A$, consider the following diagram induced by multiplication by t^{-1}:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & C_0 \arrow[d] & \\
0 \arrow[r] & A[+1] \arrow[r] \arrow[d, "t^{-1}"] & B[+1] \arrow[r] \arrow[d, "t^{-1}"] & C[+1] \arrow[r] \arrow[d, "t^{-1}"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] \arrow[d] & 0 \\
 & & & C' \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

which induces the exact sequence

$$0 \to C_0 \to \text{gr}(A) \to \text{gr}(B) \to C' \to 0.$$

Note that $\dim(C_0)_{\mathfrak{p}} = \dim C'_{\mathfrak{p}} = \dim C_{\mathfrak{p}}$ for every $\mathfrak{p} \in$ Spec (R) because the multiplication by t^{-1} is nilpotent. Also, since the **j**-multiplicity is additive with respect to exact sequences of modules of the same dimension, using the vertical exact sequence, one has $j_{\mathfrak{p}}(C_0) = j_{\mathfrak{p}}(C')$ for every $\mathfrak{p}$ since $j_{\mathfrak{p}}(C) = j_{\mathfrak{p}}(C[+1])$. From the other exact sequence we obtain $j_{\mathfrak{p}}(\text{gr}(A)) = j_{\mathfrak{p}}(\text{gr}(B))$ for every $\mathfrak{p} \in$ Spec R. It follows that $\text{jdeg}(\text{gr}(A)) = \text{jdeg}(\text{gr}(B))$. $\square$

3 Normalization and jdeg

In this section we make use of the jdeg of appropriate modules in order to give estimates for the complexity of normalization processes.

Let R be a universally catenary ring and A a standard graded integral domain over R. We examine the character of the chains of graded subalgebras between A and its integral closure $\overline{A}$. We will show that the lengths of certain chains are bounded by jdeg$(\overline{A}/A)$.

To prepare the way we first make the following observation.

Proposition 5. *Let A be an equidimensional Noetherian local ring of dimension $d > 0$ and M a finitely generated module of dimension d. Assume that N is a submodule of M such that* $\dim(M/N) = d - 1$. *If M satisfies the condition S_1 of Serre and N the condition S_2, then M/N satisfies S_1 and in particular it is equidimensional.*

Proof. Let $\mathfrak{p}$ be an associated prime of M/N. It is enough to prove that $\mathfrak{p}$ has codimension 1. Applying $\mathrm{Hom}_A(A/\mathfrak{p}, \cdot)$ to the exact sequence

$$0 \to N \longrightarrow M \longrightarrow M/N \to 0,$$

gives the exact sequence

$$\mathrm{Hom}_A(A/\mathfrak{p}, M) \longrightarrow \mathrm{Hom}_A(A/\mathfrak{p}, M/N) \longrightarrow \mathrm{Ext}^1_A(A/\mathfrak{p}, N).$$

If height $\mathfrak{p} > 1$, the modules at the ends vanish, and $\mathrm{Hom}_A(A/\mathfrak{p}, M/N)$ would be 0. □

Proposition 6. *Let $A \subset B \subset B' \subset \overline{A}$ be finitely generated graded R-algebras all satisfying the S_2 condition of Serre. Then*

$$\mathrm{jdeg}(B'/A) = \mathrm{jdeg}(B/A) + \mathrm{jdeg}(B'/B).$$

Proof. Because B, B' satisfy the condition S_2 and have the same total ring of fractions

$$\dim B/A = \dim B'/A = \dim B'/B = \dim A - 1.$$

Noting that the quotients are equidimensional, for every $\mathfrak{p} \in \mathrm{Spec}\ R$, consider the short exact sequence

$$0 \to (B/A)_{\mathfrak{p}} \to (B'/A)_{\mathfrak{p}} \to (B'/B)_{\mathfrak{p}} \to 0.$$

By Proposition 2,

$$j_{\mathfrak{p}}(B'/A) = j_{\mathfrak{p}}(B/A) + j_{\mathfrak{p}}(B'/B)$$

for every $\mathfrak{p}$, therefore

$$\mathrm{jdeg}(B'/A) = \mathrm{jdeg}(B/A) + \mathrm{jdeg}(B'/B).$$

□.

Theorem 2. *Let R be a Noetherian domain and A a semistandard graded R-algebra of finite integral closure $\overline{A}$. Consider a sequence of distinct integral graded extensions*

$$A \subseteq A_0 \to A_1 \to A_2 \to \cdots \to A_n = \overline{A},$$

where the A_i satisfy the S_2 condition of Serre. Then $n \leq \text{jdeg}(\overline{A}/A)$.

Proof. By Proposition 6 and Remark 2, we have

$$\text{jdeg}(\overline{A}/A) > \text{jdeg}(A_{n-1}/A) > \cdots > \text{jdeg}(A_1/A_0) > 0.$$

Therefore, $n \leq \text{jdeg}(\overline{A}/A)$. $\square$

To illustrate the values of $\text{jdeg}(\overline{A}/A)$ may take we consider two classes of applications. First, let A be a semistandard graded domain over a field k and let $\overline{A}$ be its integral closure. If A satisfies the condition S_2 and $A \neq \overline{A}$, the graded module $\overline{A}/A$ has dimension $\dim A - 1$. Its Hilbert polynomial satisfies

$$H_{\overline{A}/A}(n) = H_{\overline{A}}(n) - H_A(n).$$

Therefore

$$\text{jdeg}(\overline{A}/A) = \deg(\overline{A}/A) = e_1(A) - e_1(\overline{A}),$$

where $e_1(\cdot)$ is the next to the leading coefficient of the Hilbert polynomial of the graded algebra. These coefficients are positive and

$$e_1(A) - e_1(\overline{A}) \leq \binom{e_0(A)}{2},$$

according to [3], [15, Theorem 6.90].

Let now $(R, \mathfrak{m})$ be an analytically unramified Cohen–Macaulay local ring of dimension d and I an $\mathfrak{m}$–primary ideal. If $A = R[It]$,

$$\text{jdeg}(\overline{A}/A) = \deg(\overline{A}/A) = e_1(\overline{A}) - e_1(A),$$

the switch in signs explained by the character of the Hilbert-Samuel function of A. A focus is to bound $e_1(\overline{A})$, the main control of normalization algorithms for $R[It]$. Several bounds are treated in [10], in particular

$$e_1(\overline{A}) \leq \frac{d-1}{2} e_0(A),$$

if R is a regular local ring.

As we are going to see, these cases can be vastly generalized, to arbitrary Rees algebras of ideals or modules, and to algebras which are finite over a polynomial subring.

4 Rees algebras

For the wealth of its constructions, Rees algebras gives origin to several techniques extensible to others. We treat first ideals then modules.

Normalization of ideals

Let R be a commutative ring and let I be an R-ideal. The *integral closure* of I is the ideal $\overline{I}$ consisting of all $z \in R$ which are solutions of equations of the form

$$z^n + a_1 z^{n-1} + \ldots + a_n = 0, a_i \in I^i.$$

We refer to $\overline{R[It]}$ as the *normalization* of I. We are going to show how jdeg can be used to bound the number of iterations of any algorithm that builds $\overline{R[It]}$ by a succession of graded extensions

$$R[It] = A \subseteq A_0 \to A_1 \to A_2 \to \cdots \to A_n = \overline{A} = \overline{R[It]}$$

satisfying the condition S_2 of Serre. Recall that if one chooses A_0 to be the S_2-ification of $R[It]$, then the algorithm of [14] indeed produces such chains of S_2 algebras provided R is a domain essentially of finite type over a perfect field.

Let us recall the notion of the Briançon-Skoda number of an ideal. If I is an ideal of a Noetherian ring R, the *Briançon-Skoda number* $c(I)$ of I is the smallest integer c such that $\overline{I^{n+c}} \subset J^n$ for every n and every reduction J of I. The motivation for this definition is a result of [1] asserting that for the rings of convergent power series over $\mathbb{C}^n$ (later extended to all regular local rings) $c(I) < \dim R$. The existence of uniform values for $c(I)$ has been established for much broader classes of rings (and modules) by Huneke ([7]). Examination of their proofs should yield effective bounds. Here we will derive a related invariant.

The following extends [10, Theorems 2.2 and 4.2], which dealt with equimultiple ideals.

Theorem 3. *Let R be a reduced quasi-unmixed ring and let I be an ideal of Briançon-Skoda number $c(I)$. Then*

$$\text{jdeg}(\overline{R[It]}/R[It]) \leq c(I) \cdot \text{jdeg}(\text{gr}_I(R)).$$

Proof. Let J be a minimal reduction of I. We consider the $R[Jt]$-module $C = \overline{R[It]}/R[Jt]$. Then jdeg$(C) \geq$ jdeg$(\overline{R[It]}/R[It])$. By definition of $c = c(I)$, C is a submodule of the graded $R[Jt]$-module

$$D = \bigoplus_{n \geq 0} J^{n-c}/J^n.$$

The inclusions $J^{n-c} \supset J^{n-c+1} \supset \cdots \supset J^n$ induce a filtration of D. Because local cohomology commutes with direct sums, from the Hilbert functions of the factors in this filtration it follows that

$$\text{jdeg}(D) = c \cdot \text{jdeg}(\text{gr}_J(R)) = c \cdot \text{jdeg}(\text{gr}_I(R)).$$

Moreover, jdeg$(C) \leq$ jdeg(D), so jdeg$(\overline{R[It]}/R[It]) \leq c(I) \cdot \text{jdeg}(\text{gr}_I(R))$. □

There are several extensions of this result if we broaden the notion of Briançon-Skoda number of an ideal.

Theorem 4. *Let k be a perfect field, let R be a reduced Cohen–Macaulay k-algebra essentially of finite type. Then for any ideal I with a reduction J generated by ℓ elements, and every integer n,*

$$\mathrm{Jac}_k(R)\overline{I^{n+\ell(I)-1}} \subset D_n$$

where D is the S_2-ification of $R[It]$. In particular,

$$\mathrm{Jac}_k(R)I^{\ell-1} \subset \text{ann}\ (\overline{R[It]}/D).$$

Proof. The argument is essentially that of [10, Theorem 3.1]. The calculation of the Jacobian ideal of the extended Rees algebra $R[Jt, t^{-1}]$ gives, by [8],

$$\mathrm{Jac}_k(R)\overline{I^{n+\ell(I)-1}} \subset I^{n+1} : I$$

for all n. The assertion follows from

$$I^{n+1} : I \subset D_{n+1} : I = D_n,$$

the last equality because the grade of $\text{gr}(D)_+$ is at least 1. □

Theorem 5. *Let k be a perfect field, let $(R, \mathfrak{m})$ be a reduced Cohen–Macaulay k-algebra essentially of finite type. Suppose that R has isolated singularities. If L is the Jacobian ideal of R, then for any ideal I of analytic spread $\ell = \dim R$,*

$$\text{jdeg}(\overline{R[It]}/R[It]) \leq (\ell + \lambda(R/L) - 1) \cdot \text{jdeg}(\text{gr}_I(R)).$$

Proof. Let D be a S_2-ification of $R[It]$. From Theorem 4, setting $C = \overline{R[It]}$ and $c = \ell - 1$, we have $L\overline{I^{n+c}} \subset D_n$. Consider the diagram

$$\begin{tikzcd}[column sep=small] & D_n/D_{n+c} & & C_{n+c}/LC_{n+c} \arrow[d, two heads] & \\ 0 \arrow[r] & (LC_{n+c} + D_{n+c})/D_{n+c} \arrow[u, hook] \arrow[r] & C_{n+c}/D_{n+c} \arrow[r] & C_{n+c}/(LC_{n+c} + D_{n+c}) \arrow[r] & 0. \end{tikzcd}$$

Considering that the modules in the short exact sequence have dimension $\dim R$ and that the module on the right is supported in $\mathfrak{m}$ alone, it follows that

$$\text{jdeg}(C/D) = \text{jdeg}((LC{+}D)/D){+}\text{jdeg}(C/LC{+}D) = \text{jdeg}((LC{+}D)/D){+}\deg(C/LC{+}D).$$

Given the embedding on the left and the surjection on the right, we have that

$$\text{jdeg}((LC + D)/D) \leq \text{jdeg}(D/D[-c]) = c \cdot \text{jdeg}(\text{gr}(D)),$$

and

$$\deg(C/LC + D) \leq \deg(C/LC) \leq \lambda(R/L)\deg(C/\mathfrak{m}C).$$

Since $\text{jdeg}(\text{gr}(D)) = \text{jdeg}(\text{gr}_I(R))$ by Proposition 1, we obtain the estimate of multiplicities

$$\text{jdeg}(C/D) \leq c \cdot \text{jdeg}(\text{gr}_I(R)) + \lambda(R/L) \cdot f_0(C),$$

where $f_0(C) = \deg(C/\mathfrak{m}C)$.

Finally, we consider the exact sequence

$$0 \to \mathfrak{m}C_n/C_{n+1} \longrightarrow C_n/C_{n+1} \longrightarrow C_n/\mathfrak{m}C_n \to 0.$$

Taking into account that gr(C) is a ring which has the condition S_1, $\bigoplus_n \mathfrak{m}C_n/C_{n+1}$ either vanishes or has the same dimension as the ring. Arguing as above, we have

$$\text{jdeg}(\text{gr}(C)) = \text{jdeg}(\bigoplus_n \mathfrak{m}C_n/C_{n+1}) + f_0(C).$$

A final application of Proposition 1, yields either

$$\text{jdeg}(\overline{R[It]}/R[It]) \leq (\ell + \lambda(R/L) - 1) \cdot \text{jdeg}(\text{gr}_I(R))$$

as in the assertion of the theorem, or

$$\text{jdeg}(\overline{R[It]}/R[It]) \leq (\ell + \lambda(R/L) - 2) \cdot \text{jdeg}(\text{gr}_I(R)),$$

if $\overline{I} \neq \mathfrak{m}$. $\square$

Normalization of modules

Let us recall the notion of the Rees algebra of a module. Let R be a Noetherian ring, let E be a finitely generated torsionfree R–module having a rank, and choose an embedding $\varphi : E \hookrightarrow R^r$. The *Rees algebra* $\mathcal{R}(E)$ of E is the subalgebra of the polynomial ring $R[t_1, \ldots, t_r]$ generated by all linear forms $a_1t_1 + \cdots + a_rt_r$, where $(a_1, \ldots, a_r)$ is the image of an element of E in R^r under the embedding φ. The Rees algebra $\mathcal{R}(E)$ is a standard graded algebra whose nth component is denoted by E^n and is independent of the embedding φ since E is torsionfree and has a rank.

The algebra $\mathcal{R}(E)$ is a subring of the polynomial ring $S = R[t_1, \ldots, t_r]$. Following [6], we consider the ideal (E) of S generated by the forms in E. Denote by G the associated graded ring $\text{gr}_{(E)}(S)$. Let us list some of its basic properties. This portion of our exposition is highly dependent on [6].

Proposition 7. *Let $(R, \mathfrak{m})$ be a Noetherian integral domain of dimension d and let E be a torsionfree R–module of rank r with a fixed embedding $E \hookrightarrow R^r$. Then*

(a) *The components of $G = \bigoplus_{n\geq 0} E^nS/E^{n+1}$ have a natural grading*

$$G_n = E^n + E^nS_1/E^{n+1} + E^nS_2/E^{n+1}S_1 + \cdots.$$

(b) *There is a decomposition*

$$G = \mathcal{R}(E) + H,$$

where $\mathcal{R}(E)$ is the Rees algebra of E and H is the R-torsion submodule of G.

(c) *If $E \subset \mathfrak{m}R^r$ and $\lambda(R^r/E) < \infty$, $H = H^0_{\mathfrak{m}}(G)$ has dimension $d + r$ and multiplicity equal to the Buchsbaum-Rim multiplicity of E.*

Proof. Only Part (c) requires attention. The module H has a natural bigraded structure as an $\mathcal{R}(E)[y_1, \ldots, y_d]$-module which we redeploy in terms of total degrees as follows

$$H = \bigoplus_{n \geq 0} H_n, \quad H_n = \bigoplus_{i=0}^{n} E^i S_{n-i}/E^{i+1} S_{n-i-1}.$$

Observe that $\lambda(H_n) = \lambda(S_n/E^n)$, the Buchsbaum-Rim Hilbert function of the module E (see [2]). For large n,

$$\lambda(S_n/E^n) = \text{br}(E) \binom{n+d+r-2}{d+r-1} + \text{lower terms.}$$

(The nonzero integer br(E) is the Buchsbaum-Rim multiplicity of E (see [15, Section 8.3] for details).) The degree of this polynomial shows that $\dim H = d + r - 1$. $\square$

We could show directly, using properties of Rees algebras, that $\dim H = d + r - 1$ and thereby prove the existence of the polynomial above.

Corollary 1. *Let E be a module as above. Then* jdeg$(G) =$ br$(E) + 1$.

Proof. It suffices to note that $\text{Ass}_R(G) = \{0, \mathfrak{m}\}$, and use the calculation above for $j_{\mathfrak{m}}(G)$ and take into account that $j_{(0)}(G) = \deg(\mathcal{R}(E)_{(0)}) = \deg(K[t_1, \ldots, t_r]) = 1$. $\square$

One application of these estimations is to provide a module version of Theorem 3. It will have a surprising development. For that we recall the notion of the Briançon-Skoda number of a module following [6]. If E is a submodule of rank r of R^r, the *Briançon-Skoda number* $c(E)$ of E is the smallest integer c such that $\overline{E^{n+c}} \subset F^n S_c$ for every n and every reduction F of E. In contrast, the Briançon-Skoda number of the ideal (E) of S is the smallest integer $c = c((E))$ such that $\overline{E^{n+c} S} \subset F^n S$ for every n and every reduction F of E. Note that $c(E) \leq c((E))$ but one has equality of analytic spreads $\ell(E) = \ell((E))$. We shall refer to both $c(E)$ and $c((E))$ as the Briançon-Skoda numbers of E. When R is a regular local ring, applying the Briançon-Skoda theorem to S, gives $c((E)) \leq \ell(E) - 1$.

Combining Theorem 3 and Corollary 1 one has:

Theorem 6. *Let R be a reduced quasi-unmixed ring and let E be a module of Briançon-Skoda number $c((E))$ and Buchsbaum-Rim multiplicity* br(E). *Then*

$$\text{jdeg}(\overline{S[(E)t]}/S[(E)t]) \leq c((E)) \cdot (\text{br}(E) + 1).$$

Let us apply this result to the complexity of the normalization of the Rees algebra of a module $E \subset R^r$, with $\lambda(R^r/E) < \infty$. Let A be the Rees algebra of the S-ideal (E). By Theorem 2, the chain of intermediate graded algebras D,

$$A \subseteq D_0 \subset D_1 \subset \cdots \subset D_s \subset \overline{A},$$

satisfying the condition S_2, has length bounded by $\text{jdeg}(\overline{A}/A)$. Each one of these algebras D_i contains the subalgebra $D' = D \cap \overline{\mathcal{R}(E)}$, inducing the chain of subalgebras

$$\mathcal{R}(E) \subset D'_1 \subset \cdots \subset D'_s \subset \overline{\mathcal{R}(E)}.$$

Therefore, if $\overline{S[(E)t]}$ can be built in s such steps, $\overline{\mathcal{R}(E)}$ will also be built in s steps, although we cannot guarantee the algebras D'_i will satisfy the condition S_2.

The significance here comes when we compare the estimates for s which come from Theorem 6,

$$s \leq c((E)) \cdot \text{jdeg}(\text{gr}_{(E)}(S)) = c((E)) \cdot (\text{br}(E) + 1),$$

with

$$s \leq \binom{r + c(E) - 1}{r} \cdot \text{br}(E),$$

given by [6, Theorem 2.3], using chains of subalgebras of $\overline{\mathcal{R}(E)}$ satisfying the condition S_2. Despite the inequality $c(E) \leq c((E))$, one is likely to obtain in most cases significant enhancements.

5 Algebras with Noether normalization

We now treat a very different venue of applications of Theorem 2. Let R be a Noetherian domain and let $B = R[x_1, \ldots, x_n]$ be a finitely generated R–algebra. We will assume that B is R-torsionfree. By abuse of terminology, by *arithmetic Noether normalizations* we shall mean statements asserting the existence of extensions

$$R \subset S = R[y_1, \ldots, y_r] \subset B,$$

B finite over S, where r depends uniformly on the Krull dimensions of B and R, or on some of the fibers. We shall refer to S as a Noether normalization of B.

Part of the significance of the existence of S lies with the fact that it gives rise to a fibration Spec $(B) \to$ Spec (S), that may add to the understanding of the geometry of B. The 'simpler' S would bring greater benefit. Another motivation lies in that S may serve as a platform for certain construction on B, such as of its integral closure.

What are expected values for r? The answers tend to come in two groups, depending on B being a standard graded algebra or not, or whether it is an integral domain or not. (The values for r varies by one accordingly.) For instance, if $B = R + B_+$, where R is an integral domain of field of fractions K, by a standard dimension formula (see [13, Lemma 1.1.1]),

$$\dim B = \dim R + \text{height } Q = \dim R + \dim K \otimes_R B,$$

$\dim K \otimes_R B = K[x_1, \ldots, x_n] = n$ is the lower bound for r.

In one of his first papers, Shimura ([11]) proves such a result when R is a ring of algebraic integers.

Theorem 7 ([11]). *Let R be a ring of algebraic integers of field of fractions K and let B be a homogeneous R–algebra that is a domain. Then there is a graded subalgebra $S = R[y_1, \ldots, y_r]$, $r = \dim K \otimes_R B$, over which B is integral.*

Note that S is actually a ring of polynomials over R, since $\dim K \otimes_R B = \dim B - \dim R$. Shimura also remarked that this bound is not achieved if $R = \mathbb{C}[t]$.

Tracking numbers

Let R be a normal domain of field of fractions K, and let A be a semistandard graded algebra finite over the polynomial graded subring $S = R[y_1, \ldots, y_r]$. One can define the *determinant* of A as an S-module

$$\det{}_S(A) = (\wedge^r A)^{**},$$

where r is the rank of A as an S-module.

In case R is a field, the graded module $\det{}_S(A)$ isomorphic to a unique module $S[-n]$, so the integer n can be employed as a marker for A. It was used in [3] to define the *tracking number* of A, $\mathsf{tn}_R(A) = n$. If R is not a field, the integer n is still well-defined (as $\mathsf{tn}_K(K \otimes_R A)$), but it will not suffice to fix $\det{}_S(A)$ entirely, particularly when given two such S-algebras $A \subset B$ we want to compare $\det{}_S(A)$ and $\det{}_S(B)$.

To address this issue, we proceed as follows. To the rank 1 reflexive module $\det{}_S(A)$, one can attach several homogeneous divisorial ideals of S, all in the same divisor class group. Suppose $I \subset S$ is one of these, and

$$I = (\bigcap \mathfrak{p}_i^{(r_i)}) \cap (\bigcap \mathfrak{q}_{\mathrm{j}}{}^{(s_j)}),$$

is its primary decomposition, where we denote by $\mathfrak{p}_i$ the primes that are extended from R, and $\mathfrak{q}_i$ those that are not. This means that $\mathfrak{q}_j \cap R = (0)$ and thus $\mathfrak{q}_j KS = (f_j)KS$, $\deg(f_j) > 0$. We associate a *degree* to I by setting

$$\deg(I) = \sum_i r_i + \sum_j s_j \cdot \deg(f_j).$$

Definition 4. Let R and S as above and let $A \subset B \subset \overline{A}$ be finitely generated R-algebras. Set $\det_S(B) = (\wedge^r B)^{**}$ and consider the natural image of $(\wedge^r A)^{**}$ in $\det_S(B)$. Let

$$I = \text{ann}\ (\det_S(B)/\det_S(A)) = \det_S(A) :_S \det_S(B).$$

The *relative tracking number* of A in B is the integer

$$\mathsf{tn}(A, B) = \deg(I).$$

Example 1. If $A = \mathbb{Z}[x, y, z]/(z^3 + xz^2 + x^2y)$, $S = \mathbb{Z}[x, y]$, then $\overline{A} = A[z^2/x]$. It follows that $\mathsf{tn}(A, \overline{A}) = 1$.

It is clear that tn(A, B) is well-defined. We note that in [3], in case R is a field, one defines an absolute tracking number simply by declaring tn(A) as the degree of the free rank one S-module $(\wedge^r A)^{**}$. It turns out that tn$(A, B) =$ tn$(A) -$ tn(B).

Theorem 8. *Let R be a normal domain and let A be a semistandard graded R-algebra with finite integral closure $\overline{A}$ that admits a Noether normalization S. If B is a graded subalgebra, $A \subset B \subset \overline{A}$ and A satisfies the S_2 condition of Serre, then*

$$\text{jdeg}(B/A) = \text{tn}(A, B).$$

Proof. We first clarify $\text{Ass}_R(B/A)$. If $\mathfrak{p}$ is one of these primes and $\mathfrak{p} \neq (0)$, the ideal $P = \mathfrak{p}S$ has the same height as $\mathfrak{p}$ and therefore cannot consist of zerodivisors of B/A if height $\mathfrak{p} \geq 2$ since B/A has the condition S_1 of Serre (as an A–module or as an S-module). Thus $\mathfrak{p}$ has height 1.

We now argue the asserted equality. Let K be the field of fractions of R. If $\mathfrak{p} = (0)$, we note that jdeg(KB/KA) is just the difference $e_1(KA) - e_1(KB)$ of Hilbert coefficients of the graded modules KA and KB. According to [3, Proposition 3.1], for modules such as KA and KB, these values are the tracking numbers tn(KA) and tn(KB), respectively. The difference tn(A) $-$ tn(B) clearly accounts for the second summand in the definition of $\deg(I)$, for $I =$ ann $(\det_S(B)/\det_S(A))$.

Now we must account for the j-multiplicity of B/A for a prime $\mathfrak{p}$ of height 1. Localizing at $\mathfrak{p}$, we may assume that R is a discrete valuation domain and $\mathfrak{p}$ is its maximal ideal. Let H be the submodule of B/A of support in $\mathfrak{p}$. We claim that the multiplicity of H is the corresponding integer r_i for $\mathfrak{p}_i = \mathfrak{p}$ in the expression of $\deg(I)$. Consider the exact sequences of S-modules

$$0 \to A \longrightarrow B \longrightarrow B/A \to 0,$$

$$0 \to H \longrightarrow B/A \longrightarrow C \to 0,$$

and the prime ideal $P = \mathfrak{p}S$. Note that P is the only associated prime ideal of H as an S-module and no associated prime of C is contained in it, and in particular $(B/A)_P = H_P$. Applying the associativity formula for multiplicities to H and noting that $\deg(S/P) = 1$, we have $\deg(H) = \lambda(H_P)$. Since S_P is a discrete valuation domain the length of the torsion module $(B/A)_P$ is given by r_i.

We have therefore matched each summand that occurs in $\deg(I)$ to one in definition of jdeg(B/A). $\square$

Remark 3. It is not difficult to see that for the context here, the assertion of Theorem 2 could be improved by saying that the number of extensions between A and $\overline{A}$ is bounded by

$$\sum_i r_i + \sum_j s_j,$$

rather than the whole of $\deg(I)$. (A similar observation is given in [10].)

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