The Chern Coefficients of Local Rings

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Outline

Abstract

The Chern numbers of the title are the first coefficients (after the multiplicities) of the Hilbert functions of various filtrations of ideals of a local ring $(\mathbf{R}, \mathfrak{m})$. For a Noetherian (good) filtration $\mathcal{A}$ of $\mathfrak{m}$-primary ideals, the positivity and bounds for $e_1(\mathcal{A})$ are well-studied if **R** is Cohen-Macaulay, or more broadly, if **R** is a Buchsbaum ring or mild generalizations thereof.

In this talk, for arbitrary geometric local domains, we introduce techniques based on the theory of maximal Cohen-Macaulay modules and of extended multiplicity functions to establish the meaning of the positivity of $e_1(\mathcal{A})$, and to derive lower and upper bounds for $e_1(\mathcal{A})$.

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Introduction

Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$, and let I be an $\mathfrak{m}$-primary ideal. One of our goals is to study the set of I-good filtrations of $\mathbf{R}$. More concretely, we will consider the set of multiplicative, decreasing filtrations of R ideals,

$$\mathcal{A} = \{I_n, I_0 = \mathbf{R}, I_{n+1} = II_n, n \gg 0\},$$

integral over the I-adic filtration, conveniently coded in the corresponding Rees algebra and its associated graded ring

$$\mathcal{R}(\mathcal{A}) = \sum_{n \geq 0} I_n t^n, \quad \mathrm{gr}_{\mathcal{A}}(\mathbf{R}) = \sum_{n \geq 0} I_n / I_{n+1}.$$

$\mathfrak{S}(I)$

We will study certain strata of these algebras. For that we will focus on the role of the Hilbert polynomial of the Hilbert function $\lambda(\mathbf{R}/I_{n+1})$,

$$H^1_{\mathcal{A}}(n) = P^1_{\mathcal{A}}(n) \equiv \sum_{i=0}^{d} (-1)^i e_i(\mathcal{A}) \binom{n+d-i}{d-i},$$

particularly of its coefficients $e_0(\mathcal{A})$ and $e_1(\mathcal{A})$.Two of our main issues are to establish relationships between the coefficients $e_i(\mathcal{A})$, for $i = 0, 1$, and marginally $e_2(\mathcal{A})$.

If $\mathbf{R}$ is a Cohen-Macaulay ring, there are numerous related developments, noteworthy ones being given and discussed by Eliasand Rossi-Valla for adic filtrations and Polini-Ulrich-V for the integral closure filtration.

$CM \neq nCM$

The situation is very distinct in the non Cohen-Macaulay case. Just to illustrate the issue, suppose $d > 2$ and consider a comparison between $e_0(I)$ and $e_1(I)$. It is often possible to pass to a reduction $\mathbf{R} \to S$, with $\dim S = 2$, or even $\dim S = 1$, so that $e_0(I) = e_0(IS)$ and $e_1(I) = e_1(IS)$. If **R** is Cohen-Macaulay, this is straightforward.

In general the relationship between $e_0(IS)$ and $e_1(IS)$ may involve other invariants of S, some of which may not be easily traceable all the way to **R**.

Viewpoint

Our perspective is partly influenced by the interpretation of the coefficient e_1 as a *tracking number*, that is, as a numerical positional tag of the algebra $\mathcal{R}(\mathcal{A})$ in the set of all such algebras with the same multiplicity. The coefficient e_1 under various circumstances is also called the *Chern number or coefficient* of the algebra.

This talk is organized around a list of questions and conjectural statements about the values of e_1 for very general filtrations associated to the $\mathfrak{m}$–primary ideals of a local Noetherian ring $(\mathbf{R}, \mathfrak{m})$. For a filtration $\mathcal{A}$ integral over the adic filtration defined by an ideal J generated by a system of parameters, we consider:

Specific Goals

1. (Conjecture 1: the negativity conjecture) For every ideal J, generated by a system of parameters, $e_1(J) < 0$ if and only if $\mathbf{R}$ is not Cohen-Macaulay.
2. (Conjecture 2: the positivity conjecture) For every $\mathfrak{m}$-primary ideal I, for its integral closure filtration $\mathcal{A}$

$$e_1(\mathcal{A}) \geq 0.$$

3. (Conjecture 3: the uniformity conjecture) There exist two multiplicity based functions, $\mathbf{f}_l(\cdot)$, $\mathbf{f}_u(\cdot)$, such that for each $\mathfrak{m}$-primary ideal I and any I-good filtration $\mathcal{A}$,

$$\mathbf{f}_l(I) \leq e_1(\mathcal{A}) \leq \mathbf{f}_u(I).$$

4. (Question 4: baseline) $e_1(J)$ is independent of the minimal reduction J.

Results

- For general local rings the conjectures may fail for reasons that will be illustrated by examples.
- We will settle Conjecture 1 for domains that are essentially of finite type over fields by making use of the existence of special maximal Cohen-Macaulay modules (Theorem 11).
- The lower bound in Conjecture 3 is also settled for general rings through the use of extended degree functions (Theorem 13). The upper bound uses the technique of the Briançon-Skoda theorem on perfect fields (Theorem 12).
- We bring no real understanding to the last question, and just preliminary observations on Conjecture 2.

Determinant of a graded module

Let $S = k[z_1, \ldots, z_d]$ be a ring of polynomials with the standard grading, and let E be a f.g. graded module of rank (multiplicity) r.

The module $(\wedge^r E)^{**}$ is graded, reflexive of rank 1 and therefore it is isomorphic (as a graded module) to $S[-n]$, for some integer n.

Definition

Consider a graded $\mathbf{R} = k[z_1, \ldots, z_d]$–module E: Suppose E has torsionfree rank r. The *determinant* of E is the graded module

$$\det_{\mathbf{R}}(E) = (\wedge^r E)^{**}.$$

$$\det_{\mathbf{R}}(E) \simeq \mathbf{R}[-c(E)], \quad \mathrm{tn}(E) := c(E)$$

The integer $c(E)$ is the **tracking number** of E.

Theorem

If E has no associated primes of codimension 1, $\mathrm{tn}(E)$ can be read off the Hilbert series $H_E(t)$ of E:

$$H_E(t) = \frac{\mathbf{h}(t)}{(1-t)^d},$$

$$\mathrm{tn}(E) = \mathbf{h}'(1),$$

which shows that $\mathrm{tn}(E)$ is independent of $\mathbf{R}$.

Theorem

Let

$$E_0 \subset E_1 \subset \cdots \subset E_n$$

be a sequence of distinct f.g. graded $\mathbf{R}$-modules of the same rank, with the property S_2 of Serre. Then

$$n \leq \mathrm{tn}(E_0) - \mathrm{tn}(E_n).$$

e_1 as a tracking number

Let $(\mathbf{R}, \mathfrak{m})$ be a local ring of dimension $d > 0$, with a coefficient (infinite) field $k \subset \mathbf{R}$, $k \simeq \mathbf{R}/\mathfrak{m}$. For a filtration $\mathcal{A}$ as above, the associated graded ring

$$G = \mathrm{gr}_{\mathcal{A}}(\mathbf{R}) = \sum_{n \geq 0} I_n/I_{n+1}$$

admits a Noether normalization $S = k[z_1, \ldots, z_d] \to G$, $\deg z_i = 1$. The multiplicity $e_0(\mathcal{A})$ is the torsionfree rank of G as a S-module.

Definition

The *determinant* of G is

$$\det(G) := (\wedge^r(G))^{**} \simeq S[-\delta], \quad r = e_0(\mathcal{A}).$$

δ is the **tracking number** of G as a S-module, $\mathrm{tn}_S(G) = \delta$.

Proposition

Let G_0 be the torsion S-submodule of G. If $\dim G_0 < d - 1$, then

$$e_1(\mathcal{A}) = \mathrm{tn}_S(G).$$

Otherwise,

$$e_1(\mathcal{A}) = \mathrm{tn}_S(G) + \deg(G_0).$$

Both equalities suggest that relationships between $e_0(\mathcal{A})$ and $e_1(\mathcal{A})$ are to be expected, a fact that has been repeatedly borne out.

Remark

The coefficient $e_2(\mathcal{A})$ can often be understood as the degree of a *determinant*. Suppose R is a Cohen-Macaulay local of dimension $d > 0$ with a field of coefficients as above, and J is a minimal reduction for the filtration $\mathcal{A}$. Let $T = S_{\mathcal{A}/J}$ be the corresponding Sally module ([13, Theorem 2.11]). If $T \neq 0$, it is a module of dimension d with the condition S_1 of Serre, and therefore torsionfree as a subring as $S = k[z_1, \ldots, z_d]$. Another calculation gives

$$\det(S_{\mathcal{A}/J}) := (\wedge^r(T))^{**} \simeq S[-\delta], \quad r = e_1(\mathcal{A}) - e_0(\mathcal{A}) + \lambda(R/A_1).$$

The degree δ is now $e_2(\mathcal{A})$, according to [13, Theorem 2.11].

Normalization and e_1

The second interpretation of $e_1(\mathcal{A})$ uses the setting of normalization of ideals. Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local domain of dimension d, which is a quotient of a Gorenstein ring. For an $\mathfrak{m}$-primary ideal I, we are going to consider the set of all graded subalgebras $\mathbf{A}$ of the integral closure of $\mathbf{R}[It]$,

$$\mathbf{R}[It] \subset \mathbf{A} \subset \overline{\mathbf{A}} = \overline{\mathbf{R}[It]}.$$

We will assume that $\overline{\mathbf{A}}$ is a finite $\mathbf{R}[It]$-algebra. We denote the set of these algebras by $\mathfrak{S}(I)$. If the algebra

$$\mathbf{A} = \sum I_n t^n$$

comes with a filtration that is decreasing $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, it has an associated graded ring

$$\mathrm{gr}\,(\mathbf{A}) = \sum_{n=0}^{\infty} I_n/I_{n+1}.$$

Proposition

Let $(\mathbf{R}, \mathfrak{m})$ *be a normal, Noetherian local domain which is a quotient of a Gorenstein ring, and let* I *be an* $\mathfrak{m}$*-primary ideal. For algebras* $\mathbf{A}$, $\mathbf{B}$ *of* $\mathfrak{S}(I)$*:*

1. *If the algebras* $\mathbf{A}$ *and* $\mathbf{B}$ *satisfy* $\mathbf{A} \subset \mathbf{B}$*, then* $e_1(\mathbf{A}) \leq e_1(\mathbf{B})$.
2. *If* $\mathbf{B}$ *is the* S_2*-ification of* $\mathbf{A}$*, then* $e_1(\mathbf{A}) = e_1(\mathbf{B})$.
3. *If the algebras* $\mathbf{A}$ *and* $\mathbf{B}$ *satisfy the condition* S_2 *of Serre and* $\mathbf{A} \subset \mathbf{B}$*, then* $e_1(\mathbf{A}) = e_1(\mathbf{B})$ *if and only if* $\mathbf{A} = \mathbf{B}$.

Corollary

Given a sequence of distinct algebras in $\mathfrak{S}(I)$*,*

$$\mathbf{A}_0 \subset \mathbf{A}_1 \subset \cdots \subset \mathbf{A}_n = \overline{\mathbf{R}[It]},$$

that satisfy the condition S_2 *of Serre, then*

$$n \leq e_1(\overline{\mathbf{R}[It]}) - e_1(\mathbf{A}_0) \leq e_1(\overline{\mathbf{R}[It]}) - e_1(I).$$

Cohen-Macaulayness and the Negativity of e_1

Given the role of the Hilbert coefficient e_1 as a tracking number in the normalization of blowup algebras, it is of interest to know its signature.

Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local ring of dimension d. If $\mathbf{R}$ is Cohen-Macaulay, for an ideal J generated by a system of parameters, $J = (x_1, \ldots, x_d)$, $e_1(J) = 0$. As a consequence, for any $\mathfrak{m}$-primary ideal I, $e_1(I) \geq 0$. If $d = 1$, the property $e_1(J) = 0$ is characteristic of Cohen-Macaulayness.

For $d \geq 2$, the situation is somewhat different. Consider the ring $\mathbf{R} = k[x, y, z]/(z(x, y, z))$. Then for $T = H^0_{\mathfrak{m}}(\mathbf{R})$ and $S = k[x, y] = \mathbf{R}/T$, $e_1(\mathbf{R}) = e_1(S) = 0$.

Conjecture 1

We are going to argue that the negativity of $e_1(J)$ is an expression of the lack of Cohen-Macaulayness of $\mathbf{R}$ in numerous classes of rings. To provide a framework, we state:

Conjecture

Let $\mathbf{R}$ be a Noetherian local ring that admits an embedding into a big Cohen-Macaulay module. Then for a parameter ideal J, $e_1(J) < 0$ if and only if $\mathbf{R}$ is not Cohen-Macaulay.

We next establish the **small** version of the conjecture. Its proof will point to **larger** versions.

Theorem

Let $(\mathbf{R}, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 2$*. Suppose there is an embedding*

$$0 \to \mathbf{R} \longrightarrow E \longrightarrow C \to 0,$$

where E *is a finitely generated maximal Cohen-Macaulay* $\mathbf{R}$*-module. If* $\mathbf{R}$ *is not Cohen-Macaulay, then* $e_1(J) < 0$ *for any parameter ideal* J*.*

Proof. We may assume that the residue field of $\mathbf{R}$ is infinite. We are going to argue by induction on d. For $d = 2$, let $J = (x, y)$. in

$$0 \to \mathbf{R} \longrightarrow E \longrightarrow C \to 0$$

if $\mathbf{R}$ is not Cohen-Macaulay, $\operatorname{depth} C = 0$. We may assume that x is a superficial element for the purpose of computing $e_1(J)$, and also superficial relative to C, that is, x is not contained in any associated prime of C distinct from $\mathfrak{m}$. Tensoring the exact sequence above by $\mathbf{R}/(x)$, we get the exact complex

$$0 \to T = \operatorname{Tor}_1^{\mathbf{R}}(\mathbf{R}/(x), C) \longrightarrow \mathbf{R}/(x) \longrightarrow E/xE \longrightarrow C/xC \to 0,$$

where T is a nonzero module of finite support. Denote by S the image of $\mathbf{R}' = \mathbf{R}/(x)$ in E/xE. S is a Cohen-Macaulay ring of dimension 1.

By the Artin-Rees Theorem, for $n \gg 0$, $T \cap (y^n)\mathbf{R}' = 0$, and therefore from the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T \cap (y^n)\mathbf{R}' & \longrightarrow & (y^n)\mathbf{R}' & \longrightarrow & (y^n)S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T & \longrightarrow & \mathbf{R}' & \longrightarrow & S & \longrightarrow & 0 \end{array}$$

the Hilbert polynomial of the ideal $y\mathbf{R}'$ is

$$e_0 n - e_1 = e_0(yS)n + \lambda(T).$$

Thus

$$e_1(J) = -\lambda(T) < 0, \tag{1}$$

as claimed.

Assume now that $d \geq 3$, and let x be a superficial element for J, and the modules E and C. In the exact sequence

$$0 \to T = \mathrm{Tor}_1^{\mathbf{R}}(\mathbf{R}/(x), C) \longrightarrow \mathbf{R}' = \mathbf{R}/(x) \longrightarrow E/xE \longrightarrow C/xC \to 0, \quad (2)$$

T is either zero, and we would go on with the induction procedure, or T is a nonzero module of finite support.

Let us point out first an elementary observation for the calculation of Hilbert coefficients. Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local ring, and let $\mathcal{A} = \{I_n, n \geq 0\}$ be a filtration as above. For a finitely generated $\mathbf{R}$-module M, denote by $e_i(M)$ the Hilbert coefficients of M for the filtration $\mathcal{A}M = \{I_nM, n \geq 0\}$. Note the elementary observation:

Proposition

Let

$$0 \to A \longrightarrow B \longrightarrow C \to 0$$

be an exact sequence of finitely generated $\mathbf{R}$*-modules. If* $r = \dim A < s = \dim B$*, then* $e_i(B) = e_i(C)$ *for* $i < s - r$*.*

To continue with the proof, if in the exact sequence

$$0 \to T = \mathrm{Tor}_1^{\mathbf{R}}(\mathbf{R}/(x), C) \longrightarrow \mathbf{R}' = \mathbf{R}/(x) \longrightarrow E/xE \longrightarrow C/xC \to 0,$$

$T \neq 0$, by previous observation, we have that $e_1(J\mathbf{R}') = e_1(J(\mathbf{R}'/T))$, and the resulting embedding $\mathbf{R}'/T \hookrightarrow E/xE$. By the induction hypothesis, it suffices to prove that if $\mathbf{R}'/T$ is Cohen-Macaulay then $\mathbf{R}'$, and therefore $\mathbf{R}$, will be Cohen-Macaulay. This is a result of Huneke-Ulrich ([8, Proposition 2.1]).

We may assume that $\mathbf{R}$ is a complete local ring. Since $\mathbf{R}$ is embedded in a maximal Cohen-Macaulay module, any associated prime of $\mathbf{R}$ is an associated prime of E and therefore it is equidimensional.

Big Cohen-Macaulay modules questions

We now analyze what is required to extend the proof to big Cohen-Macaulay cases. We are going to assume that $\mathbf{R}$ is an integral domain and that E is a big **balanced** Cohen-Macaulay module. Embed $\mathbf{R}$ into E,

$$0 \to \mathbf{R} \longrightarrow E \longrightarrow C \to 0.$$

The argument above ($d \geq 3$) will work if in the induction argument we can pick $x \in J$ superficial for the Hilbert polynomial of J, avoiding the finite set of associated primes of E and all associated primes of C different from $\mathfrak{m}$. It is this last condition that is the most troublesome. There is one case when this can be overcome, to wit, when $\mathbf{R}$ is a complete local ring and E is countably generated. Indeed, C will be countably generated and $\mathrm{Ass}(C)$ will be a countable set. The prime avoidance result of Burch allows for the choice of x. Let us apply these ideas in an important case.

On Conjecture 1

Theorem

Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local integral domain essentially of finite type over a field. If $\mathbf{R}$ is not Cohen-Macaulay, then $e_1(J) < 0$ for any parameter ideal J.

Proof. Let A be the integral closure of $\mathbf{R}$ and $\widehat{\mathbf{R}}$ its completion. Tensor the embedding $\mathbf{R} \subset A$ to obtain

$$0 \to \widehat{\mathbf{R}} \longrightarrow \widehat{\mathbf{R}} \otimes_{\mathbf{R}} A = \widehat{A}.$$

From the properties of pseudo-geometric local rings, $\widehat{A}$ is a reduced semi-local ring with a decomposition

$$\widehat{A} = A_1 \times \cdots \times A_r,$$

where each A_i is a complete local domain, of dimension $\dim \mathbf{R}$ and finite over $\widehat{\mathbf{R}}$.

For each A_i we make use of results of Griffith ([5, Theorem 3.1] and [6, Proposition 1.4]) and pick a countably generated big balanced Cohen-Macaulay A_i-module and therefore $\widehat{\mathbf{R}}$-module. Collecting the E_i we have an embedding

$$\widehat{\mathbf{R}} \longrightarrow A_1 \times \cdots \times A_r \longrightarrow E = E_1 \oplus \cdots \oplus E_r.$$

As E is a countably generated big balanced Cohen-Macaulay $\widehat{\mathbf{R}}$-module, the argument above shows that if $\widehat{\mathbf{R}}$ is not Cohen-Macaulay then $e_1(J\widehat{\mathbf{R}}) < 0$. This suffices to prove the assertion about $\mathbf{R}$. □

Comment/Query

In the exact sequence

$$0 \to \mathbf{R} \longrightarrow E \longrightarrow C \to 0,$$

where E is a big balanced Cohen-Macaulay module, the argument made use of the fact that we could pick superficial elements relative to C. If the cardinality of $\mathbf{R}/\mathfrak{m}$ is larger than the cardinality of a generating set for E, it would be fine.

A naive move is to pass $\mathbf{R} \to \mathbf{R}(\mathbb{X})$, where $\mathbb{X}$ is a large set of indeterminates, and pass to $E \to E \otimes_{\mathbf{R}} \mathbf{R}(\mathbb{X})$. The apparent difficulty is that we don't know whether $E \otimes_{\mathbf{R}} \mathbf{R}(\mathbb{X})$ remains balanced. (It suffices to prove for 1 indeterminate.)

Examples

(i) Let $(\mathbf{R}, \mathfrak{m})$ be a regular local ring and let F be a nonzero (f.g.) free $\mathbf{R}$-module. For any non-free submodule N of F, the idealization (trivial extension) of $\mathbf{R}$ by N, $S = \mathbf{R} \oplus N$ is a non Cohen-Macaulay local ring. Picking $E = \mathbf{R} \oplus F$, the theorem above implies that for any parameter ideal $J \subset S$, $e_1(J) < 0$. It is not difficult to give an explicit formula for $e_1(J)$ in this case.

(ii) Let $\mathbf{R} = \mathbb{R} + (x, y)\mathbb{C}[x, y] \subset \mathbb{C}[x, y]$, for x, y distinct indeterminates. $\mathbf{R}$ is not Cohen-Macaulay but its localization S at the maximal irrelevant ideal is a Buchsbaum ring. It is easy to verify that $e_1(x, y) = -1$ and that $e_1(S) = 0$ for the $\mathfrak{m}$-adic filtration of S.

Note that $\mathbf{R}$ has an isolated singularity. For these rings, [9, Theorem 5] can be extended (does not require the Cohen-Macaulay condition), and therefore describes bounds for $e_1(\overline{\mathbf{A}})$ of integral closures. Thus, if $\mathbf{A}$ is the Rees algebra of the parameter ideal J, one has

$$e_1(\overline{\mathbf{A}}) - e_1(J) \leq (d - 1 + \lambda(\mathbf{R}/L))e_0(J),$$

where L is the Jacobian of $\mathbf{R}$.

In the example above, one has $d = 2$, $\lambda(\mathbf{R}/L) = 1$,

$$e_1(\overline{\mathbf{A}}) - e_1(J) \leq 2e_0(J).$$

(iii) Let k be a field of characteristic zero and let $f = x^3 + y^3 + z^3$ be a polynomial of $k[x, y, z]$. Set $A = k[x, y, z]/(f)$ and let $\mathbf{R}$ be the Rees algebra of the maximal irrelevant ideal $\mathfrak{m}$ of A. Using the Jacobian criterion, $\mathbf{R}$ is normal.Because the reduction number of $\mathfrak{m}$ is 2, R is not Cohen-Macaulay. Furthermore, it is easy to verify that $\mathbf{R}$ is not contained in any Cohen-Macaulay domain that is finite over $\mathbf{R}$. Let $S = \mathbf{R}_{\mathcal{M}}$, where $\mathcal{M}$ is the irrelevant maximal ideal of $\mathbf{R}$. The first superficial element (in the reduction to dimension two) can be chosen to be prime. Now one takes the integral closure of S, which will be a maximal Cohen-Macaulay module.

Cohen-Macaulay Rings

If $(\mathbf{R}, \mathfrak{m})$ is a Cohen-Macaulay local ring, there are two situations when bounds are well delineated:

- [Elias and Rossi-Valla]: The filtration $\mathcal{A}$ is I-adic:

$$e_1(I) \leq \binom{e_0(I)}{2} - \binom{\nu(I) - d}{2} - \lambda(\mathbf{R}/I) + 1$$

- [Polini-Ulrich-V]: Let $(\mathbf{R}, \mathfrak{m})$ be a reduced local ring of dimension d, essentially of finite type over a perfect field, and let I be an $\mathfrak{m}$-primary ideal. If δ is a nonzero element of the Jacobian ideal of $\mathbf{R}$, then for any I-good filtration $\mathcal{A}$,

$$\begin{aligned} e_1(\mathcal{A}) &< (d-1)e_0(I) + e_0((I, \delta)/(\delta)) \\ e_1(\mathcal{A}) &< \frac{d-1}{2} e_0(I), \quad \mathbf{R} \text{ is regular} \end{aligned}$$

Comment

It is not clear what the bounds are if **R** is not Cohen-Macaulay. At first glance, one might look for a modification of these bounds in the form:

$$e_1(\mathcal{A} \leq \text{Cohen-Macaulay Bound}) - T,$$

where $T \geq 0$ is a **non-Cohen-Macaulay tax or penalty**.

We next explore some examples:

Uniform lower bounds for e_1 are rare but still exist in special cases. For example, if **R** is a generalized Cohen-Macaulay ring, then according to Goto-Nishida ([4, Theorem 5.4]),

$$e_1(J) \geq -\sum_{i=1}^{d-1} \binom{d-2}{i-1} \lambda(H^i_{\mathfrak{m}}(\mathbf{R})),$$

with equality if **R** is Buchsbaum.

It should be observed that uniform lower bounds may not always exist. For instance, if $A = k[x, y, z]$, and **R** the idealization of (x, y), then for the ideal $J = (x, y, z^n)$, $e_1(J) = -n$.

The Koszul homology modules $H_i(J)$ of J is a first place where to look for bounds for $e_1(J)$. We recall [1, Theorem 4.6.10], that the multiplicity of J is given by the formula

$$e_0(J) = \lambda(\mathbf{R}/J) - \sum_{i=1}^{d}(-1)^{i-1}h_i(J),$$

where $h_i(J)$ is the length of $H_i(J)$. The summation term is non-negative and only vanishes if $\mathbf{R}$ is Cohen-Macaulay. Unfortunately it does not gives a bounds for $e_1(J)$.There is a formula involving these terms in the special case when J is generated by a d-sequence. Then the corresponding *approximation complex* is acyclic, and the Hilbert-Poincaré series of J ([7, Corollary 4.6]) is

$$\frac{\sum_{i=0}^{d}(-1)^i h_i(J)t^i}{(1-t)^d},$$

and therefore

$$e_1(J) = \sum_{i=1}^{d}(-1)^i i h_i(J).$$

Upper Bounds

Theorem (Existence of Bounds)

Let $(\mathbf{R}, \mathfrak{m})$ *be a local integral domain of dimension* d*, essentially of finite type over a perfect field, and let* I *be an* $\mathfrak{m}$*-primary ideal. If* $\mathbf{R}$ *is a generalized Cohen-Macaulay ring and* δ *is a nonzero element of the Jacobian ideal of* $\mathbf{R}$ *then for any* I*-good filtration* $\mathcal{A}$*,*

$$e_1(\mathcal{A}) < (d-1)e_0(I) + e_0((I,\delta)/(\delta)) - T,$$

where

$$T = \sum_{i=1}^{d-1} \binom{d-2}{i-1} \lambda(H^i_{\mathfrak{m}}(\mathbf{R})).$$

There is a version of this theorem without the generalized Cohen-Macaulay hypothesis. T is replaced by an expression involving homological degrees.

Lower Bounds

The following establishes lower bounds for e_1 for arbitrary Noetherian rings.

Theorem (Lower Bound for $e_1(I)$)

Let $(\mathbf{R}, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 1$*. If* I *is an* $\mathfrak{m}$*-primary ideal, then*

$$e_1(I) \geq -\mathrm{hdeg}_I(\mathbf{R}) + e_0(I).$$

(*Note that* $\mathrm{hdeg}_I(\mathbf{R}) - e_0(I)$ *is the Cohen-Macaulay defficiency of* $\mathbf{R}$ *relative to the degree function* hdeg_I.)

Bibliography

- W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, 1993.
- K. Dalili and W. V. Vasconcelos, The tracking number of an algebra, *American J. Math.* **127** (2005), 697–708.
- J. Elias, Upper bounds of Hilbert coefficients and Hilbert functions, *Math. Proc. Camb. Phil. Soc.*, to appear.
- S. Goto and K. Nishida, Hilbert coefficients and Buchsbaumness of associated graded rings, *J. Pure & Applied Algebra* **181** (2003), 61–74.
- P. Griffith, A representation theorem for complete local rings, *J. Pure & Applied Algebra* **7** (1976), 303–315.
- P. Griffith, Maximal Cohen-Macaulay modules and representation theory, *J. Pure & Applied Algebra* **13** (1978), 321–334.
- J. Herzog, A. Simis and W. V. Vasconcelos, Approximation complexes of blowing-up rings, *J. Algebra* **74** (1982), 466–493.
- C. Huneke and B. Ulrich, General hyperplane sections of algebraic varieties, *J. Algebraic Geom.* **2** (1993), 487–505.
- T. Pham and W. V. Vasconcelos, Complexity of the normalization of algebras, *Math. Zeit.* **258** (2008), 729–743.
- C. Polini, B. Ulrich and W. V. Vasconcelos, Normalization of ideals and Briançon-Skoda numbers, *Math. Research Letters* **12** (2005), 827–842.
- M. E. Rossi and G. Valla, *Hilbert Function of Filtered Modules*, arXiv: 0710.2346.
- B. Ulrich and W. V. Vasconcelos, On the complexity of the integral closure, *Trans. Amer. Math. Soc.* **357** (2005), 425–442.
- W. V. Vasconcelos, *Integral Closure*, Springer Monographs in Mathematics, New York, 2005.
- W. V. Vasconcelos, The Chern coefficients of local rings, *Michigan Mathematical J.*, to appear, 18 pgs.