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Cohen–Macaulayness versus the vanishing of the first Hilbert coefficient of parameter ideals

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Abstract

The conjecture of Wolmer Vasconcelos on the vanishing of the first Hilbert coefficient $e_1(Q)$ is solved affirmatively, where Q is a parameter ideal in a Noetherian local ring. Basic properties of the rings for which $e_1(Q)$ vanishes are derived. The invariance of $e_1(Q)$ for parameter ideals Q and its relationship to Buchsbaum rings are studied.

1. *Introduction*

Let A be a Noetherian local ring with a maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $\ell_A(M)$ denote, for an A-module M, the length of M. Then, for each $\mathfrak{m}$-primary ideal I in A, we have integers $\{e_i(I)\}_{0 \leqslant i \leqslant d}$ such that the equality

$$\ell_A(A/I^{n+1}) = e_0(I)\binom{n+d}{d} - e_1(I)\binom{n+d-1}{d-1} + \ldots + (-1)^d e_d(I)$$

holds for all integers $n \gg 0$. We call $\{e_i(I)\}_{0 \leqslant i \leqslant d}$ the Hilbert coefficients of A with respect to I. These integers carry a great deal of information about the ideal I. We argue that $e_1(Q)$, for parameter ideals Q, codes structural information about the ring A itself. Noteworthy properties of A associated to values of $e_1(Q)$ are the Cohen–Macaulay, the generalized Cohen–Macaulay and the Buchsbaum conditions.

We say that A is unmixed if $\dim \hat{A}/\mathfrak{p} = d$ for every $\mathfrak{p} \in \mathrm{Ass}(\hat{A})$, where $\hat{A}$ denotes the $\mathfrak{m}$-adic completion of A. With this notation Vasconcelos, exploring the vanishing of $e_1(Q)$ for parameter ideals Q, posed the following conjecture in his lecture at the conference in Yokohama in March 2008.

Conjecture 1.1 [**18**]. Assume that A is unmixed. Then A is a Cohen–Macaulay local ring, once $e_1(Q) = 0$ for some parameter ideal Q of A.

In Section 2 of the present paper we shall settle Conjecture 1.1 affirmatively (Theorem 2.1). Here we should note that Conjecture 1.1 is already solved partially by [**5**, **12**]. In fact, Ghezzi, Hong and Vasconcelos [**5**, Theorem 3.3] proved that the conjecture holds if A is an integral domain which is a homomorphic image of a Cohen–Macaulay ring. Mandal, Singh and Verma [**12**] proved that $e_1(Q) \leqslant 0$ for every parameter ideal Q in an arbitrary Noetherian local ring A and showed that $e_1(Q) < 0$, if depth $A = d - 1$.

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Theorem (Theorem 2.1). *Let A be a Noetherian local ring with $d = \dim A > 0$ and let Q be a parameter ideal in A. Then the following are equivalent*:
(a) *A is Cohen–Macaulay*;
(b) *A is unmixed and $e_1(Q) = 0$*;
(c) *A is unmixed and $e_1(Q) \geqslant 0$.*

Let $\mathrm{Assh}(A) = \{\mathfrak{p} \in \mathrm{Ass}(A) \mid \dim A/\mathfrak{p} = d\}$ and let $(0) = \bigcap_{\mathfrak{p}\in \mathrm{Ass}(A)} I(\mathfrak{p})$ be a primary decomposition of (0) in A with $\mathfrak{p}$-primary ideals $I(\mathfrak{p})$ in A. We consider

$$U_A(0) = \bigcap_{\mathfrak{p}\in \mathrm{Assh}(A)} I(\mathfrak{p})$$

and call it the *unmixed component* of (0) in A.

Let us call those local rings A with $e_1(Q) = 0$ for some parameter ideal Q of A *Vasconcelos*[†] rings. In Section 3 we shall explore basic properties of Vasconcelos rings. Certain sequentially Cohen–Macaulay rings are good examples of Vasconcelos rings. A basic characterization of some of these rings is the following theorem.

Theorem (Theorem 2.6). *Let A be a Noetherian local ring of dimension $d \geqslant 2$. Let $U = U_A(0)$ and let Q be a parameter ideal of A. Suppose that A is a homomorphic image of a Cohen–Macaulay ring. Then the following are equivalent*:
(a) *$e_1(Q) = 0$;*
(b) *A/U is Cohen–Macaulay and $\dim U \leqslant d - 2$.*

Note that unless A is a homomorphic image of a Cohen–Macaulay ring, the implication (a) $\Rightarrow$ (b) is not true in general (Remark 2).

In Section 4 we study the problem of when $e_1(Q)$ is independent of the choice of the parameter ideal Q in A. We shall show that A is a quasi-Buchsbaum ring, if A is unmixed and $e_1(Q)$ is constant (Corollary 4.3). The authors conjecture that A is furthermore a Buchsbaum ring, if A is unmixed and $e_1(Q)$ is independent of the choice of parameter ideals Q of A. We will show that this is the case, at least when $e_1(Q) = -1$ or -2 (by Theorem 4.8 or Theorem 4.10, respectively). Goto and Ozeki [**9**] recently solved the conjecture affirmatively.

Another important issue is that of the variability of $e_1(Q)$, sometimes for Q in a same integral closure class, and its role in the structure of the ring. This will be pursued in a sequel paper.

In what follows, unless otherwise specified, let $(A, \mathfrak{m})$ denote a Noetherian local ring with a maximal ideal $\mathfrak{m}$ and $d = \dim A$. Let $\{\mathrm{H}^i_{\mathfrak{m}}(*)\}_{i\in\mathbb{Z}}$ be the local cohomology functors of A with respect to the maximal ideal $\mathfrak{m}$.

2. *The vanishing conjecture*

The purpose of this section is to prove the following, which settles Conjecture 1.1 affirmatively. Throughout let $A = (A, \mathfrak{m})$ be a Noetherian local ring with a maximal ideal $\mathfrak{m}$ and $d = \dim A$.

Theorem 2.1. *Let A be a Noetherian local ring with $d = \dim A > 0$ and let Q be a parameter ideal in A. Then the following are equivalent*:
(a) *A is Cohen–Macaulay*;
(b) *A is unmixed and $e_1(Q) = 0$*;
(c) *A is unmixed and $e_1(Q) \geqslant 0$.*

[†]The terminology is due to the first five authors.

In our proof of Theorem 2.1 the following facts are the key. See [7, Section 3] for the proof.

PROPOSITION 2.2 [7]. *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring with* $d = \dim A \geqslant 2$, *possessing the canonical module* K_A. *Assume that* $\dim A/\mathfrak{p} = d$ *for every* $\mathfrak{p} \in \mathrm{Ass}(A) \setminus \{\mathfrak{m}\}$. *Then the following assertions hold.*

(a) *The local cohomology module* $\mathrm{H}^1_{\mathfrak{m}}(A)$ *is finitely generated.*
(b) *The set* $\mathcal{F} = \{\mathfrak{p} \in \operatorname{Spec} A \mid \dim A_{\mathfrak{p}} > \operatorname{depth}(A_{\mathfrak{p}}) = 1\}$ *is finite.*
(c) *Suppose that the residue class field* $k = A/\mathfrak{m}$ *of* A *is infinite and let* I *be an* $\mathfrak{m}$*-primary ideal in* A. *Then one can choose an element* $a \in I$ *so that* a *is superficial for* I *and* $\dim A/\mathfrak{p} = d - 1$ *for every* $\mathfrak{p} \in \mathrm{Ass}_A(A/aA) \setminus \{\mathfrak{m}\}$.

REMARK 1. Let $(A, \mathfrak{m})$ be a Noetherian local ring with $\dim A = d > 0$. Recall that the unmixed component of (0) in A is $U_A(0) = \bigcap_{\mathfrak{p} \in \mathrm{Assh}(A)} I(\mathfrak{p})$. Since $\mathrm{H}^0_{\mathfrak{m}}(A) = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(A)\setminus\{\mathfrak{m}\}} I(\mathfrak{p})$, we have that $\mathrm{H}^0_{\mathfrak{m}}(A) \subseteq U_A(0)$. If $\mathrm{Ass}(A) \setminus \{\mathfrak{m}\} = \mathrm{Assh}(A)$, then $\mathrm{H}^0_{\mathfrak{m}}(A) = U_A(0)$.

Proof of Theorem 2.1. (a) ⇒ (b) ⇒ (c) are clear. In order to show that (c) ⇒ (a), we may assume that A is a complete unmixed local ring with $d \geqslant 2$ and infinite residue field. Let $Q = (a_1, \ldots, a_d)$. We use induction on d.

Let $d = 2$. Then $Q = (a_1, a_2)$, where we may assume that $a = a_1$ is a superficial element. Let $S = A/aA$ and let $q = QS$. Then $\dim S = 1$ and by [8, Lemma 2.2] we have

$$-\ell_A(\mathrm{H}^0_{\mathfrak{m}}(S)) = e_1(q) = e_1(Q) - \ell_A(0 :_A a) = e_1(Q) \geqslant 0.$$

Hence $\mathrm{H}^0_{\mathfrak{m}}(S) = (0)$. Therefore S is Cohen–Macaulay and so is A.

Suppose that $d \geqslant 3$. Then there exists $a \in Q$ such that $\mathrm{Ass}(A/aA) \subseteq \mathrm{Assh}(A/aA) \cup \{\mathfrak{m}\}$ (Proposition 2.2(c)). Let $S = A/aA$ and $q = QS$. Note that S is not necessarily unmixed. Let $U = U_S(0)$, $\overline{S} = S/U$ and $\overline{q} = q\overline{S}$. Then $\overline{S}$ is unmixed of dimension $d - 1$. Since $e_1(\overline{q}) = e_1(q) = e_1(Q) \geqslant 0$, by the induction hypothesis $\overline{S}$ is Cohen–Macaulay, that is, $\mathrm{H}^i_{\mathfrak{m}}(\overline{S}) = (0)$ for all $0 \leqslant i \leqslant d - 2$.

From the exact sequence $0 \to U \to S \to \overline{S} \to 0$, we get a long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{H}^0_{\mathfrak{m}}(U) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(S) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(\overline{S}) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(U) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(S) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(\overline{S}) \longrightarrow \cdots \\ &\longrightarrow \mathrm{H}^{d-2}_{\mathfrak{m}}(U) \longrightarrow \mathrm{H}^{d-2}_{\mathfrak{m}}(S) \longrightarrow \mathrm{H}^{d-2}_{\mathfrak{m}}(\overline{S}). \end{aligned}$$

Since $\mathrm{H}^0_{\mathfrak{m}}(S) = U$ (Remark 1), we have $\mathrm{H}^i_{\mathfrak{m}}(U) = (0)$ for all $i \geqslant 1$. Therefore

$$\mathrm{H}^i_{\mathfrak{m}}(S) = (0) \quad \text{for all } 1 \leqslant i \leqslant d - 2.$$

From the exact sequence $0 \to A \xrightarrow{\cdot a} A \to S \to 0$, we get

$$\begin{aligned} 0 &\longrightarrow \mathrm{H}^0_{\mathfrak{m}}(A) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(A) \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(S) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(A) \xrightarrow{\cdot a} \mathrm{H}^1_{\mathfrak{m}}(A) \longrightarrow 0 \longrightarrow \cdots \\ &\longrightarrow 0 \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(A) \xrightarrow{\cdot a} \mathrm{H}^i_{\mathfrak{m}}(A) \longrightarrow 0 \longrightarrow \cdots \longrightarrow \mathrm{H}^{d-2}_{\mathfrak{m}}(S) = 0 \longrightarrow \mathrm{H}^{d-1}_{\mathfrak{m}}(A) \xrightarrow{\cdot a} \mathrm{H}^{d-1}_{\mathfrak{m}}(A). \end{aligned}$$

Since A has a non-zero divisor, it follows that $\mathrm{H}^0_{\mathfrak{m}}(A) = (0)$. The epimorphism $\mathrm{H}^1_{\mathfrak{m}}(A) \to \mathrm{H}^1_{\mathfrak{m}}(A) \to 0$ implies that $\mathrm{H}^1_{\mathfrak{m}}(A) = a\mathrm{H}^1_{\mathfrak{m}}(A)$. Since $\mathrm{H}^1_{\mathfrak{m}}(A)$ is finitely generated (Proposition 2.2(a)), it follows that $\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$. For $2 \leqslant i \leqslant d - 1$ we obtain $\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ because, for every $x \in \mathrm{H}^i_{\mathfrak{m}}(A)$, some power of a annihilates x. □

Let us discuss some consequences of Theorem 2.1.

Lemma 2.3. *Let A be a Noetherian local ring of dimension $d > 0$. Let Q be a parameter ideal of A. Suppose that $U = U_A(0) \neq (0)$. Let $C = A/U$. Then the following assertions hold*:

(a) $\dim U < \dim A$;

(b) *we have*

$$e_1(Q) = \begin{cases} e_1(QC) & \text{if } \dim U \leqslant d-2 \\ e_1(QC) - s_0 & \text{if } \dim U = d-1, \end{cases}$$

where $s_0 \geqslant 1$ is the multiplicity of $\bigoplus_n U/(Q^{n+1} \cap U)$;

(c) $e_1(Q) \leqslant e_1(QC)$ *with equality if and only if* $\dim U \leqslant d-2$.

Proof. (b) We write

$$\ell_A(U/(Q^{n+1} \cap U)) = s_0\binom{n+t}{t} - s_1\binom{n+t-1}{t-1} + \ldots + (-1)^t s_t$$

for $n \gg 0$ with integers $\{s_i\}_{0 \leqslant i \leqslant t}$, where $t = \dim U$. Then the claim follows from the exact sequence $0 \to U \to A \to C \to 0$ of A–modules, which gives

$$\ell_A(A/Q^{n+1}) = \ell_A(C/Q^{n+1}C) + \ell_A(U/(Q^{n+1} \cap U))$$

for all $n \geqslant 0$.

(c) follows from (b) and the fact that $s_0 \geqslant 1$. □

The following results are due to [12]. We include an independent proof.

Corollary 2.4 [12]. *Let A be a Noetherian local ring of dimension $d > 0$. Let Q be a parameter ideal of A. Then the following assertions hold*:

(a) $e_1(Q) \leqslant 0$;

(b) *if* $\operatorname{depth}(A) = d-1$, *then* $e_1(Q) < 0$.

Proof. (a) We may assume that A is complete. Let $U = U_A(0)$ and $C = A/U$. Then, by Lemma 2.3, we have $e_1(Q) \leqslant e_1(QC)$. Thus we may also assume that A is unmixed. Hence the claim follows from Theorem 2.1.

(b) We may assume that the residue field $A/\mathfrak{m}$ is infinite. If $d = 1$, by [8, Lemma 2.4(1)], we have $e_1(Q) = -\ell_A(H^0_{\mathfrak{m}}(A)) < 0$, where the last inequality follows from the fact that $\operatorname{depth}(A) = 0$. Suppose that $d \geqslant 2$. Let $a_1, \ldots, a_{d-1} \in Q$ be a superficial sequence and let $\overline{A} = A/(a_1, \ldots, a_{d-1})$ and $\overline{Q} = Q/(a_1, \ldots, a_{d-1})$. Since $\operatorname{depth}(A) = d-1$, we have that $e_1(Q) = e_1(\overline{Q}) = -\ell_A(H^0_{\mathfrak{m}}(\overline{A})) < 0$. □

Proposition 2.5. *Let A be a Noetherian local ring of dimension $d \geqslant 2$. Let $U = U_A(0)$. Suppose that A/U is Cohen–Macaulay and* $\dim U \leqslant d-2$. *Then $e_1(Q) = 0$ for every parameter ideal Q of A.*

Proof. Let $C = A/U$. Since $\dim U \leqslant d-2$, we get $e_1(Q) = e_1(QC)$ (Lemma 2.3). Since C is Cohen–Macaulay, we have $e_1(QC) = 0$ (Theorem 2.1), which completes the claim. □

Theorem 2.6. *Let A be a Noetherian local ring of dimension $d \geqslant 2$. Suppose that A is a homomorphic image of a Cohen–Macaulay ring. Let $U = U_A(0)$ and let Q be a parameter ideal* *of A. Then the following are equivalent*:

(a) $e_1(Q) = 0$;
(b) *A/U is Cohen–Macaulay and* $\dim U \leqslant d - 2$.

Proof. It is enough to prove that (a) $\Rightarrow$ (b). Let $C = A/U$. Then C is an unmixed local ring because C is a homomorphic image of a Cohen–Macaulay ring and $\dim C/P = d$ for all $P \in \mathrm{Ass}(C)$. If $U = (0)$, then A is unmixed. Therefore A is Cohen–Macaulay by Theorem 2.1. Suppose that $U \neq (0)$. We show that $\dim U \leqslant d - 2$; suppose not, that is, $\dim U = d - 1$. By Lemma 2.3, we get

$$e_1(Q) = e_1(QC) - s_0,$$

where $s_0 \geqslant 1$. On the other hand, $e_1(QC) \leqslant 0$ by Corollary 2.4(a). Hence $e_1(Q) < 0$, which is a contradiction. Therefore $\dim U \leqslant d - 2$. Now Lemma 2.3 implies that $e_1(QC) = e_1(Q) = 0$. By Theorem 2.1, we see that C is a Cohen–Macaulay ring. □

The following corollary gives a characterization of Cohen–Macaulayness.

COROLLARY 2.7. *Let A be a Noetherian local ring of dimension $d > 0$. Let Q be a parameter ideal in A. Suppose that $e_i(Q) = 0$ for all $1 \leqslant i \leqslant d$. Then A is a Cohen–Macaulay ring.*

Proof. We may assume that A is complete. Let $U = U_A(0)$. By Theorem 2.1 it is enough to show that $U = (0)$. Suppose that $U \neq (0)$ and let $C = A/U$. Since $e_1(Q) = 0$, we have that C is Cohen–Macaulay and $\dim U \leqslant d - 2$ (Theorem 2.6). From the exact sequence $0 \to U \to A \to C \to 0$ of A-modules, we get

$$\ell_A(A/Q^{n+1}) = \ell_A(C/Q^{n+1}C) + \ell_A(U/(Q^{n+1} \cap U))$$

for all $n \geqslant 0$. By assumption, for $n \gg 0$ we have

$$\ell_A(A/Q^{n+1}) = e_0(Q)\binom{n+d}{d}.$$

Also, since C is Cohen–Macaulay, for $n \gg 0$ we have

$$\ell_A(C/Q^{n+1}C) = e_0(QC)\binom{n+d}{d}.$$

Since $e_0(Q) = e_0(QC)$, for $n \gg 0$ we obtain

$$\ell_A(U/(Q^{n+1} \cap U)) = 0,$$

that is, $U \subseteq Q^{n+1}$ for $n \gg 0$. Thus $U = (0)$, which is a contradiction. □

The implication (a) $\Rightarrow$ (b) in Theorem 2.6 is not true in general without the assumption that A is a homomorphic image of a Cohen–Macaulay ring.

REMARK 2. Let R be a Noetherian equicharacteristic complete local ring and assume that $\mathrm{depth}(R) > 0$. Then one can construct a Noetherian local integral domain $(A, \mathfrak{m})$ such that $R = \hat{A}$, where $\hat{A}$ denotes the $\mathfrak{m}$-adic completion of A (see [**11**]). For example, let $k[[X, Y, Z, W]]$ be the formal power series ring over a field k and consider the local ring

$$R = k[[X, Y, Z, W]]/(X) \cap (Y, Z, W).$$

We can choose a Noetherian local integral domain $(A, \mathfrak{m})$ so that $R = \hat{A}$. Then $e_1(Q) = 0$ for every parameter ideal Q in A, since $e_1(Q) = e_1(QR) = 0$; but A is not Cohen–Macaulay because $R = \hat{A}$ is not Cohen–Macaulay.

3. *Vasconcelos rings*

Throughout this section let $A = (A, \mathfrak{m})$ be a Noetherian local ring with a maximal ideal $\mathfrak{m}$ and $d = \dim A$. Our purpose is to develop a theory of Vasconcelos rings. Let us begin with the definition.

Definition 3.1. A Noetherian local ring A is a *Vasconcelos* ring if either $\dim A = 0$, or $\dim A > 0$ and $e_1(Q) = 0$ for some parameter ideal Q in A.

A Cohen–Macaulay local ring is a Vasconcelos ring.

Proposition 3.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with $\dim A = d$. Then we have the following.*

(a) *A 1-dimensional Vasconcelos ring is Cohen–Macaulay.*

(b) *An unmixed Vasconcelos ring is Cohen–Macaulay.*

(c) *Suppose that $d \geqslant 2$. Then A is a Vasconcelos ring if and only if $A/\mathrm{H}^0_{\mathfrak{m}}(A)$ is a Vasconcelos ring.*

Proof. (a) There exists a parameter ideal Q of A such that $e_1(Q) = 0$. By [8, 2.4 (1)], we have $0 = e_1(Q) = -\ell_A(\mathrm{H}^0_{\mathfrak{m}}(A))$, which shows that A is Cohen–Macaulay.

(b) This follows from Theorem 2.1.

(c) Let $B = A/\mathrm{H}^0_{\mathfrak{m}}(A)$. Then $e_1(QB) = e_1(Q)$ for every parameter ideal Q in A. Hence A is a Vasconcelos ring if and only if B is a Vasconcelos ring. □

Let A be a Noetherian local ring of dimension $d \geqslant 1$ and let M be a finite A–module. We define

$$\mathrm{Assh}(M) = \{\mathfrak{p} \in \mathrm{Ass}(M) \mid \dim A/\mathfrak{p} = \dim M\}.$$

Let $(0_M) = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(M)} M(\mathfrak{p})$ be a primary decomposition, where $M(\mathfrak{p})$ is $\mathfrak{p}$-primary. The *unmixed component* $U_M(0)$ of (0) in M is given by

$$U_M(0) = \bigcap_{\mathfrak{p} \in \mathrm{Assh}(M)} M(\mathfrak{p}).$$

Note that, for any $\mathfrak{p} \in \mathrm{Assh}(M)$, we have $U_M(0)_{\mathfrak{p}} = (0)$.

Lemma 3.3. *Let A be a Noetherian local ring of dimension $d \geqslant 1$. Let M be a finite A-module and let L be a submodule of M. Suppose that $\dim L \leqslant \dim M - 1$ and $\mathrm{Ass}(M/L) \subseteq \mathrm{Assh}(M)$. Then $L = U_M(0)$.*

Proof. Let $\dim M = g$ and $U = U_M(0)$. Since $\dim L \leqslant g - 1$, we have $L_{\mathfrak{p}} = (0)$ for every $\mathfrak{p} \in \mathrm{Assh}(M)$. Note that, for each $\mathfrak{p} \in \mathrm{Assh}(M)$, there exists $s \in A \setminus \mathfrak{p}$ such that $sL = (0)$. Then $L \subseteq M(\mathfrak{p})$ for every $\mathfrak{p} \in \mathrm{Assh}(M)$ because s is a non-zero divisor on $M/M(\mathfrak{p})$ and $M(\mathfrak{p})$

is $\mathfrak{p}$-primary. This means that

$$L \subseteq \bigcap_{\mathfrak{p} \in \mathrm{Assh}(M)} M(\mathfrak{p}) = U.$$

Now suppose that $L \subsetneq U$. Then exists $\mathfrak{q} \in \mathrm{Ass}(U/L)$. By assumption $\mathfrak{q} \in \mathrm{Assh}(M)$. Hence $L_{\mathfrak{q}} = (0) = U_{\mathfrak{q}}$, so that $(U/L)_{\mathfrak{q}} = (0)$, which is a contradiction. □

Here is a basic characterization of Vasconcelos rings.

Theorem 3.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geqslant 2$. Let $\hat{A}$ be the $\mathfrak{m}$-adic completion of A. The following are equivalent*:

(a) *A is a Vasconcelos ring*;
(b) *$e_1(Q) = 0$ for every parameter ideal Q in A*;
(c) *$\hat{A}/U_{\hat{A}}(0)$ is a Cohen–Macaulay ring and $\dim_{\hat{A}} U_{\hat{A}}(0) \leqslant d-2$*;
(d) *there exists a proper ideal I of $\hat{A}$ such that $\hat{A}/I$ is a Cohen–Macaulay ring of dimension d and $\dim_{\hat{A}} I \leqslant d-2$.*

When this is the case, $\hat{A}$ is a Vasconcelos ring, $\mathrm{H}^{d-1}_{\mathfrak{m}}(A) = (0)$, and the canonical module $\mathrm{K}_{\hat{A}}$ of $\hat{A}$ is a Cohen–Macaulay $\hat{A}$-module.

Proof. (c) ⇒ (b) follows from Proposition 2.5, and (a) ⇒ (c) follows from Theorem 2.6. The implications (b) ⇒ (a) and (c) ⇒ (d) are trivial. Finally (d) ⇒ (c) follows from Lemma 3.3.

To see the last assertions, let $U = U_{\hat{A}}(0)$ and let $\hat{\mathfrak{m}} = \mathfrak{m}\hat{A}$ be the maximal ideal of $\hat{A}$. Then $\hat{A}/U$ is a Cohen–Macaulay ring and $\dim_{\hat{A}} U \leqslant d-2$. In particular, $\mathrm{H}^{d-1}_{\hat{\mathfrak{m}}}(\hat{A}/U) = (0) = \mathrm{H}^{d-1}_{\hat{\mathfrak{m}}}(U)$. Hence $\mathrm{H}^{d-1}_{\hat{\mathfrak{m}}}(\hat{A}) = (0)$. Moreover, we have $\mathrm{H}^{d}_{\hat{\mathfrak{m}}}(\hat{A}) \cong \mathrm{H}^{d}_{\hat{\mathfrak{m}}}(\hat{A}/U)$, which means that $\mathrm{K}_{\hat{A}} \cong \mathrm{K}_{\hat{A}/U}$. Since $\mathrm{K}_{\hat{A}/U}$ is Cohen–Macaulay, so is $\mathrm{K}_{\hat{A}}$. □

Corollary 3.5. *Let A be a Vasconcelos ring of dimension $d \geqslant 1$. If x is a non-zero divisor in A, then A/xA is a Vasconcelos ring.*

Proof. We may assume that A is complete and that $d \geqslant 2$. Let $U = U_A(0)$ and $C = A/U$. By Theorem 3.4, we see that C is a d-dimensional Cohen–Macaulay ring and $\dim U \leqslant d-2$. Then $\dim U/xU \leqslant d-3$. Since x is C-regular, it follow that C/xC is Cohen–Macaulay. By Theorem 3.4, we see that A/xA is a Vasconcelos ring. □

Example 3.6. *Let $(R, \mathfrak{n})$ be a Cohen–Macaulay local ring of dimension $d \geqslant 3$. Let x be a non-zero divisor of R and let $\mathfrak{a}$ be an R-ideal such that $\mathrm{ht}(\mathfrak{a}) \geqslant 3$ and $x \notin \mathfrak{a}$. Let $A = R/(xR \cap \mathfrak{a})$. Then A is a non-Cohen–Macaulay Vasconcelos ring of dimension $d-1 \geqslant 2$.*

Proof. Consider the exact sequence $0 \to R/(\mathfrak{a} : x) \to A \to R/xR \to 0$. Then A/xA is Cohen–Macaulay. Also we have

$$\dim_A(xA) = \dim R/(\mathfrak{a} : x) \leqslant \dim R/\mathfrak{a} \leqslant d-3 = \dim A - 2.$$

By Theorem 3.4, we see that A is a non-Cohen–Macaulay Vasconcelos ring with $\dim A = d-1 \geqslant 2$. □

Example 3.7. *Let $(R, \mathfrak{n})$ be a Cohen–Macaulay local ring with $\dim R = d \geqslant 2$. Let M be a finitely generated R-module with $\dim_R M \leqslant d-2$. Then the idealization $A = R \ltimes M$ is a*

Vasconcelos ring with $\dim A = d$*. Also we have* $\mathrm{Assh}(A) = \mathrm{Min}(A)$*. Moreover, if* $R = k[[x, y]]$ *and* $M = R/(x, y)^2$*, then* $A = R \ltimes M$ *is not quasi-Buchsbaum.*

Proof. Let $N = (0) \times M$. Consider the exact sequence $0 \to N \to A \xrightarrow{\varepsilon} R \to 0$, where $\varepsilon(x) = r$ for each $x = (r, m) \in A$. Then $A/N \simeq R$ is Cohen–Macaulay. Also $\dim_A N = \dim_R M \leqslant d - 2$. By Theorem 3.4, we see that A is a Vasconcelos ring with $\dim A = d$. Moreover, we have

$$\mathrm{Assh}(A) = \{\mathfrak{p} \times M \mid \mathfrak{p} \in \mathrm{Assh}(R)\} = \mathrm{Min}(A). \qquad \square$$

We are now interested in the question of how Vasconcelos rings are preserved under flat base changes.

THEOREM 3.8. *Let* $(A, \mathfrak{m})$ *and* $(B, \mathfrak{n})$ *be Noetherian local rings and let* $\varphi : A \to B$ *be a flat local homomorphism. Then the following assertions hold true.*

(a) *Suppose that* A *is a Vasconcelos ring and* $B/\mathfrak{m}B$ *is a Cohen–Macaulay ring. Then* B *is a Vasconcelos ring.*

(b) *Suppose that* B *is a homomorphic image of a Cohen–Macaulay ring. If* B *is a Vasconcelos ring and* $\mathrm{Ass}_B(B/\mathfrak{p}B) = \mathrm{Assh}_B(B/\mathfrak{p}B)$ *for every* $\mathfrak{p} \in \mathrm{Assh}(A)$*, then* A *is a Vasconcelos ring and* $B/\mathfrak{m}B$ *is a Cohen–Macaulay ring.*

Proof. Let $U = U_A(0)$. Note that $(A/U) \otimes_A B$ is Cohen–Macaulay if and only if both A/U and $B/\mathfrak{m}B$ are Cohen–Macaulay.

(a) We may assume that A is complete and $\dim A = d > 0$. Since A is a Vasconcelos ring, by Theorem 3.4, we see that A/U is Cohen–Macaulay and $\dim_A U \leqslant d - 2$. Since $B/\mathfrak{m}B$ is also Cohen–Macaulay by assumption, it follows that $B/UB \simeq (A/U) \otimes_A B$ is Cohen–Macaulay. Moreover, we have

$$\begin{aligned}\dim_B(UB) &= \dim_B(U \otimes_A B) = \dim_A U + \dim_B(B/\mathfrak{m}B)\\ &\leqslant d - 2 + \dim_B(B/\mathfrak{m}B) = \dim B - 2.\end{aligned}$$

Hence, by Proposition 2.5, we have that B is a Vasconcelos ring.

(b) Suppose that A is Cohen–Macaulay. It is enough to show that B is Cohen–Macaulay. Let $\mathfrak{P} \in \mathrm{Ass}(B)$ and $\mathfrak{p} = \mathfrak{P} \cap A$. Then $\mathfrak{p} \in \mathrm{Ass}(A) = \mathrm{Assh}(A)$. We have $\mathfrak{P} \in \mathrm{Ass}(B/\mathfrak{p}B) = \mathrm{Assh}(B/\mathfrak{p}B)$ by assumption. This means that

$$\dim B/\mathfrak{P} = \dim B/\mathfrak{p}B = \dim_A(A/\mathfrak{p}) + \dim_B(B/\mathfrak{m}B) = d + \dim_B(B/\mathfrak{m}B) = \dim B.$$

Therefore $\mathrm{Ass}(B) = \mathrm{Assh}(B)$, which shows that B is an unmixed Vasconcelos ring. By Proposition 3.2(b), we see that B is Cohen–Macaulay.

Now suppose that A is not Cohen–Macaulay. We claim that $UB = U_B(0)$, the unmixed component of (0) in B. Let $\mathfrak{P} \in \mathrm{Ass}(B/UB)$. Then $\mathfrak{P} \in \mathrm{Ass}(B/\mathfrak{p}B)$ for some $\mathfrak{p} \in \mathrm{Ass}(A/U) = \mathrm{Assh}(A)$. By assumption we have $\mathfrak{P} \in \mathrm{Assh}(B/\mathfrak{p}B)$. This means that $\dim B/\mathfrak{P} = \dim B/\mathfrak{p}B = \dim B$. Hence $\mathrm{Ass}(B/UB) = \mathrm{Assh}(B)$, which proves the claim. Now since B is a Vasconcelos ring, by Theorem 2.6 we have that B/UB is Cohen–Macaulay and $\dim_B UB \leqslant \dim B - 2$. Therefore both A/U and $B/\mathfrak{m}B$ are Cohen–Macaulay because $(A/U) \otimes_A B \simeq B/UB$ is Cohen–Macaulay. Also $\dim_B UB \leqslant \dim B - 2$ implies that $\dim U \leqslant d - 2$. In particular, A is a Vasconcelos ring by Proposition 2.5. $\square$

COROLLARY 3.9. *Let* $(A, \mathfrak{m})$ *be a Vasconcelos ring and let* $R = A[X_1, X_2, \ldots, X_n]$ *be the polynomial ring. Then* R_P *is a Vasconcelos ring for every* $P \in \mathrm{V}(\mathfrak{m}R)$*.*

Corollary 3.10. *Let $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ be Noetherian local rings that are homomorphic images of Cohen–Macaulay rings. Let $\varphi : A \to B$ be a flat local homomorphism. Then the following two conditions are equivalent*:

(a) *A is a Vasconcelos ring and $B/\mathfrak{m}B$ is a Cohen–Macaulay ring*;

(b) *B is a Vasconcelos ring and $\mathrm{Ass}_B(B/\mathfrak{p}B) = \mathrm{Assh}_B(B/\mathfrak{p}B)$ for every $\mathfrak{p} \in \mathrm{Assh}(A)$.*

Proof. (a) $\Rightarrow$ (b) Let $\mathfrak{p} \in \mathrm{Assh}(A)$. Then, since $\mathfrak{p} \in \mathrm{Ass}(A/U)$, we have that $\mathrm{Ass}_B(B/\mathfrak{p}B) \subseteq \mathrm{Ass}_B(B \otimes_A A/U)$. Hence $\dim B/\mathfrak{P} = \dim B$ for every $\mathfrak{P} \in \mathrm{Ass}_B(B/\mathfrak{p}B)$, since $B \otimes_A A/U$ is a Cohen–Macaulay ring with $\dim(B \otimes_A A/U) = \dim B$. □

Next we show that quasi-unmixed Vasconcelos rings behave well under localization.

Proposition 3.11. *Let A be a Noetherian local ring of dimension d. Suppose that A is a homomorphic image of a Cohen–Macaulay ring and $\mathrm{Assh}(A) = \mathrm{Min}(A)$. If A is a Vasconcelos ring, then $A_\mathfrak{p}$ is a Vasconcelos ring for every $\mathfrak{p} \in \mathrm{Spec}\, A$.*

Proof. We may assume that A is not a Cohen–Macaulay ring. Let $U = U_A(0)$. Then $U \neq (0)$ by Theorem 2.6. Let $\mathfrak{p} \in \mathrm{Spec}\, A$. Note that $U \subseteq \mathfrak{p}$. Since $(A/U)_\mathfrak{p}$ is a Cohen–Macaulay ring, we may assume that $U_\mathfrak{p} \neq (0)$. Let $\mathfrak{a} = (0) : U$. Then $\mathfrak{a} \neq A$ and $\mathrm{ht}_A \mathfrak{a} \geqslant 2$, since $\dim U \leqslant d - 2$ and A is catenary and equidimensional. Hence $\mathrm{ht}_{A_\mathfrak{p}} \mathfrak{a}A_\mathfrak{p} \geqslant 2$ and $\dim A_\mathfrak{p}/\mathfrak{a}A_\mathfrak{p} \leqslant \dim A_\mathfrak{p} - 2$. Therefore $A_\mathfrak{p}$ is a Vasconcelos ring by Theorem 2.6. □

Corollary 3.12. *Suppose that A is a Vasconcelos ring and $\mathrm{Assh}(\hat{A}) = \mathrm{Min}(\hat{A})$. Then $\hat{A}_\mathfrak{p}$ is a Vasconcelos ring for every $\mathfrak{p} \in \mathrm{Spec}\, \hat{A}$.*

Suppose that $d > 0$ and let Q be a parameter ideal in A. We denote by $R = \mathcal{R}(Q) = A[Qt]$ and $G = \mathrm{G}(Q)$ the Rees algebra and the associated graded ring of Q, respectively. Let $\mathfrak{M} = \mathfrak{m}R + R_+$ be the graded maximal ideal in R.

Theorem 3.13. *Let A be a Noetherian local ring with dimension $d > 0$. With the above notation we have the following*:

(a) *A is a Vasconcelos ring if and only if $G_\mathfrak{M}$ is a Vasconcelos ring.*

(b) *Suppose that A is a homomorphic image of a Cohen–Macaulay ring. If A is a Vasconcelos ring, then $R_\mathfrak{M}$ is a Vasconcelos ring.*

Proof. (a) Let $Q = (a_1, a_2, \ldots, a_d)$ be a parameter ideal in A and let $f_i = a_i t$ for each $1 \leqslant i \leqslant d$. Then $G_+ = (f_1, f_2, \ldots, f_d)G$ and $f_1, f_2, \ldots, f_d$ forms a linear system of parameters in the graded ring G. Furthermore, we have

$$\ell_G(G/[(f_1, f_2, \ldots, f_d)G]^{n+1}) = \ell_A(A/Q^{n+1})$$

for all $n \geqslant 0$. Hence $e_1(Q) = e_1((f_1, f_2, \ldots, f_d)G_\mathfrak{M})$ and the conclusion follows from Theorem 3.4.

(b) We may assume that A is not a Cohen–Macaulay ring. Let $U = U_A(0)$ and $B = A/U$. Then, by Theorem 2.6, we see that B is a Cohen–Macaulay ring and $\dim U \leqslant d - 2$. Consider the canonical epimorphism $\varphi : R \to \mathcal{R}(QB)$ and let $K = \mathrm{Ker}\ \varphi$. Then $K \subseteq UA[t]$. Let $\mathfrak{a} = (0) : U$. Let $P \in \mathrm{Ass}_R(K)$ and let $\mathfrak{p} = P \cap A$. Then $\mathfrak{a} \subseteq \mathfrak{p}$, since $\mathfrak{a}K = (0)$. Note that P is the kernel of the canonical epimorphism $\phi : R \to \mathcal{R}([Q + \mathfrak{p}]/\mathfrak{p})$. If $Q \subseteq \mathfrak{p}$, then $\mathfrak{p} = \mathfrak{m}$ and

$\dim R/P = \dim \mathcal{R}([Q+\mathfrak{p}]/\mathfrak{p}) = 0$. If $Q \nsubseteq \mathfrak{p}$, then we have $\dim R/P = \dim A/\mathfrak{p} + 1 \leqslant \dim A/\mathfrak{a} + 1 \leqslant d-1$. Therefore $\dim R/P \leqslant d-1$ for every $P \in \mathrm{Ass}_R(K)$, which shows that $\dim_R K \leqslant d-1 = (d+1)-2$. Since $\mathcal{R}(QB)$ is a Cohen–Macaulay ring, we have that $R_{\mathfrak{M}}$ is a Vasconcelos ring by Theorem 2.6. □

We close this section with an application to sequentially Cohen–Macaulay rings. We refer the reader to [**6**] for the definition and details. See also [**2**, **14**, **15**]. We use the following characterization.

Proposition 3.14 [**14**]. *Let A be a Noetherian local ring. Then M is a sequentially Cohen–Macaulay A-module if and only if M admits a Cohen–Macaulay filtration, that is, a family $\mathcal{M} = \{M_i\}_{0 \leqslant i \leqslant t}$ $(t > 0)$ of A-submodules of M with*

$$M_0 = (0) \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M_t = M,$$

such that we have the following:
(i) $\dim_A M_{i-1} < \dim_A M_i$;
(ii) M_i/M_{i-1} *is a Cohen–Macaulay A-module for all* $1 \leqslant i \leqslant t$.

The result below follows from Proposition 2.5.

Corollary 3.15. *Let A be a Noetherian local ring with dimension $d > 0$. Suppose that A is a sequentially Cohen–Macaulay ring. If $\dim A/\mathfrak{p} \neq d-1$ for every $\mathfrak{p} \in \mathrm{Ass}(A)$, then A is a Vasconcelos ring.*

Proof. We may assume that A is not a Cohen–Macaulay ring. Let $U = U_A(0)$. In the notation of Proposition 3.14, we have that $t \geqslant 2$ and $M_{t-1} = U$ (see [**6**, Theorem 2.3]). Therefore $\dim U \leqslant d-2$, since $\dim A/\mathfrak{p} \neq d-1$ for all $\mathfrak{p} \in \mathrm{Ass}(A)$. Thus A is a Vasconcelos ring by Proposition 2.5, because the ring A/U is Cohen–Macaulay. □

Lastly we show that the converse of Corollary 3.15 holds, when $\dim A = 3$ and A is a homomorphic image of a Cohen–Macaulay ring.

Proposition 3.16. *Let A be a Noetherian local ring of dimension 3, which is a homomorphic image of a Cohen–Macaulay ring. If A is a Vasconcelos ring, then A is a sequentially Cohen–Macaulay ring with $\dim A/\mathfrak{p} \neq 2$ for every $\mathfrak{p} \in \mathrm{Ass}(A)$.*

Proof. Let $U = U_A(0)$. We may assume that $U \neq (0)$. Then A/U is a Cohen–Macaulay ring and $\dim U \leqslant 1$ by Theorem 2.6. Hence $\dim A/\mathfrak{p} \neq 2$ for all $\mathfrak{p} \in \mathrm{Ass}(A)$. Let $W = \mathrm{H}^0_{\mathfrak{m}}(A)$. Then $W \subseteq U$ and $W = \mathrm{H}^0_{\mathfrak{m}}(U)$ because $\mathrm{depth}(A/U) > 0$. Therefore, if $\dim U = 1$ and $\mathrm{depth}(A) = 1$, then $(0) \subsetneq U \subsetneq A$ is a Cohen–Macaulay filtration of A. If $\dim U = 1$ but $\mathrm{depth}(A) = 0$, then $(0) \subsetneq W \subsetneq U \subsetneq A$ is a Cohen–Macaulay filtration of A. If $\dim U = 0$, then $(0) \subsetneq U \subsetneq A$ is a Cohen–Macaulay filtration of A. Thus A is a sequentially Cohen–Macaulay ring by Proposition 3.14. □

Corollary 3.17. *Suppose that A is a Vasconcelos ring of dimension 3. Then the completion $\hat{A}$ of A is a sequentially Cohen–Macaulay ring.*

4. *Rings with $e_1(Q)$ constant*

In this section we study the problem of when $e_1(Q)$ is independent of the choice of parameter ideals Q in A. Part of the motivation comes from the fact that Buchsbaum rings have this property. We establish here that when $e_1(Q) = -1$ or $e_1(Q) = -2$ and A is unmixed, then A is indeed Buchsbaum. (The question of the variability of $e_1(Q)$ will be considered in another paper.)

Let $(A, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and let $d = \dim A > 0$. Assume that A is a homomorphic image of a Gorenstein ring. Then A contains a system of parameters $x_1, x_2, \ldots, x_d$ that forms a strong d-sequence in A, that is, the sequence $x_1^{n_1}, x_2^{n_2}, \ldots, x_d^{n_d}$ is a d-sequence in A for all integers $n_1, n_2, \ldots, n_d \geqslant 1$ (see [**3**, Theorem 2.6] or [**10**, Theorem 4.2] for the existence of such systems of parameters). For each integer $q \geqslant 1$ let $\Lambda_q(A)$ be the set of values $e_1(Q)$, where Q runs over the parameter ideals of A such that $Q \subseteq \mathfrak{m}^q$ and $Q = (a_1, a_2, \ldots, a_d)$ with $a_1, a_2, \ldots, a_d$ a d-sequence. We then have $\Lambda_q(A) \neq \emptyset$, $\Lambda_{q+1}(A) \subseteq \Lambda_q(A)$ for all $q \geqslant 1$ and $\alpha \leqslant 0$ for every $\alpha \in \Lambda_q(A)$ (Corollary 2.4(a)).

LEMMA 4.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geqslant 2$, which is a homomorphic image of a Gorenstein ring. Assume that $\Lambda_q(A)$ is a finite set for some integer $q \geqslant 1$ and let $\ell = -\min \Lambda_q(A)$. Suppose that* $\operatorname{Ass}(A) = \operatorname{Assh}(A)$. *Then $\mathfrak{m}^{\ell}\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for all $i \neq d$, whence all the local cohomology modules $\{\mathrm{H}^i_{\mathfrak{m}}(A)\}_{0 \leqslant i < d}$ of A are finitely generated.*

Proof. Let $C = \mathrm{H}^1_{\mathfrak{m}}(A)$. Then C is a finitely generated A-module (Proposition 2.2(a)). Suppose that $d = 2$ and let $\ell' = \ell_A(C)$. Let a, b be a system of parameters of A and assume that a, b is a d-sequence in A. Then the element a is superficial for the ideal $Q = (a, b)$, so that $e_1(Q) = e_1(Q/(a)) = -\ell_A((0) :_C a)$. Therefore, choosing $a, b \in \mathfrak{m}^q$ with $aC = (0)$, we get $-\ell' = e_1(Q) \in \Lambda_q(A)$, whence $\ell' \leqslant \ell$. Thus $\mathfrak{m}^{\ell}C = (0)$, because $\mathfrak{m}^{\ell'}C = (0)$.

Suppose now that $d \geqslant 3$ and that our assertion holds true for $d - 1$. Let

$$\mathcal{F}' = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \neq \mathfrak{m}, \dim A_{\mathfrak{p}} > \operatorname{depth}(A_{\mathfrak{p}}) = 1\}.$$

Then $\mathcal{F}'$ is a finite set (Proposition 2.2(b)). We choose $x \in \mathfrak{m}$ so that

$$x \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass}(A)} \mathfrak{p} \cup \bigcup_{\mathfrak{p} \in \mathcal{F}'} \mathfrak{p}.$$

Let $n \geqslant q$ be an integer such that $x^n \mathrm{H}^1_{\mathfrak{m}}(A) = (0)$ and consider $y = x^n$. Let $B = A/yA$. Then $\dim B = d - 1$ and $\operatorname{Ass}_A(B) \setminus \{\mathfrak{m}\} = \operatorname{Assh}_A(B)$. It follows that $U_B(0) = \mathrm{H}^0_{\mathfrak{m}}(B)$ (see Remark 1). Let $\tilde{B} = B/\mathrm{H}^0_{\mathfrak{m}}(B)$.

Let $q' \geqslant q$ be an integer such that $\mathfrak{n}^{q'} \cap \mathrm{H}^0_{\mathfrak{m}}(B) = (0)$, where $\mathfrak{n}$ denotes the maximal ideal of B. Let $y_2, y_3, \ldots, y_d \in \mathfrak{m}^{q'}$ be a system of parameters for the A-module $\tilde{B}$ and assume that $y_2, y_3, \ldots, y_d$ is a d-sequence in $\tilde{B}$. Since $(y_2, y_3, \ldots, y_d)B \cap \mathrm{H}^0_{\mathfrak{m}}(B) = (0)$, we have that $y_2, y_3, \ldots, y_d$ forms a d-sequence in B also. Then, since y is A-regular, the sequence $y_1 = y, y_2, \ldots, y_d$ forms a d-sequence in A, whence y_1 is superficial for the parameter ideal $Q = (y_1, y_2, \ldots, y_d)$ of A. Consequently

$$e_1((y_2, y_3, \ldots, y_d)\tilde{B}) = e_1((y_2, y_3, \ldots, y_d)B) = e_1(Q) \in \Lambda_q(A),$$

so that $\Lambda_{q'}(\tilde{B}) \subseteq \Lambda_q(A)$. Therefore $\Lambda_{q'}(\tilde{B})$ is a finite set, whence the hypothesis of induction on d yields that $\mathfrak{m}^{\ell''}\mathrm{H}^i_{\mathfrak{m}}(\tilde{B}) = (0)$ for all $i \neq d - 1$, where $\ell'' = -\min \Lambda_{q'}(\tilde{B})$. Hence $\mathfrak{m}^{\ell}\mathrm{H}^i_{\mathfrak{m}}(\tilde{B}) = (0)$ for all $i \neq d - 1$, because $\ell'' \leqslant \ell$.

Now consider the exact sequence

$$\cdots \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(A) \xrightarrow{x^n} \mathrm{H}^1_{\mathfrak{m}}(A) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(B) \longrightarrow \cdots \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(B) \longrightarrow \mathrm{H}^{i+1}_{\mathfrak{m}}(A) \xrightarrow{x^n} \mathrm{H}^{i+1}_{\mathfrak{m}}(A) \longrightarrow \cdots$$

of local cohomology modules. We then have

$$\mathfrak{m}^\ell[(0) :_{\mathrm{H}^{i+1}_\mathfrak{m}(A)} x^n] = (0)$$

for all integers $1 \leqslant i \leqslant d-2$ and $n \geqslant q$, because $\mathfrak{m}^\ell \mathrm{H}^i_\mathfrak{m}(B) = (0)$ for all $1 \leqslant i \leqslant d-2$. Hence $\mathfrak{m}^\ell \mathrm{H}^{i+1}_\mathfrak{m}(A) = (0)$, because

$$\mathrm{H}^{i+1}_\mathfrak{m}(A) = \bigcup_{n \geqslant 1} [(0) :_{\mathrm{H}^{i+1}_\mathfrak{m}(A)} \mathfrak{m}^n].$$

On the other hand we have the embedding $\mathrm{H}^1_\mathfrak{m}(A) \subseteq \mathrm{H}^1_\mathfrak{m}(B)$, since $x^n \mathrm{H}^1_\mathfrak{m}(A) = (0)$. Thus $\mathfrak{m}^\ell \mathrm{H}^1_\mathfrak{m}(A) = (0)$, which completes the proof of the lemma. □

Now let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $\Lambda(A) = \{e_1(Q) \mid Q$ is a parameter ideal in $A\}$. Passing to the completion $\hat{A}$ of A and applying Lemma 4.1, we obtain the following.

PROPOSITION 4.2. *Let $(A, \mathfrak{m})$ be an unmixed Noetherian local ring of dimension $d \geqslant 2$. Assume that $\Lambda(A)$ is a finite set and consider $\ell = -\min \Lambda(A)$. Then $\mathfrak{m}^\ell \mathrm{H}^i_\mathfrak{m}(A) = (0)$ for every $i \neq d$, whence $\mathrm{H}^i_\mathfrak{m}(A)$ is a finitely generated A-module for every $i \neq d$.*

A system of parameters $a_1, a_2, \dots, a_d$ of A is said to be *standard* if it forms a d^+-sequence, that is, $a_1, a_2, \dots, a_d$ forms a strong d-sequence in any order. We have that A possesses a standard system of parameters if and only if A is a generalized Cohen–Macaulay ring, that is, all the local cohomology modules $\{\mathrm{H}^i_\mathfrak{m}(A)\}_{0 \leqslant i < d}$ are finitely generated (see [**17**]).

We say that a parameter ideal Q of A is standard, if it is generated by a standard system of parameters. We have that Q is standard if and only if the equality

$$\ell_A(A/Q) - e_0(Q) = \sum_{i=0}^{d-1} \binom{d-1}{i} h^i(A) := \mathbb{I}(A)$$

holds, where $h^i(A) = \ell_A(\mathrm{H}^i_\mathfrak{m}(A))$ for each $i \in \mathbb{Z}$ (cf. [**17**, Theorem 2.1]). See [**4**, **17**] for details, where the notion of generalized Cohen–Macaulay module is also given and various basic properties of generalized Cohen–Macaulay rings and modules are explored.

Assume that A is a generalized Cohen–Macaulay ring with $d \geqslant 2$ and let

$$s = \sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A).$$

If Q is a parameter ideal of A, then by [**8**, Lemma 2.4] we have that

$$e_1(Q) \geqslant -s,$$

where the equality holds, if Q is standard [**13**, Korollar 3.2].

Therefore, if A is unmixed, $d \geqslant 2$ and $\Lambda(A)$ is a finite set, by Proposition 4.2 we have that

$$\mathfrak{m}^s \mathrm{H}^i_\mathfrak{m}(A) = (0)$$

for all $i \neq d$. This exponent is, however, never the best possible, as we show in the following.

COROLLARY 4.3. *Let $(A, \mathfrak{m})$ be an unmixed Noetherian local ring of dimension $d \geqslant 2$. If $\#\Lambda(A) = 1$, then A is a quasi-Buchsbaum ring, that is, $\mathfrak{m}\mathrm{H}^i_\mathfrak{m}(A) = (0)$ for all $i \neq d$.*

To prove Corollary 4.3 we need the lemmas below.

LEMMA 4.4. *Suppose that* $(A, \mathfrak{m})$ *is a Noetherian local ring of dimension* $d = 2$ *and* $\operatorname{depth}(A) = 1$. *Assume that* $\mathrm{H}^1_{\mathfrak{m}}(A)$ *is finitely generated. Let* Q *be a parameter ideal of* A. *Then the following three conditions are equivalent*:

(a) $e_1(Q) = -\ell_A(\mathrm{H}^1_{\mathfrak{m}}(A))$;
(b) $Q\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$;
(c) Q *is standard.*

Proof. (c) ⇒ (a) See [**8**, Lemma 2.4(2)].
(b) ⇔ (c) See [**17**, Corollary 3.7].
(a) ⇒ (b) We may assume that the field $A/\mathfrak{m}$ is infinite. Let $Q = (a, b)$ be such that each one of a, b is a superficial element of Q. Then

$$-\ell_A(\mathrm{H}^1_{\mathfrak{m}}(A)) = e_1(Q) = e_1(Q/(a)) = -\ell_A(\mathrm{H}^0_{\mathfrak{m}}(A/(a)))$$

by [**8**, Lemma 2.1(1)]. Hence $\ell_A(\mathrm{H}^1_{\mathfrak{m}}(A)) = \ell_A((0) :_{\mathrm{H}^1_{\mathfrak{m}}(A)} a)$, and so $a\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$. Similarly we get $b\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$, whence $Q\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$. □

LEMMA 4.5. *Suppose that* $(A, \mathfrak{m})$ *is a generalized Cohen–Macaulay local ring of dimension* $d \geqslant 2$ *and* $\operatorname{depth}(A) > 0$. *Let* Q *be a parameter ideal of* A *such that* $e_1(Q) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$. *Then* $Q\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ *for all* $i \neq d$.

Proof. If $d = 2$, then the conclusion follows from Lemma 4.4. Assume that $d \geqslant 3$. Let $Q = (a_1, a_2, \ldots, a_d)$, where each a_i is superficial for the ideal Q, and let $a = a_i$. Let $B = A/aA$. We have that $e_1(QB) = e_1(Q)$.

Consider the exact sequence of local cohomology modules

$$\ldots \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(A) \xrightarrow{a} \mathrm{H}^i_{\mathfrak{m}}(A) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(B) \longrightarrow \mathrm{H}^{i+1}_{\mathfrak{m}}(A) \xrightarrow{a} \mathrm{H}^{i+1}_{\mathfrak{m}}(A) \longrightarrow \mathrm{H}^{i+1}_{\mathfrak{m}}(B) \xrightarrow{a} \cdots.$$

We then have

$$h^i(B) = \ell_A(\mathrm{H}^i_{\mathfrak{m}}(A)/a\mathrm{H}^i_{\mathfrak{m}}(A)) + \ell_A((0) :_{\mathrm{H}^{i+1}_{\mathfrak{m}}(A)} a) \leqslant h^i(A) + h^{i+1}(A)$$

for all $0 \leqslant i \leqslant d-2$. Hence

$$\begin{aligned} e_1(QB) \geqslant -\sum_{i=1}^{d-2} \binom{d-3}{i-1} h^i(B) &\geqslant -\sum_{i=1}^{d-2} \binom{d-3}{i-1} [h^i(A) + h^{i+1}(A)] \\ &= -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A) \\ &= e_1(QB). \end{aligned}$$

It follows that $h^i(B) = h^i(A) + h^{i+1}(A)$ for every $1 \leqslant i \leqslant d-2$, whence $a\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for all $1 \leqslant i \leqslant d-1$. Thus $Q\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$, if $i \neq d$. □

Proof of Corollary 4.3. By Proposition 4.3 we see that A is a generalized Cohen–Macaulay ring. Hence we have $\Lambda(A) = \{-\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)\}$ by [**13**, Korollar 3.2]. Let $a \in \mathfrak{m}$ be such that $\dim A/aA = d-1$. It is enough to show that $a\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for all $i \neq d$. This follows from Lemma 4.5. □

Since every quasi-Buchsbaum ring A is Buchsbaum when $\operatorname{depth} A \geqslant d-1$ (see [**16**, Corollary 1.1]), we readily get the following.

Corollary 4.6. *Suppose that A is an unmixed Noetherian local ring of dimension $d \geqslant 2$ and* $\operatorname{depth} A \geqslant d-1$. *Then $\#\Lambda(A) = 1$ if and only if A is a Buchsbaum ring.*

The authors expect that A is a Buchsbaum ring if A is unmixed and $e_1(Q)$ is independent of the choice of parameter ideals Q of A. We will show that this is the case when $e_1(Q) = -1$ and when $e_1(Q) = -2$.

Proposition 4.7. *Let A be a Noetherian local ring of dimension $d \geqslant 2$ and suppose that, for all parameter ideals Q of A, we have $e_1(Q) = -t$ for some $t \geqslant 0$. Let $U = U_A(0)$. Then $\dim U \leqslant d-2$, and for all parameter ideals $\mathfrak{q}$ of A/U we have that $e_1(\mathfrak{q}) = -t$.*

Proof. Let $B = A/U$. Assume that $\dim U = d-1$. Choose a system of parameters $a_1, a_2, \ldots, a_d$ of A so that $(a_d) \cap U = (0)$ (cf. [**2**]). Let $\ell > t$ be an integer and let $Q = (a_1^\ell, a_2, \ldots, a_d)$. For all $n \geqslant 0$ we have the exact sequence of A-modules

$$0 \longrightarrow U/(Q^{n+1} \cap U) \longrightarrow A/Q^{n+1} \longrightarrow B/Q^{n+1}B \longrightarrow 0.$$

Let $k \geqslant 0$ be an integer such that

$$Q^n \cap U = Q^{n-k}(Q^k \cap U)$$

for all $n \geqslant k$. We consider $U' = Q^k \cap U$ and $\mathfrak{q} = (a_1^\ell, a_2, \ldots, a_{d-1})$. Then $Q^{n-k}U' = \mathfrak{q}^{n-k}U'$, because $a_dU = (0)$. Therefore, for all $n \geqslant k$ we have

$$\ell_A(A/Q^{n+1}) = \ell_A(B/Q^{n+1}B) + \ell_A(U'/\mathfrak{q}^{n-k+1}U') + \ell_A(U/U'),$$

which implies that

$$-t = e_1(Q) = e_1(QB) - e_0(\mathfrak{q}U').$$

Consequently, since $e_1(QB) \leqslant 0$, we have

$$\ell \leqslant \ell e_0((a_1, a_2, \ldots, a_{d-1})U') = e_0(\mathfrak{q}U') = e_1(QB) + t \leqslant t,$$

which is impossible. Thus $\dim U \leqslant d-2$.

To see the second assertion, let $\mathfrak{q}$ be a parameter ideal of B. Then, choosing a parameter ideal Q of A so that $QB = \mathfrak{q}$, we get $e_1(\mathfrak{q}) = e_1(Q) = -t$, since $\dim U \leqslant d-2$. □

Theorem 4.8. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geqslant 2$. Then the following two conditions are equivalent.*

(a) *We have $e_1(Q) = -1$ for every parameter ideal Q of A,*

(b) *Let $U = U_{\hat{A}}(0)$ be the unmixed component of (0) in the $\mathfrak{m}$-adic completion $\hat{A}$ of A. Then $\dim_{\hat{A}} U \leqslant d-2$ and $\hat{A}/U$ is a Buchsbaum ring such that we have one of the following*:

(i) $\mathrm{H}^i_{\hat{\mathfrak{m}}}(\hat{A}/U) = (0)$ *for all $i \neq 1, d$ and* $\ell_{\hat{A}}(\mathrm{H}^1_{\hat{\mathfrak{m}}}(\hat{A}/U)) = 1$;

(ii) $\mathrm{H}^i_{\hat{\mathfrak{m}}}(\hat{A}/U) = (0)$ *for all $i \neq d-1, d$ and* $\ell_{\hat{A}}(\mathrm{H}^{d-1}_{\hat{\mathfrak{m}}}(\hat{A}/U)) = 1$;

where $\widehat{\mathfrak{m}}$ denotes the maximal ideal of $\hat{A}$.

Proof. We may assume that A is complete.

(b) ⇒ (a) Let Q be a parameter ideal of A and let $B = A/U$. Then $e_1(Q) = e_1(QB)$, since $\dim_A U \leqslant d-2$. Consequently $e_1(Q) = -1$, because $e_1(QB) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(B) = -1$ by [**13**, Korollar 3.2] and condition (i) or (ii).

(a) ⇒ (b) By Proposition 4.7 we may assume that A is unmixed. Then A is a quasi-Buchsbaum ring by Corollary 4.3. We have condition (i) or (ii), because

$e_1(Q) = -\sum_{i=1}^{d-1}\binom{d-2}{i-1}h^i(A)$. Hence A is a Buchsbaum ring, because A is a quasi-Buchsbaum ring with $\mathrm{H}^i_\mathfrak{m}(A) = (0)$ for all $i \neq \operatorname{depth} A, \dim A$ (see [16, Corollary 1.1]). □

To treat the case where $e_1(Q) = -2$ we need the following lemma.

LEMMA 4.9. *Suppose that $(A, \mathfrak{m})$ is a generalized Cohen–Macaulay local ring of dimension $d \geqslant 2$ and $\operatorname{depth}(A) > 0$. Assume that $h^1(A) = 1$ and $h^i(A) = 0$ for all $2 \leqslant i \leqslant d-2$. Let Q be a parameter ideal in A such that $e_1(Q) = -\sum_{i=1}^{d-1}\binom{d-2}{i-1}h^i(A)$. Then Q is standard.*

Proof. We may assume that the field $A/\mathfrak{m}$ is infinite. If $d = 2$, the conclusion follows from Lemma 4.4. Let $d \geqslant 3$ and let $Q = (a_1, a_2, \ldots, a_d)$, where each a_i is superficial for the ideal Q. Let $a = a_i$ and $\mathrm{U}(a) = (a) : \mathfrak{m}$, and consider $B = A/aA$ and $\tilde{B} = B/\mathrm{H}^0_\mathfrak{m}(B)$. Then

$$e_1(Q) = e_1(QB) = e_1(Q\tilde{B}).$$

Consider the exact sequence of local cohomology modules

$$\cdots \longrightarrow \mathrm{H}^i_\mathfrak{m}(A) \xrightarrow{a} \mathrm{H}^i_\mathfrak{m}(A) \longrightarrow \mathrm{H}^i_\mathfrak{m}(B) \longrightarrow \mathrm{H}^{i+1}_\mathfrak{m}(A) \xrightarrow{a} \mathrm{H}^{i+1}_\mathfrak{m}(A) \longrightarrow \mathrm{H}^{i+1}_\mathfrak{m}(B) \longrightarrow \cdots.$$

Since $h^1(A) = 1$, we have $h^0(B) = 1$, whence $\mathrm{H}^0_\mathfrak{m}(B) = \mathrm{U}(a)/(a)$ and $\tilde{B} = A/\mathrm{U}(a)$.

By Lemma 4.5 we have that $a\mathrm{H}^{d-1}_\mathfrak{m}(A) = (0)$. Therefore, if $d = 3$, then we get an exact sequence

$$0 \longrightarrow \mathrm{H}^1_\mathfrak{m}(A) \longrightarrow \mathrm{H}^1_\mathfrak{m}(B) \longrightarrow \mathrm{H}^2_\mathfrak{m}(A) \longrightarrow 0,$$

whence $h^1(\tilde{B}) = h^1(B) = 1 + h^2(A)$.

If $d \geqslant 4$, then we get $\mathrm{H}^1_\mathfrak{m}(B) \cong \mathrm{H}^1_\mathfrak{m}(A)$, $\mathrm{H}^i_\mathfrak{m}(B) = (0)$ if $2 \leqslant i \leqslant d-3$ and $\mathrm{H}^{d-2}_\mathfrak{m}(B) \cong \mathrm{H}^{d-1}_\mathfrak{m}(A)$.

Consequently, if $d \geqslant 3$, then we have that $e_1(Q\tilde{B}) = -\sum_{i=1}^{d-2}\binom{d-3}{i-1}h^i(\tilde{B})$, and so by induction on d the parameter ideal $Q\tilde{B}$ is standard. Therefore

$$\ell_{\tilde{B}}(\tilde{B}/Q\tilde{B}) - e_0(Q\tilde{B}) = \mathbb{I}(\tilde{B}) = (d-2) + h^{d-1}(A).$$

Now assume by contradiction that Q is not a standard parameter ideal in A. Then

$$\begin{aligned}
(d-2) + h^{d-1}(A) &= \ell_{\tilde{B}}(\tilde{B}/Q\tilde{B}) - e_0(Q\tilde{B}) \\
&= \ell_A(A/(\mathrm{U}(a) + Q)) - e_0(A) \\
&= [\ell_A(A/Q) - e_0(A)] - \ell_A((Q + \mathrm{U}(a))/Q) \\
&< \mathbb{I}(A) - \ell_A((Q + \mathrm{U}(a))/Q) \\
&= [(d-1) + h^{d-1}(A)] - \ell_A((Q + \mathrm{U}(a))/Q).
\end{aligned}$$

Consequently, $\ell_A((Q + \mathrm{U}(a))/Q) = 0$, whence $\mathrm{U}(a) \subseteq Q$. Therefore

$$\sum_{i=1}^{d} \mathrm{U}(a_i) = Q$$

by the symmetry among the elements a_i. Let $\tilde{A}$ denote the (S_2)-fication of A and look at the canonical exact sequence

$$0 \longrightarrow A \longrightarrow \tilde{A} \longrightarrow \mathrm{H}^1_\mathfrak{m}(A) \longrightarrow 0$$

(see [1, Theorem 1.6]). Then $\dim \tilde{A} = d$, $\operatorname{depth} \tilde{A} \geqslant d-1$ and $\mathrm{H}^{d-1}_\mathfrak{m}(\tilde{A}) \cong \mathrm{H}^{d-1}_\mathfrak{m}(A)$. Hence Q is a standard parameter ideal for the generalized Cohen–Macaulay A-module $\tilde{A}$ by [17, Corollary 3.7], because $Q\mathrm{H}^{d-1}_\mathfrak{m}(\tilde{A}) = (0)$. Therefore, since $\operatorname{depth}_A \tilde{A} \geqslant 2$, any two of $a_1, a_2, \ldots, a_d$ form an $\tilde{A}$-regular sequence, whence $\mathrm{U}(a_i) \subseteq a_i\tilde{A}$ for all $1 \leqslant i \leqslant d$. Consequently

$\mathrm{U}(a_i) = a_i\tilde{A}$, because $a_iA \subsetneq \mathrm{U}(a_i) \subseteq a_i\tilde{A}$ and $\ell_A(a_i\tilde{A}/a_iA) = \ell_A(\tilde{A}/A) = 1$. Thus $Q = Q\tilde{A}$, so that we have $Q^{n+1} = Q^{n+1}\tilde{A}$ for all $n \geqslant 0$. Since

$$\ell_A(A/Q^{n+1}) = \ell_A(\tilde{A}/Q^{n+1}\tilde{A}) - 1,$$

we get

$$e_1(Q) = e_1(Q\tilde{A}) = -h^{d-1}(\tilde{A}) = -h^{d-1}(A)$$

by [**13**, Korollar 3.2], which is a contradiction. Thus the parameter ideal Q is standard in A. □

THEOREM 4.10. *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geqslant 2$. Then the following two conditions are equivalent.*

(a) *We have $e_1(Q) = -2$ for every parameter ideal Q of A.*

(b) *Let $U = U_{\hat{A}}(0)$ be the unmixed component of (0) in the $\mathfrak{m}$-adic completion $\hat{A}$ of A. Then $\dim_{\hat{A}} U \leqslant d - 2$ and $\hat{A}/U$ is a Buchsbaum ring with*

$$\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(\hat{A}/U) = 2.$$

Proof. (b) ⇒ (a) The assertion follows from [**13**, Korollar 3.2].

(a) ⇒ (b) By Proposition 4.7, we may assume that A is an unmixed complete local ring. Hence A is a quasi-Buchsbaum ring by Corollary 4.3 and $\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A) = 2$ by [**13**, Korollar 3.2]. By [**16**, Corollary 1.1] we may assume that $d \geqslant 3$, $h^1(A) = h^{d-1}(A) = 1$ and $h^i(A) = 0$ if $2 \leqslant i \leqslant d - 2$. Then by Lemma 4.9 every parameter ideal Q of A is standard, so that A is a Buchsbaum ring. □

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