THE CHERN COEFFICIENTS OF LOCAL RINGS

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Abstract. The Chern numbers of the title are the first coefficients (after the multiplicities) of the Hilbert functions of various filtrations of ideals of a local ring $(R, \mathfrak{m})$. For a Noetherian (good) filtration $\mathcal{A}$ of $\mathfrak{m}$-primary ideals, the positivity and bounds for $e_1(\mathcal{A})$ are well-studied if R is Cohen-Macaulay, or more broadly, if R is a Buchsbaum ring or mild generalizations thereof. For arbitrary geometric local domains, we introduce techniques based on the theory of maximal Cohen-Macaulay modules and of extended multiplicity functions to establish the meaning of the positivity of $e_1(\mathcal{A})$, and to derive lower and upper bounds for $e_1(\mathcal{A})$.

Dedicated to Professor Melvin Hochster on the occasion of his 65th birthday

1. Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$, and let I be an $\mathfrak{m}$-primary ideal. One of our goals is to study the set of I-good filtrations of R. More concretely, we will consider the set of multiplicative, decreasing filtrations of R ideals, $\mathcal{A} = \{I_n, I_0 = R, I_{n+1} = II_n, n \gg 0\}$, integral over the I-adic filtration, conveniently coded in the corresponding Rees algebra and its associated graded ring

$$\mathcal{R}(\mathcal{A}) = \sum_{n\geq 0} I_n t^n, \quad \mathrm{gr}_{\mathcal{A}}(R) = \sum_{n\geq 0} I_n/I_{n+1}.$$

We will study certain strata of these algebras. For that we will focus on the role of the Hilbert polynomial of the Hilbert function $\lambda(R/I_{n+1})$, $n \gg 0$,

$$H^1_{\mathcal{A}}(n) = P^1_{\mathcal{A}}(n) = \sum_{i=0}^{d} (-1)^i e_i(\mathcal{A}) \binom{n+d-i}{d-i},$$

particularly of its coefficients $e_0(\mathcal{A})$ and $e_1(\mathcal{A})$. Two of our main issues are to establish relationships between the coefficients $e_i(\mathcal{A})$, for $i = 0, 1$, and marginally $e_2(\mathcal{A})$. If R is a Cohen-Macaulay ring, there are numerous related developments, noteworthy ones being given and discussed in [8] and [28]. The situation is very distinct in the non Cohen-Macaulay case. Just to illustrate the issue, suppose $d > 2$ and consider a comparison between $e_0(I)$ and $e_1(I)$, a subject that has received considerable attention. It is often possible to pass to a reduction $R \to S$, with $\dim S = 2$, or even

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$\dim S = 1$, so that $e_0(I) = e_0(IS)$ and $e_1(I) = e_1(IS)$. If R is Cohen-Macaulay, this is straightforward. However, in general the relationship between $e_0(IS)$ and $e_1(IS)$ may involve other invariants of S, some of which may not be easily traceable all the way to R.

Our perspective is partly influenced by the interpretation of the coefficient e_1 as a *tracking number* (see [6] for the original terminology and [26]), that is, as a numerical positional tag of the algebra $\mathcal{R}(\mathcal{A})$ in the set of all such algebras with the same multiplicity. The coefficient e_1 under various circumstances is also called the *Chern number or Chern coefficient* of the algebra.

This paper is organized around a list of questions and conjectural statements about the values of e_1 for very general filtrations associated to the $\mathfrak{m}$–primary ideals of a local Noetherian ring $(R, \mathfrak{m})$:

(1) (Conjecture 1: the negativity conjecture) For every ideal J, generated by a system of parameters, $e_1(J) < 0$ if and only if R is not Cohen-Macaulay.
(2) (Conjecture 2: the positivity conjecture) For every $\mathfrak{m}$-primary ideal I, for its integral closure filtration $\mathcal{A}$
$$e_1(\mathcal{A}) \geq 0.$$
(3) (Conjecture 3: the uniformity conjecture) For each Noetherian local ring R, there exist two functions $\mathbf{f}_l(\cdot)$, $\mathbf{f}_u(\cdot)$ defined with some extended multiplicity degree over R, such that for each $\mathfrak{m}$-primary ideal I and any I-good filtration $\mathcal{A}$
$$\mathbf{f}_l(I) \leq e_1(\mathcal{A}) \leq \mathbf{f}_u(I).$$
(4) For any two minimal reductions J_1, J_2 of an $\mathfrak{m}$-primary ideal I,
$$e_1(J_1) = e_1(J_2).$$

For general local rings the conjectures may fail for reasons that will be illustrated by examples. We will settle Conjecture 1 for domains that are essentially of finite type over fields by making use of the existence of special maximal Cohen-Macaulay modules (Theorem 3.2). The lower bound in Conjecture 3 is also settled for general rings through the use of extended degree functions (Corollary 7.7). The upper bound uses the technique of the Briançon-Skoda theorem on perfect fields (Theorem 6.1). We bring no real understanding to the last question.

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2. e_1 as a tracking number

Let $(R, \mathfrak{m})$ be a Noetherian local domain of dimension d, which is a quotient of a Gorenstein ring. For an $\mathfrak{m}$-primary ideal I, we are going to consider the set of all graded subalgebras $\mathbf{A}$ of the integral closure of $R[It]$,

$$R[It] \subset \mathbf{A} \subset \bar{\mathbf{A}} = \bar{R[It]}.$$

We will assume that $\bar{\mathbf{A}}$ is a finite $R[It]$-algebra. We denote the set of these algebras by $\mathfrak{S}(I)$. If the algebra

$$\mathbf{A} = \sum I_n t^n$$

comes with a filtration that is decreasing $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, it has an associated graded ring

$$\mathrm{gr}(\mathbf{A}) = \sum_{n=0}^{\infty} I_n/I_{n+1}.$$

Proposition 2.1. *If $\mathbf{A}$ has the condition S_2 of Serre, then $\{I_n, n \geq 0\}$ is a decreasing filtration.*

Proof. See [34, Proposition 4.6].

We are going to describe the role of the Hilbert coefficient $e_1(\cdot)$ in the study of normalization of $R[It]$, following [26] and [25]. For each $\mathbf{A} = \sum A_n t^n \in \mathfrak{S}(I)$, we consider the Hilbert polynomial (for $n \gg 0$)

$$\lambda(R/A_{n+1}) = \sum_{i=0}^{d} (-1)^i e_i(\mathbf{A}) \binom{n+d-i}{d-i}.$$

The multiplicity $e_0(\mathbf{A})$ is constant across the whole set $\mathfrak{S}(I)$, $e_0(\mathbf{A}) = e_0(I)$. The next proposition shows the role of $e_1(\mathbf{A})$ in tracking $\mathbf{A}$ in the set $\mathfrak{S}(I)$.

Proposition 2.2. *Let $(R, \mathfrak{m})$ be a normal, Noetherian local domain which is a quotient of a Gorenstein ring, and let I be an $\mathfrak{m}$-primary ideal. For algebras $\mathbf{A}$, $\mathbf{B}$ of $\mathfrak{S}(I)$:*

(1) *If the algebras $\mathbf{A}$ and $\mathbf{B}$ satisfy $\mathbf{A} \subset \mathbf{B}$, then $e_1(\mathbf{A}) \leq e_1(\mathbf{B})$.*
(2) *If $\mathbf{B}$ is the S_2-ification of $\mathbf{A}$, then $e_1(\mathbf{A}) = e_1(\mathbf{B})$.*
(3) *If the algebras $\mathbf{A}$ and $\mathbf{B}$ satisfy the condition S_2 of Serre and $\mathbf{A} \subset \mathbf{B}$, then $e_1(\mathbf{A}) = e_1(\mathbf{B})$ if and only if $\mathbf{A} = \mathbf{B}$.*

Proof. The first two assertions follow directly from the relationship between Krull dimension and the degree of Hilbert polynomials. The exact sequence of graded $R[It]$-modules

$$0 \rightarrow \mathbf{A} \longrightarrow \mathbf{B} \longrightarrow \mathbf{B}/\mathbf{A} \rightarrow 0$$

gives that the dimension of $\mathbf{B}/\mathbf{A}$ is at most $d-1$. Moreover, $\dim \mathbf{B}/\mathbf{A} = d-1$ if and only if its multiplicity is

$$\deg(\mathbf{B}/\mathbf{A}) = e_1(\mathbf{B}) - e_1(\mathbf{A}) > 0.$$

The last assertion follows because with $\mathbf{A}$ and $\mathbf{B}$ satisfying S_2, the quotient $\mathbf{B}/\mathbf{A}$ is nonzero, will satisfy S_1 and therefore has Krull dimension $d-1$. □

Corollary 2.3. *Given a sequence of distinct algebras in* $\mathfrak{S}(I)$*,*

$$\mathbf{A}_0 \subset \mathbf{A}_1 \subset \cdots \subset \mathbf{A}_n = R[\bar{It}],$$

that satisfy the condition S_2 *of Serre, then*

$$n \leq e_1(R[\bar{It}]) - e_1(\mathbf{A}_0) \leq e_1(R[\bar{It}]) - e_1(I).$$

This highlights the importance of having lower bounds for $e_1(I)$ and upper bounds for $e_1(R[\bar{It}])$. For simplicity we denote the last coefficient as $\bar{e}_1(I)$. In the Cohen-Macaulay case, for any parameter ideal J, $e_1(J) = 0$. Upper bounds for $\bar{e}_1(I)$ were given in [26]. For instance, [26, Theorem 3.2(a),(b)] shows that if R is a Cohen-Macaulay algebra of type t, essentially of finite type over a perfect field k and δ is a non zerodivisor in the Jacobian ideal $\text{Jac}_k(R)$, then

$$\bar{e}_1(I) \leq \frac{t}{t+1}\big[(d-1)e_0(I) + e_0(I + \delta R/\delta R)\big]$$

and

$$\bar{e}_1(I) \leq (d-1)\big[e_0(I) - \lambda(R/\bar{I})\big] + e_0(I + \delta R/\delta R).$$

3. Cohen-Macaulayness and the negativity of e_1

Given the role of the Hilbert coefficient e_1 as a tracking number in the normalization of blowup algebras, it is of interest to know its signature.

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. If R is Cohen-Macaulay, then $e_1(J) < 0$ for an ideal J generated by a system of parameters $x_1, \ldots, x_d$. As a consequence, for any $\mathfrak{m}$-primary ideal I, $e_1(I) \geq 0$. If $d = 1$, the property $e_1(J) = 0$ is characteristic of Cohen-Macaulayness. For $d \geq 2$, the situation is somewhat different. Consider the ring $R = k[x, y, z]/(z(x, y, z))$. Then for $T = H^0_{\mathfrak{m}}(R)$ and $S = k[x, y] = R/T$, $e_1(R) = e_1(S) = 0$.

We are going to argue that the negativity of $e_1(J)$ is an expression of the lack of Cohen-Macaulayness of R in numerous classes of rings. To provide a framework, we state:

Conjecture 3.1. Let R be a Noetherian local ring that admits an embedding into a big Cohen-Macaulay module. Then for a parameter ideal J, $e_1(J) < 0$ if and only if R is not Cohen-Macaulay.

This places restrictions on R, in particular R must be unmixed and equidimensional. As a matter of fact, we will be concerned almost exclusively with integral domains that are essentially of finite type over a field.

We next establish the *small* version of the conjecture.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 2$. Suppose there is an embedding*

$$0 \to R \longrightarrow E \longrightarrow C \to 0,$$

where E is a finitely generated maximal Cohen-Macaulay R-module. If R is not Cohen-Macaulay, then $e_1(J) < 0$ for any parameter ideal J.

Proof. We may assume that the residue field of R is infinite. We are going to argue by induction on d. For $d = 2$, let J be a parameter ideal. If R is not Cohen-Macaulay, depth $C = 0$.

Let $J = (x, y)$; we may assume that x is a superficial element for the purpose of computing $e_1(J)$ and is also superficial relative to C, that is, x is not contained in any associated prime of C distinct from $\mathfrak{m}$.

Tensoring the exact sequence above by $R/(x)$, we get the exact complex

$$0 \to T = \mathrm{Tor}_1^R(R/(x), C) \longrightarrow R/(x) \longrightarrow E/xE \longrightarrow C/xC \to 0,$$

where T is a nonzero module of finite support. Denote by S the image of $R' = R/(x)$ in E/xE. S is a Cohen-Macaulay ring of dimension 1. By the Artin-Rees Theorem, for $n \gg 0$, $T \cap (y^n)R' = 0$, and therefore from the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T \cap (y^n)R' & \longrightarrow & (y^n)R' & \longrightarrow & (y^n)S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T & \longrightarrow & R' & \longrightarrow & S & \longrightarrow & 0 \end{array}$$

the Hilbert polynomial of the ideal yR' is

$$e_0 n - e_1 = e_0(yS)n + \lambda(T).$$

Thus

$$e_1(J) = -\lambda(T) < 0, \quad (1)$$

as claimed.

Assume now that $d \geq 3$, and let x be a superficial element for J and for the modules E and C. In the exact sequence

$$0 \to T = \mathrm{Tor}_1^R(R/(x), C) \longrightarrow R' = R/(x) \longrightarrow E/xE \longrightarrow C/xC \to 0, \quad (2)$$

T is either zero, and we would go on with the induction procedure, or T is a nonzero module of finite support.

Let us point out first an elementary rule for the calculation of Hilbert coefficients. Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let $\mathcal{A} = \{I_n, n \geq 0\}$ be a filtration as above. For a finitely generated R-module M, denote by $e_i(M)$ the Hilbert coefficients of M for the filtration $\mathcal{A}M = \{I_n M, n \geq 0\}$.

Proposition 3.2. *Let*

$$0 \to A \longrightarrow B \longrightarrow C \to 0$$

be an exact sequence of finitely generated R-modules. If $r = \dim A < s = \dim B$, then $e_i(B) = e_i(C)$ for $i < s - r$.

To continue with the proof, if in the exact sequence (2) $T \neq 0$, by Proposition 3.2 we have that $e_1(JR') = e_1(J(R'/T))$, and the resulting embedding $R'/T \hookrightarrow E/xE$. By the induction hypothesis, it suffices to prove that if R'/T is Cohen-Macaulay then R', and therefore R, will be Cohen-Macaulay. This is the content of [18, Proposition 2.1]. For convenience, we give the details.

We may assume that R is a complete local ring. Since R is embedded in a maximal Cohen-Macaulay module, any associated prime of R is an associated prime of E and therefore it is equidimensional. Consider the exact sequences

$$0 \to T \to R' \to S = R'/T \to 0,$$

$$0 \to R \xrightarrow{x} R \longrightarrow R' \to 0.$$

From the first sequence, taking local cohomology,

$$0 \to H^0_{\mathfrak{m}}(T) = T \longrightarrow H^0_{\mathfrak{m}}(R') \longrightarrow H^0_{\mathfrak{m}}(S) = 0,$$

since $H^i_{\mathfrak{m}}(T) = 0$ for $i > 0$ and S is Cohen-Macaulay of dimension ≥ 2; one also has $H^1_{\mathfrak{m}}(S) = H^1_{\mathfrak{m}}(R') = 0$. From the second sequence, since the associated primes of R have dimension d, $H^1_{\mathfrak{m}}(R)$ is a finitely generated R-module. Finally, by Nakayama Lemma $H^1_{\mathfrak{m}}(R) = 0$, and therefore $T = H^0_{\mathfrak{m}}(R') = 0$. □

We now analyze what is required to extend the proof to big Cohen-Macaulay cases. We are going to assume that R is an integral domain and that E is a big balanced Cohen-Macaulay module (see [3, Chapter 8], [30]). Embed R into E,

$$0 \to R \longrightarrow E \longrightarrow C \to 0.$$

The argument above ($d \geq 3$) will work if in the induction argument we can pick $x \in J$ superficial for the Hilbert polynomial of J, avoiding the finite set of associated primes of E and all associated primes of C different from $\mathfrak{m}$. It is this last condition that is the most troublesome.

There is one case when this can be overcome, to wit, when R is a complete local ring and E is countably generated. Indeed, C will be countably generated and $\text{Ass}(C)$ will be a countable set. The prime avoidance result of [4, Lemma 3] allows for the choice of x. Let us apply these ideas in an important case.

Theorem 3.2. *Let $(R, \mathfrak{m})$ be a Noetherian local integral domain essentially of finite type over a field. If R is not Cohen-Macaulay, then $e_1(J) < 0$ for any parameter ideal J.*

Proof. Let A be the integral closure of R and $\widehat{R}$ its completion. Tensor the embedding $R \subset A$ to obtain

$$0 \to \widehat{R} \longrightarrow \widehat{R} \otimes_R A = \widehat{A}.$$

From the properties of pseudo-geometric local rings ([21, Section 37]), $\widehat{A}$ is a reduced semi-local ring with a decomposition

$$\widehat{A} = A_1 \times \cdots \times A_r,$$

where each A_i is a complete local domain, of dimension $\dim R$ and finite over $\widehat{R}$.

For each A_i we make use of [11, Theorem 3.1] and [12, Proposition 1.4] and pick a countably generated big balanced Cohen-Macaulay A_i-module and therefore $\widehat{R}$-module. Collecting the E_i we have an embedding

$$\widehat{R} \longrightarrow A_1 \times \cdots \times A_r \longrightarrow E = E_1 \oplus \cdots \oplus E_r.$$

As E is a countably generated big balanced Cohen-Macaulay $\widehat{R}$-module, the argument above shows that if $\widehat{R}$ is not Cohen-Macaulay then $e_1(J\widehat{R}) < 0$. This suffices to prove the assertion about R. □

Remark 3.3. There are other classes of local rings admitting big balanced Cohen-Macaulay modules. A crude way to handle it would be: Let E be such a module and assume it has a set of generators of cardinality s. Let X be a set of indeterminates of cardinality larger than s. The local ring $S = R[X]_{\mathfrak{m}[X]}$ is R-flat and has the same depth as R. If $E' = S \otimes_R E$ is a big balanced Cohen-Macaulay S-module, with its residue field of cardinality larger than that of the corresponding module C, prime avoidance would again work. Experts have cautioned that E' may not be balanced.

Example 3.4. We will consider some classes of examples.

(i) Let $(R, \mathfrak{m})$ be a regular local ring and let F be a nonzero (finitely generated) free R-module. For any non-free submodule N of F, the idealization (trivial extension) of R by N, $S = R \oplus N$ is a non Cohen-Macaulay local ring. Picking $E = R \oplus F$, Theorem 3.2 implies that for any parameter ideal $J \subset S$, $e_1(J) < 0$. It is not difficult to give an explicit formula for $e_1(J)$ in this case.

(ii) Let $R = \mathbb{R} + (x, y)\mathbb{C}[x, y] \subset \mathbb{C}[x, y]$, for x, y distinct indeterminates. R is not Cohen-Macaulay but its localization S at the maximal irrelevant ideal is a Buchsbaum ring. It is easy to verify that $e_1(x, y) = -1$ and that $e_1(S) = 0$ for the $\mathfrak{m}$-adic filtration of S.

Note that R has an isolated singularity. For these rings, [25, Theorem 5] can be extended (does not require the Cohen-Macaulay condition), and therefore describes bounds for $e_1(\bar{\mathbf{A}})$ of integral closures. Thus, if $\mathbf{A}$ is the Rees algebra of the parameter ideal J, one has

$$e_1(\bar{\mathbf{A}}) - e_1(J) \leq (d - 1 + \lambda(R/L))e_0(J),$$

where L is the Jacobian of R.

In this example, one has $d = 2$, $\lambda(R/L) = 1$,

$$e_1(\bar{\mathbf{A}}) - e_1(J) \leq 2e_0(J).$$

(iii) Let k be a field of characteristic zero and let $f = x^3 + y^3 + z^3$ be a polynomial of $k[x, y, z]$. Set $A = k[x, y, z]/(f)$ and let R be the Rees algebra of the maximal irrelevant ideal $\mathfrak{m}$ of A. Using the Jacobian criterion, R is normal. Because the reduction number of $\mathfrak{m}$ is 2, R is not Cohen-Macaulay. Furthermore, it is easy to verify that R is not contained in any Cohen-Macaulay domain that is finite over R. Let $S = R_{\mathcal{M}}$, where $\mathcal{M}$ is the irrelevant maximal ideal of R. The first superficial element (in the reduction to dimension two) can be chosen to be prime. Now one takes the integral closure of S, which will be a maximal Cohen-Macaulay module.

The argument extends to geometric domains in any characteristic if depth $R = d - 1$.

Remark 3.5. Uniform lower bounds for e_1 are rare but still exist in special cases. For example, if R is a generalized Cohen-Macaulay ring, then according to [10, Theorem 5.4],

$$e_1(J) \geq -\sum_{i=1}^{d-1} \binom{d-2}{i-1} \lambda(H^i_{\mathfrak{m}}(R)),$$

with equality if R is Buchsbaum.

It should be observed that uniform lower bounds may not always exist. For instance, if $A = k[x, y, z]$, and R the idealization of (x, y), then for the ideal $J = (x, y, z^n)$, $e_1(J) = -n$.

The Koszul homology modules $H_i(J)$ of J is a first place where to look for bounds for $e_1(J)$. We recall [3, Theorem 4.6.10], that the multiplicity of J is given by the formula

$$e_0(J) = \lambda(R/J) - \sum_{i=1}^{d} (-1)^{i-1} h_i(J),$$

where $h_i(J)$ is the length of $H_i(J)$. The summation term is non-negative and only vanishes if R is Cohen-Macaulay. Unfortunately it does not gives a bounds for $e_1(J)$. There is a formula involving these terms in the special case when J is generated by a d-sequence. Then the corresponding *approximation complex* is acyclic, and the Hilbert-Poincaré series of J ([14, Corollary 4.6]) is

$$\frac{\sum_{i=0}^{d} (-1)^i h_i(J) t^i}{(1-t)^d},$$

and therefore

$$e_1(J) = \sum_{i=1}^{d} (-1)^i i h_i(J).$$

Later we shall prove the existence of lower bounds more generally, by making use of extended degree functions.

4. Bounds on $\overline{e}_1(I)$ via the Briançon–Skoda number

We discuss the role of Briançon–Skoda type theorems (see [1], [20]) in determining some relationships between the coefficients $e_0(I)$ and $\overline{e}_1(I)$. We follow the treatment given in [26, Theorem 3.1] and [25, Theorem 5], but formulated for the non Cohen-Macaulay case. We are going to provide a short proof along the lines of [20] for the special case we need: $\mathfrak{m}$–primary ideals in a local ring. The general case is treated by Hochster and Huneke in [16, 1.5.5 and 4.1.5]. Let k be a perfect field, let R be a reduced and equidimensional k–algebra essentially of finite type, and assume that R is affine with $d = \dim R$ or $(R, \mathfrak{m})$ is local with $d = \dim R + \operatorname{trdeg}_k(R/\mathfrak{m})$. Recall that the *Jacobian ideal* $\operatorname{Jac}_k(R)$ of R is defined as the d–th Fitting ideal of the module of differentials $\Omega_k(R)$ – it can be computed explicitly from a presentation of the algebra. By varying Noether normalizations one deduces from [20, Theorem 2] that the Jacobian ideal $\operatorname{Jac}_k(R)$ is contained in the conductor $R\colon \overline{R}$ of R (see also [23], [2, 3.1] and [15, 2.1]); here $\overline{R}$ denotes the integral closure of R in its total ring of fractions.

Theorem 4.1. *Let k be a perfect field, let R be a reduced local k–algebra essentially of finite type with the property S_2 of Serre, and let I be an ideal with a minimal reduction generated by g elements. Denote by $\mathbf{D} = \sum_{n\geq 0} D_n t^n$ the S_2-ification of $R[It]$. Then for every integer n,*

$$\operatorname{Jac}_k(R)\,\overline{I^{n+g-1}} \subset D_n.$$

Proof. The proof is lifted from [26, 3.1], with the modification required by the use of $\mathbf{D}$ at the end.

We may assume that k is infinite. Then, passing to a minimal reduction, we may suppose that I is generated by g generators. Let S be a finitely generated k–subalgebra of R so that $R = S_{\mathfrak{p}}$ for some $\mathfrak{p} \in \operatorname{Spec}(S)$, and write $S = k[x_1, \ldots, x_e] = k[X_1, \ldots, X_e]/\mathfrak{a}$ with $\mathfrak{a} = (h_1, \ldots, h_t)$ an ideal of height c. Notice that S is reduced and equidimensional. Let $K = (f_1, \ldots, f_g)$ be an S–ideal with $K_{\mathfrak{p}} = I$, and consider the extended Rees ring $B = S[Kt, t^{-1}]$. Now B is a reduced and equidimensional affine k–algebra of dimension $e - c + 1$.

Let $\varphi\colon k[X_1, \ldots, X_e, T_1, \ldots, T_g, U] \twoheadrightarrow B$ be the k–epimorphism mapping X_i to x_i, T_i to $f_i t$ and U to t^{-1}. Its kernel has height $c+g$ and contains the ideal $\mathfrak{b}$ generated by $\{h_i, T_j U - f_j | 1 \leq i \leq t, 1 \leq j \leq g\}$. Consider the Jacobian matrix of these generators,

$$\Theta = \left(\begin{array}{c|cccc} \dfrac{\partial h_i}{\partial X_j} & & 0 & & \\ \hline & U & & & T_1 \\ * & & \ddots & & \vdots \\ & & & U & T_g \end{array}\right).$$

Notice that $I_{c+g}(\Theta) \supset I_c(\left(\frac{\partial h_i}{\partial X_j}\right))U^{g-1}(T_1,\ldots,T_g)$. Applying φ we obtain $\text{Jac}_k(B) \supset I_{c+g}(\Theta)B \supset \text{Jac}_k(S)Kt^{-g+2}$. Thus $\text{Jac}_k(S)Kt^{-g+2}$ is contained in the conductor of B. Localizing at $\mathfrak{p}$ we see that $\text{Jac}_k(R)It^{-g+2}$ is in the conductor of the extended Rees ring $R[It,t^{-1}]$. Hence for every n, $\text{Jac}_k(R)\, I\, \overline{I^{n+g-1}} \subset I^{n+1}$, which yields

$$\text{Jac}_k(R)\, \overline{I^{n+g-1}} \subset I^{n+1} \colon I \subset D_{n+1} \colon I = D_n,$$

as $\text{gr}(\mathbf{D})_+$ has positive grade. □

This result, with an application of [25, Theorem 5], gives the following estimation.

Corollary 4.1. *Let k be a perfect field, let $(R,\mathfrak{m})$ be a normal, reduced local k–algebra essentially of finite type of dimension d and let I be an $\mathfrak{m}$-primary ideal. If the Jacobian ideal L of R is $\mathfrak{m}$-primary, then for any minimal reduction J of I,*

$$\bar{e}_1(I) - e_1(J) \leq (d + \lambda(R/L) - 1)e_0(I).$$

Moreover, if $L \neq R$ and $\bar{I} \neq \mathfrak{m}$, one replaces -1 by -2.

5. Lower bounds for $\bar{e}_1$

Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$ and let I be an $\mathfrak{m}$-primary ideal. If R is Cohen-Macaulay, the original lower bound for $e_1(I)$ was provided by Narita ([22]) and Northcott ([24]),

$$e_1(I) \geq e_0(I) - \lambda(R/I).$$

It has been improved in several ways (see a detailed discussion in [29]). For non Cohen-Macaulay rings, estimates for $e_1(I)$ are in a state of flux.

We are going to experiment with a special class of non Cohen-Macaulay rings and methods in seeking lower bounds for $\bar{e}_1(I)$. We are going to assume that R is a normal domain and the minimal reduction J of I is generated by a d-sequence. This is the case of normal Buchsbaum rings, examples of which can be constructed by a machinery developed in [9] (see also [31]).

Let $\mathbf{A} = R[Jt]$ and $\mathbf{B} = R[\bar{J}t]$. The corresponding Sally module $S_{\mathbf{B}/\mathbf{A}}$ is defined by the exact sequence

$$0 \to B_1 t\mathbf{A} \longrightarrow \mathbf{B}_+ \longrightarrow S_{\mathbf{A}}(\mathbf{B}) = \bigoplus_{n\geq 2} B_n/B_1 J^{n-1} \to 0. \tag{3}$$

Lemma 5.1. *Let $(R,\mathfrak{m})$ be an analytically unramified local ring and let J be an ideal generated by a system of parameters $x_1,\ldots,x_d$. Then*

$$R/\overline{J} \otimes_R \text{gr}_J(R) \simeq R/\overline{J}[T_1,\ldots,T_d].$$

Proof. Let $R[T_1,\ldots,T_d] \to \mathbf{A}$ be a minimal presentation of $\mathbf{A} = R[Jt]$. According to [27, Theorems 3.1, 3.6], the presentation ideal $\mathcal{L}$ has all coefficients in $\overline{J}$, that is $\mathcal{L} \subset \bar{J}R[T_1,\ldots,T_d]$. □

Proposition 5.2. *Let $(R, \mathfrak{m})$ be a normal, analytically unramified local domain and let J be an ideal generated by a system of parameters $J = (x_1, \ldots, x_d)$ of linear type. The Sally module $S_J(\mathbf{B})$ defined above is either 0 or a module of dimension d and multiplicity*

$$e_0(S_J(\mathbf{B})) = \deg(S_J(\mathbf{B})) = \bar{e}_1(I) - e_0(I) - e_1(J) + \lambda(R/\bar{I}).$$

Proof. By [14, Corollary 4.6], since R is normal of dimension $d \geq 2$, depth $\mathrm{gr}_J(R) \geq 2$. Now we make use of [17, Lemma 1.1] (see also [34, Proposition 3.11]), depth $R[Jt] \geq 2$ as well. Now using [34, Theorem 3.53], we have that $R[Jt]$ has the condition S_2 of Serre.

Consider the exact sequence

$$0 \to B_1R[Jt] \longrightarrow R[Jt] \longrightarrow R/\overline{J} \otimes_R \mathrm{gr}_J(R) \to 0.$$

By the Lemma, the last algebra is a polynomial ring in d variables. Therefore $B_1R[Jt]$ satisfies the condition S_2 of Serre. Thus in the defining sequence (3) of $S_J(\mathbf{B})$, either S vanishes (and $B_n = B_1J^n$ for $n \geq 2$) or $\dim S_J(\mathbf{B}) = d$. In the last case, the calculation of the Hilbert function (see [34, Remark 2.17]) of $S_J(\mathbf{B})$ gives the asserted expression for its multiplicity. □

6. Existence of general bounds

We shall now treat bounds for $\bar{e}_1(I)$ for several classes of geometric local rings. Suppose $\dim R = d > 0$. Let I be an $\mathfrak{m}$-primary ideal and let $\mathcal{A} = \{A_n, n \geq 0\}$ be a filtration integral over the I-adic filtration. We may assume that $I = J = (x_1, \ldots, x_d)$ is a parameter ideal. We will consider some reductions on $R \to R'$ such that $e_i(\mathcal{A}) = e_i(\mathcal{A}')$, $i = 0, 1$, for $\mathcal{A}' = \{A_nR', n \geq 0\}$.

Using superficial sequences of length $d-1$, $\mathbf{x} = \{x_1, \ldots, x_{d-1}\}$ is the technique of choice for Cohen-Macaulay rings. In general, as in the equality (1), one needs more control over $H^0_{\mathfrak{m}}(R/(\mathbf{x}))$. Let us examine first the case of generalized Cohen-Macaulay local rings by examining the effect of certain reductions.

(1) If $d \geq 2$ and $R' = R/H^0_{\mathfrak{m}}(R)$, then $e_i(\mathcal{A}) = e_i(\mathcal{A}')$, for $i = 0, 1$, by Proposition 3.2. In addition $H^i_{\mathfrak{m}}(R) = H^i_{\mathfrak{m}}(R')$ for $i \geq 1$.
(2) Another property is that if $d \geq 2$ and x_1 is a superficial element for $\mathcal{A}$ (that is R-regular if $d = 2$), then preservation will hold passing to $R_1 = R/(x_1)$. As for the lengths of the local cohomology modules, in case x_1 is R-regular, from

$$0 \to R \xrightarrow{x} R \longrightarrow R_1 \to 0$$

we have the exact sequence

$$0 \to H^0_{\mathfrak{m}}(R_1) \to H^1_{\mathfrak{m}}(R) \to H^1_{\mathfrak{m}}(R) \to H^1_{\mathfrak{m}}(R_1) \to H^2_{\mathfrak{m}}(R) \to \cdots,$$

that gives

$$\begin{array}{rcl} \lambda(H^0_{\mathfrak m}(R_1)) & \leq & \lambda(H^1_{\mathfrak m}(R)) \\ \lambda(H^1_{\mathfrak m}(R_1)) & \leq & \lambda(H^1_{\mathfrak m}(R)) + \lambda(H^2_{\mathfrak m}(R)) \\ & \vdots & \\ \lambda(H^{d-2}_{\mathfrak m}(R_1)) & \leq & \lambda(H^{d-2}_{\mathfrak m}(R)) + \lambda(H^{d-1}_{\mathfrak m}(R)). \end{array}$$

(3) Let us combine the two transformations. Let $T = H^0_{\mathfrak m}(R)$, set $R' = R/T$, let $x \in I$ be a superficial element for $\mathcal{A}R'$ and set $R_1 = R'/(x)$. As x is regular on R', we have the exact sequence

$$0 \to T/xT \longrightarrow R/(x) \longrightarrow R'/xR' \to 0,$$

and the associated exact sequence

$$0 \to T/xT \longrightarrow H^0_{\mathfrak m}(R/(x)) \longrightarrow H^0_{\mathfrak m}(R'/xR') \to 0,$$

since $H^1(T/xT) = 0$. Note that this gives

$$R/(x)/H^0_{\mathfrak m}(R/(x)) \simeq R'/xR'/H^0_{\mathfrak m}(R'/xR').$$

There are two consequences to this calculation:

$$\begin{array}{rcl} \lambda(H^0_{\mathfrak m}(R/(x))) & \leq & \lambda(H^0_{\mathfrak m}(R)) + \lambda(H^0_{\mathfrak m}(R'/xR')) \\ & \leq & \lambda(H^0_{\mathfrak m}(R)) + \lambda(H^1_{\mathfrak m}(R)) \\ \lambda(H^i_{\mathfrak m}(R/(x))) & \leq & \lambda(H^i_{\mathfrak m}(R)) + \lambda(H^{i+1}_{\mathfrak m}(R)), \quad 1 \leq i \leq d-2 \end{array}$$

Proposition 6.1. *Let $(R, \mathfrak m)$ be a generalized Cohen-Macaulay local ring of positive depth, with I and $\mathcal{A}$ as above. If $d \geq 2$, consider a sequence of $d-1$ reductions of the type $R \to R/(x)$, and denote by S the ring $R/(x_1, \ldots, x_{d-1})$. Then* $\dim S = 1$ *and*

$$\lambda(H^0_{\mathfrak m}(S)) \leq T(R) = \sum_{i=1}^{d-1} \binom{d-2}{i-1} \lambda(H^i_{\mathfrak m}(R)).$$

Moreover, if R is a Buchsbaum ring, equality holds.

Let us illustrate another elementary but useful kind of reduction. Let $(R, \mathfrak m)$ be a Noetherian local domain of dimension $d \geq 2$ and let I be an $\mathfrak m$-primary ideal. Suppose R has a finite extension S with the condition S_2 of Serre,

$$0 \to R \longrightarrow S \longrightarrow C \to 0.$$

Consider the polynomial ring $R[x, y]$ and tensor the sequence by $R' = R[x, y]_{\mathfrak m[x,y]}$. Then there are $a, b \in I$ such that the ideal generated by polynomial $\mathbf{f} = ax + by$ has the following type of primary decomposition:

$$(\mathbf{f}) = P \cap Q,$$

where P is a minimal prime of $\mathbf{f}$ and Q is $\mathfrak{m}R'$-primary. In the sequence

$$0 \to R' \longrightarrow S' = R' \otimes_R S \longrightarrow C' = R' \otimes_R C \to 0$$

we can find $a \in I$ which is superficial for C, and pick $b \in I$ so that (a, b) has codimension 2. Now reduce the second sequence modulo $\mathbf{f} = ax + by$. Noting that $\mathbf{f}$ will be superficial for C', we will have an exact sequence

$$0 \to T \longrightarrow R'/(\mathbf{f}) \longrightarrow S'/\mathbf{f}S' \longrightarrow C'/\mathbf{f}C' \to 0,$$

in which T has finite length and $S'/\mathbf{f}S'$ is an integral domain since a, b is a regular sequence in S. This suffices to establish the assertion.

Finally we consider generic reductions on R. Let $\mathbf{X}$ be $d \times (d-1)$ matrix $\mathbf{X} = (x_{ij})$ in $d(d-1)$ indeterminates and let C be the local ring $R[\mathbf{X}]_{\mathfrak{m}[\mathbf{X}]}$. The filtration $\mathcal{A}C$ has the same Hilbert polynomial as $\mathcal{A}$. If $I = (x_1, \ldots, x_d)$ we now define the ideal

$$(f_1, \ldots, f_{d-1}) = (x_1, \ldots, x_d) \cdot \mathbf{X}.$$

Proposition 6.2. *Let R be an analytically unramified, generalized Cohen-Macaulay integral domain and I and $\mathcal{A}$ defined as above. Then $S = C/(f_1, \ldots, f_{d-1})$ is a local ring of dimension one such that $\lambda(H^0_{\mathfrak{m}}(S)) \leq T(R)$ and $S/H^0_{\mathfrak{m}}(S)$ is an integral domain.*

The next result shows the existence of bounds for $\bar{e}_1(I)$ as in [26].

We outline the strategy to find bounds for $e_1(\mathcal{A})$. Suppose $(R, \mathfrak{m})$ is a local domain essentially of finite type over a field and I is an $\mathfrak{m}$-primary ideal, and denote by $\mathcal{A}$ a filtration as above. We first achieve a reduction to a one dimensional ring $R \to R'$, where $e_0(I) = e_0(IR')$, $e_1(\mathcal{A}) = e_1(\mathcal{A}R')$, and in the sequence

$$0 \to T = H^0_{\mathfrak{m}}(R') \longrightarrow R' \longrightarrow S \to 0,$$

T is a prime ideal.

Since S is a one-dimensional integral domain, we have he following result.

Proposition 6.3. *In these conditions,*

$$\begin{aligned} e_1(\mathcal{A}) = e_1(\mathcal{A}R') &= e_1(\mathcal{A}S) - \lambda(T) \\ e_1(\mathcal{A}S) &\leq \bar{e}_1(I) = \lambda(\bar{S}/S), \end{aligned}$$

where $\bar{S}$ is the integral closure of S.

This shows that to find bounds for $e_1(\mathcal{A})$ one needs to trace back to the original ring R the properties of T and S. As to T, this is realized if R is a generalized Cohen-Macaulay ring for the reductions described in Proposition 6.1.

Theorem 6.1 (Existence of Bounds)**.** *Let* $(R, \mathfrak{m})$ *be a local integral domain of dimension* d*, essentially of finite type over a perfect field, and let* I *be an* $\mathfrak{m}$*-primary ideal. If* R *is a generalized Cohen-Macaulay ring and* δ *is a nonzero element of the Jacobian ideal of* R *then for any* I*-good filtration* $\mathcal{A}$ *as above,*

$$e_1(\mathcal{A}) < (d-1)e_0(I) + e_0((I,\delta)/(\delta)) - T.$$

Proof. It is a consequence of the proof of [26, Theorem 3.2(a)]. The assertion there is that

$$\bar{e}_1(I) \leq \frac{t}{t+1}\big[(d-1)e_0(I) + e_0((I+\delta R)/\delta R)\big],$$

where t is the Cohen-Macaulay type of R. Here we apply it to the reduction in Proposition 6.3

$$\bar{e}_1(IS) < (d-1)e_0(I) + e_0((I+\delta R)/\delta R),$$

dropping the term involving t, over which we lose control in the reduction process. Key to the conclusion is the fact that the element δ survives the reduction. □

7. Extended degrees and lower bounds for e_1

The derivation of upper bounds for $e_1(\mathcal{A})$ above required that R be a generalized Cohen-Macaulay ring. Let us do away with this requirement by working with the variation of the extended degree function hdeg ([7], [33]) labelled hdeg_I (see [19], [34, p. 142]). The same method will provide lower bounds for $e_1(I)$.

Cohomological degrees. Let $(R, \mathfrak{m})$ be a Noetherian local ring (or a standard graded algebra over an Artinian local ring) of infinite residue field. We denote by $\mathcal{M}(R)$ the category of finitely generated R-module (or the corresponding category of graded R-modules).

A general class of these functions was introduced in [7] , while a prototype was defined earlier in [33]. In his thesis ([13]), T. Gunston carried out a more formal examination of such functions in order to introduce his own construction of a new cohomological degree. One of the points that must be taken care of is that of an appropriate *generic hyperplane* section. Let us recall the setting.

Definition 7.1. If $(R, \mathfrak{m})$ is a local ring, a *notion of genericity* on $\mathcal{M}(R)$ is a function

$$U : \{\text{isomorphism classes of } \mathcal{M}(R)\} \longrightarrow \{\text{non-empty subsets of } \mathfrak{m} \setminus \mathfrak{m}^2\}$$

subject to the following conditions for each $A \in \mathcal{M}(R)$:

(i) If $f - g \in \mathfrak{m}^2$ then $f \in U(A)$ if and only if $g \in U(A)$.
(ii) The set $\overline{U(A)} \subset \mathfrak{m}/\mathfrak{m}^2$ contains a non-empty Zariski-open subset.
(iii) If depth $A > 0$ and $f \in U(A)$, then f is regular on A.

There is a similar definition for graded modules. We shall usually switch notation, denoting the algebra by S.

Another extension is that associated to an $\mathfrak{m}$-primary ideal I ([19]): A notion of genericity on $\mathcal{R}$ with respect to I is a function

$$U : \{\text{isomorphism classes of } \mathcal{M}(R)\} \longrightarrow \{\text{non-empty subsets of } I \setminus \mathfrak{m}I\}$$

subject to the following conditions for each $A \in \mathcal{M}(R)$:

(i) If $f - g \in \mathfrak{m}I$ then $f \in U(A)$ if and only if $g \in U(A)$.
(ii) The set $\overline{U(A)} \subset I/\mathfrak{m}I$ contains a non-empty Zariski-open subset.
(iii) If depth $A > 0$ and $f \in U(A)$, then f is regular on A.

Fixing a notion of genericity $U(\cdot)$ one has the following extension of the classical multiplicity.

Definition 7.2. A *cohomological degree*, or *extended multiplicity function*, is a function

$$\text{Deg}(\cdot) : \mathcal{M}(R) \mapsto \mathbb{N},$$

that satisfies the following conditions.

(i) If $L = \Gamma_{\mathfrak{m}}(M)$ is the submodule of elements of M that are annihilated by a power of the maximal ideal and $\overline{M} = M/L$, then

$$\text{Deg}(M) = \text{Deg}(\overline{M}) + \lambda(L), \tag{4}$$

where $\lambda(\cdot)$ is the ordinary length function.

(ii) (Bertini's rule) If M has positive depth, there is $h \in \mathfrak{m} \setminus \mathfrak{m}^2$, such that

$$\text{Deg}(M) \geq \text{Deg}(M/hM). \tag{5}$$

(iii) (The calibration rule) If M is a Cohen-Macaulay module, then

$$\text{Deg}(M) = \text{deg}(M), \tag{6}$$

where $\text{deg}(M)$ is the ordinary multiplicity of M

In the case of a notion of genericity relative to an $\mathfrak{m}$-primary ideal I, $\text{deg}(M) = e(I; M)$, the Samuel's multiplicity of M relative to I.

The existence of cohomological degrees in arbitrary dimensions was established in [33]:

Definition 7.3. Let M be a finitely generated graded module over the graded algebra A and S a Gorenstein graded algebra mapping onto A, with maximal graded ideal $\mathfrak{m}$. Set $\dim S = r$, $\dim M = d$. The *homological degree* of M is the integer

$$\begin{aligned} \text{hdeg}(M) &= \text{deg}(M) + \\ & \sum_{i=r-d+1}^{r} \binom{d-1}{i-r+d-1} \cdot \text{hdeg}(\text{Ext}^i_S(M, S)). \end{aligned} \tag{7}$$

This expression becomes more compact when $\dim M = \dim S = d > 0$:

$$\text{hdeg}(M) = \deg(M) + \sum_{i=1}^{d} \binom{d-1}{i-1} \cdot \text{hdeg}(\text{Ext}^i_S(M,S)). \tag{8}$$

Remark 7.4. Note that this definition morphs easily into an extended degree noted hdeg_I where Samuel multiplicities relative to I are used. The definition of hdeg can be extended to any Noetherian local ring S by setting $\text{hdeg}(M) = \text{hdeg}(\widehat{S} \otimes_S M)$. On other occasions, we may also assume that the residue field of S is infinite, an assumption that can be realized by replacing $(S, \mathfrak{m})$ by the local ring $S[X]_{\mathfrak{m}S[X]}$. In fact, if X is any set of indeterminates, the localization is still a Noetherian ring, so the residue field can be assumed to have any cardinality, as we shall assume in the proofs.

Specialization and torsion. One of the uses of extended degrees is the following. Let M be a module and $\mathbf{x} = \{x_1, \ldots, x_r\}$ be a superficial sequence for the module M relative to an extended degree Deg. How to estimate the length of $H^0_{\mathfrak{m}}(M)$ in terms of M?

Let us consider the case of $r = 1$. Let $H = H^0_{\mathfrak{m}}(M)$ and write

$$0 \to H \longrightarrow M \longrightarrow M' \to 0. \tag{9}$$

Reduction modulo x_1 gives the exact sequence

$$0 \to H/x_1H \longrightarrow M/x_1M \longrightarrow M'/x_1M' \to 0. \tag{10}$$

From the first sequence we have $\text{Deg}(M) = \text{Deg}(H) + \text{Deg}(M')$, and from the second

$$\text{Deg}(M/x_1M) - \text{Deg}(H/x_1H) = \text{Deg}(M'/x_1M') \leq \text{Deg}(M').$$

Taking local cohomology of the second exact sequence yields the short exact sequence

$$0 \to H/x_1H \longrightarrow H^0_{\mathfrak{m}}(M/x_1M) \longrightarrow H^0_{\mathfrak{m}}(M'/x_1M') \to 0,$$

from we have the estimation

$$\begin{aligned} \text{Deg}(H^0_{\mathfrak{m}}(M/x_1M)) &= \text{Deg}(H/x_1H) + \text{Deg}(H^0_{\mathfrak{m}}(M'/x_1M')) \\ &\leq \text{Deg}(H/x_1H) + \text{Deg}(M'/x_1M') \\ &\leq \text{Deg}(H) + \text{Deg}(M') = \text{Deg}(M). \end{aligned}$$

We resume these observations as:

Proposition 7.5. *Let M be a module and let $\{x_1, \ldots, x_r\}$ be a superficial sequence relative to M and* Deg. *Then*

$$\lambda(H^0_{\mathfrak{m}}(M/(x_1, \ldots, x_r)M)) \leq \text{Deg}(M).$$

Now we derive a more precise formula using hdeg. It will be of use later.

Theorem 7.1. *Let M be a module of dimension $d \geq 2$ and let $\mathbf{x} = \{x_1, \ldots, x_{d-1}\}$ be a superficial sequence for M and* hdeg*. Then*

$$\lambda(H^0_{\mathfrak{m}}(M/(\mathbf{x})M)) \leq \lambda(H^0_{\mathfrak{m}}(M)) + T(M).$$

Proof. Consider the exact sequence

$$0 \to H = H^0_{\mathfrak{m}}(M) \longrightarrow M \longrightarrow M' \to 0.$$

We have $\mathrm{Ext}^i_S(M, S) = \mathrm{Ext}^i_S(M', S)$ for $d > i \geq 0$, and therefore $T(M) = T(M')$. On the other hand, reduction mod $\mathbf{x}$ gives

$$\begin{aligned} \lambda(H^0_{\mathfrak{m}}(M/(\mathbf{x})M)) &\leq \lambda(H^0_{\mathfrak{m}}(M'/(\mathbf{x})M')) + \lambda(H/(\mathbf{x})H) \\ &\leq \lambda(H^0_{\mathfrak{m}}(M'/(\mathbf{x})M')) + \lambda(H), \end{aligned}$$

which shows that it is enough to prove the assertion for M'.

If $d > 2$, the argument in the main theorem of [33] can be used to pass to M'/x_1M'. This reduces all the way to the case $d = 2$. Write $h = \mathbf{x}$. The assertion requires that $\lambda(H^0_{\mathfrak{m}}(M/hM)) \leq \mathrm{hdeg}(\mathrm{Ext}^1_S(M, S))$. We have the cohomology exact sequence

$$\mathrm{Ext}^1_S(M, S) \xrightarrow{h} \mathrm{Ext}^1_S(M, S) \longrightarrow \mathrm{Ext}^2_S(M/hM, S) \longrightarrow \mathrm{Ext}^0_S(M, S) = 0,$$

where

$$\lambda(H^0_{\mathfrak{m}}(M/hM)) = \mathrm{hdeg}(\mathrm{Ext}^2_S(M/hM, S)).$$

If $\mathrm{Ext}^1_S(M, S)$ has finite length the assertion is clear. Otherwise $L = \mathrm{Ext}^1_S(M, S)$ is a module of dimension 1 over a discrete valuation domain V with h for its parameter. By the fundamental theorem for such modules,

$$V = V^r \oplus (\bigoplus_{j=1}^{s} V/h^{e_j}V),$$

so that multiplication by h gives

$$\lambda(L/hL) = r + s \leq r + \sum_{j=1}^{s} e_j = \mathrm{hdeg}(L).$$

An alternative argument at this point is to consider the exact sequence (we may assume $\dim S = 1$)

$$0 \to L_0 \longrightarrow L \xrightarrow{h} L \longrightarrow L/hL \to 0,$$

where both L_0 and L/hL have finite length. If H denotes the image of the multiplication by h on L, dualizing we have the short exact sequence

$$0 \to \mathrm{Hom}_S(L, S) \xrightarrow{h} \mathrm{Hom}_S(L, S) \longrightarrow \mathrm{Ext}^1_S(L/hL, S) \longrightarrow \mathrm{Ext}^1_S(L, S),$$

which shows that

$$\lambda(L/hL) \leq \deg(L) + \lambda(L_0) = \mathrm{hdeg}(L),$$

as desired. □

We now employ the extended degree hdeg_I to derive lower bounds for $e_1(I)$. We begin by making a crude comparison between $\mathrm{hdeg}(M)$ and $\mathrm{hdeg}_I(M)$.

Proposition 7.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an $\mathfrak{m}$-primary ideal. Suppose $\mathfrak{m}^r \subset I$. If M is an R-module of dimension d, then*

$$\mathrm{hdeg}_I(M) \leq r^d \cdot \deg(M) + r^{d-1} \cdot (\mathrm{hdeg}(M) - \deg(M)).$$

Proof. If r is the index of nilpotency of R/I, for any R-module L of dimension s,

$$\lambda(L/(\mathfrak{m}^r)^n L) \geq \lambda(L/I^n L).$$

The Hilbert polynomial of L gives

$$\lambda(L/(\mathfrak{m}^r)^n L) = \deg(M)\frac{r^s}{s!}n^s + \text{lower terms}.$$

We now apply this estimate to the definition of $\mathrm{hdeg}(M)$, taking into account that its terms are evaluated at modules of decreasing dimension. □

Theorem 7.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an $\mathfrak{m}$-primary ideal, and let M be a finitely generated R-module of dimension $d \geq 1$. Let $\mathbf{x} = \{x_1, \ldots, x_r\}$ be a superficial sequence in I relative to M and hdeg_I. Then*

$$\mathrm{hdeg}_I(M/(\mathbf{x})M) \leq \mathrm{hdeg}_I(M).$$

Moreover, if $r < d$ then

$$H^0(M/(\mathbf{x})M) \leq \mathrm{hdeg}_I(M) - e(I; M).$$

If we apply this to R, passing to $R' = R/(x_1, \ldots, x_{d-1})$, we have the estimate for $H^0_{\mathfrak{m}}(R')$ so that the formula (1) can be used:

Corollary 7.7 (Lower Bound for $e_1(I)$)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$. If I is an $\mathfrak{m}$-primary ideal, then*

$$e_1(I) \geq -\mathrm{hdeg}_I(R) + e_0(I).$$

(*Note that* $\mathrm{hdeg}_I(R) - e_0(I)$ *is the Cohen-Macaulay defficiency of R relative to the degree function* hdeg_I.)

References

[1] I. M. Aberbach and C. Huneke, An improved Briançon–Skoda theorem with applications to the Cohen–Macaulayness of Rees algebras, *Math. Ann.* **297** (1993), 343–369.

[2] M. Auslander and D. Buchsbaum, On ramification theory in noetherian rings, *American J. Math.* **81** (1959), 749–765.

[3] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, 1993.

[4] L. Burch, Codimension and analytic spread, *Math. Proc. Camb. Phil. Soc.* **72** (1972), 369–373.

[5] A. Corso, Sally modules of $\mathfrak{m}$-primary ideals in local rings, Preprint, 2003.

[6] K. Dalili and W. V. Vasconcelos, The tracking number of an algebra, *American J. Math.* **127** (2005), 697-708.
[7] L. R. Doering, T. Gunston and W. V. Vasconcelos, Cohomological degrees and Hilbert functions of graded modules, *American J. Math.* **120** (1998), 493–504.
[8] J. Elias, Upper bounds of Hilbert coefficients and Hilbert functions, *Math. Proc. Camb. Phil. Soc.*, to appear.
[9] G. Evans and P. Griffith, Local cohomology modules for local domains, *J. London Math. Soc.* **19** (1979), 277–284.
[10] S. Goto and K. Nishida, Hilbert coefficients and Buchsbaumness of associated graded rings, *J. Pure & Applied Algebra* **181** (2003), 61–74.
[11] P. Griffith, A representation theorem for complete local rings, *J. Pure & Applied Algebra* **7** (1976), 303–315.
[12] P. Griffith, Maximal Cohen-Macaulay modules and representation theory, *J. Pure & Applied Algebra* **13** (1978), 321–334.
[13] T. Gunston, *Cohomological degrees, Dilworth numbers and linear resolution*, Ph.D. Thesis, Rutgers University, 1998.
[14] J. Herzog, A. Simis and W. V. Vasconcelos, Approximation complexes of blowing-up rings, *J. Algebra* **74** (1982), 466–493.
[15] M. Hochster, Presentation depth and the Lipman–Sathaye Jacobian theorem, in *The Roos Festschrift*, Vol. 2, Homology Homotopy Appl. **4** (2002), 295–314.
[16] M. Hochster and C. Huneke, Tight closure in equal characteristic zero, preprint.
[17] C. Huneke, On the associated graded ring of an ideal, *Illinois J. Math.* **26** (1982), 121-137.
[18] C. Huneke and B. Ulrich, General hyperplane sections of algebraic varieties, *J. Algebraic Geom.* **2** (1993), 487–505.
[19] C. H. Linh, Upper bound for the Castelnuovo-Mumford regularity of associated graded modules, *Comm. Algebra* **33** (2005), 1817–1831.
[20] J. Lipman and A. Sathaye, Jacobian ideals and a theorem of Briançon-Skoda, *Michigan Math. J.* **28** (1981), 199–222.
[21] M. Nagata, *Local Rings*, Interscience, New York, 1962.
[22] M. Narita, A note on the coefficients of Hilbert characteristic functions in semi-regular local rings, *Math. Proc. Camb. Phil. Soc.* **59** (1963), 269–275.
[23] E. Noether, Idealdifferentiation und Differente, *J. reine angew. Math.* **188** (1950), 1–21.
[24] D. G. Northcott, A note on the coefficients of the abstract Hilbert function, *J. London Math. Soc.* **35**
(1960), 209–214.
[25] T. Pham and W. V. Vasconcelos, Complexity of the normalization of algebras, *Math. Zeit.* **258** (2008), 729–743.
[26] C. Polini, B. Ulrich and W. V. Vasconcelos, Normalization of ideals and Briançon-Skoda numbers, *Math. Research Letters* **12** (2005), 827–842.
[27] D. Rees, Reductions of modules, *Math. Proc. Camb. Phil. Soc.* **101** (1987), 431–449.
[28] M. E. Rossi and G. Valla, The Hilbert function of the Ratliff-Rush filtration, *J. Pure & Applied Algebra* **201** (2005), 25–41.
[29] M. E. Rossi and G. Valla, *Hilbert Function of Filtered Modules*, arXiv:0710.2346.
[30] R. Y. Sharp, Cohen-Macaulay properties for balanced big Cohen-Macaulay modules, *Math. Proc. Camb. Phil. Soc.* **90** (1981), 229–238.
[31] J. Stückrad and W. Vogel, *Buchsbaum Rings and Applications*, Springer, New York, 1986.
[32] G. Valla, Problems and results on Hilbert functions of graded algebras, in *Six Lectures on Commutative Algebra*, Progress in Mathematics **166**, Birkhäuser, Boston, 1998, 293–344.

[33] W. V. Vasconcelos, The homological degree of a module, *Trans. Amer. Math. Soc.* **350** (1998), 1167–1179.

[34] W. V. Vasconcelos, *Integral Closure*, Springer Monographs in Mathematics, New York, 2005.

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