The Equations of Almost Complete Intersections

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Outline

Abstract

This talk uses many sources, but particularly joint work with Jooyoun Hong and Aron Simis.

The Equations of an Ideal

Let R be a Noetherian ring and let I be an ideal. By the equations of I it is meant a free presentation of the Rees algebra $R[It]$ of I,

$$0 \to \mathbf{L} \longrightarrow S = \mathbf{R}[\mathbf{T}_1, \ldots, \mathbf{T}_m] \xrightarrow{\psi} \mathbf{R}[It] \to 0, \quad \mathbf{T}_i \mapsto f_i t.$$

More precisely, $\mathbf{L}$ is the defining ideal of the Rees algebra of I but we refer to it simply as the ideal of equations of I.

The ideal $\mathbf{L}$ depends on the chosen set of generators of I, but all of its significant cohomological properties, such as the integers that bound the degrees of minimal generating sets of $\mathbf{L}$, are independent of the presentation ψ. The examination of $\mathbf{L}$ is one pathway to the unveiling of the properties of $\mathbf{R}[It]$.

Intro cont'd

Probably, the most general question one can ask about **L** is about estimates for the degrees of the generators of its generators, the so-called relation type of the ideal I: $\mathrm{reltype}(I)$. If **L** is generated by forms of degree 1, I is said to be of linear type (this is independent of the set of generators). The Rees algebra $\mathbf{R}[It]$ is then the symmetric algebra $\mathrm{Sym}(I)$ of I. Such is the case when the f_i form a regular sequence, **L** is then generated by the Koszul forms $f_iT_j - f_jT_i$, $i < j$.

The most detailed study of these equations took place in the theory of d-sequences and of approximation complexes. The class of ideals studied here falls just outside the full reach of these techniques.

The search for the equations of almost complete intersections, and their use, has attracted considerable interest by a diverse group of researchers, partly motivated by David Cox lectures on the subject. There is a small, but noteworthy, class of ideals for which full information is known about **L** ([Busé], [Cox-Hoffman-Wang], [Hong-Simis-V], [Kustin-Polini-Ulrich]).

Technical Goals

L is a graded ideal,

$$\mathbf{L} = L_1 + L_2 + \cdots + L_n + \cdots ,$$

where L_n are the syzygies of I^n. Coding all powers of I, **L** is a carrier of not just algebraic properties of I, but of analytic ones as well. It is therefore connected to estimations of various Castelnuovo-Mumford regularities.

The first of the goals: We focus on the fact that when I is homogeneous, **L** is bi-graded

$$\mathbf{L} : \quad \mathbf{f} = \sum_{\alpha} c_\alpha \mathbf{T}^\alpha$$

and we seek bounds on

- $\deg(c_\alpha)$:
- $\deg(\mathbf{T}^\alpha)$:

The other goal is to discuss/make predictions and bring classical elimination methods to the understanding of the components L_n of $\mathbf{L}$.

- L_1 is simply the module of syzygies of I. The ideal $(L_1) = S \cdot L_1$ defines the symmetric algebra $\mathrm{Sym}(I)$.
- L_2 is the ideal of syzygies of I^2. It is more practical however to consider the quotient $\delta(I) = L_2/S_1 L_1$. It is determined entirely by the ideal $I_1(\varphi)$ of entries of the syzygies of I.
- The higher L_n is *terra incognita*. In a limited number of cases, one mounts on L_1 and L_2 resultants to obtain, possibly, important equations of $\mathbf{L}$.

Goals

Our motivation here is to use it as a vehicle to understanding geometric properties of several constructions built out of I, particularly:

- When I is a homogeneous ideal, $\mathbf{L}$ is bigraded. We know a lot about general bounds for each of the two bi-degrees:

$$\text{reltype}(I) \leq (r_{\mathbf{x}}, r_{\mathbf{T}}).$$

  $r_{\mathbf{T}}$ is still too large, being not specific.
- For ideals I of finite co-length, to derive information about the Hilbert-Samuel polynomial of I.
- If I is defined by forms of the same degree, to examine the rational map between projective spaces they define, including questions of interest to geometric modelling.

Initial Data

For $\mathbf{R} = k[x_1, \ldots, x_d]$, $J = (a_1, \ldots, a_d)$ (regular sequence), $I = (J, a)$, a integral over J and presentation

$$\mathbf{R}^n \xrightarrow{\varphi} \mathbf{R}^{d+1} \longrightarrow I \to 0$$

Our underlying metaphor here is to focus on distinguished sets of equations by examining 4 Hilbert functions associated to the ideal I and to the coefficients of its syzygies: $\mathbf{R}/I$, $\mathbf{R}/I_1(\varphi)$, $\mathbf{R}/(J : a)$ and the Hilbert-Samuel function of I.

We will argue that they bring considerable effectivity to the methods by developing explicit formulas for some of the equations in **L**.

Almost Complete Intersections: Notation

- $R = k[x_1, \ldots, x_d]_{\mathfrak{m}}$ with $d \geq 2$ and $\mathfrak{m} = (x_1, \ldots, x_d)$.
- $I = (a_1, \ldots, a_d, a_{d+1})$ with $\deg(a_i) = d_i$, $\operatorname{height}(I) = d$ and I minimally generated by these forms.
- Assume that $J = (a_1, \ldots, a_d)$ is a minimal reduction of $I = (J, a)$, that is $I^{r+1} = JI^r$ for some natural number r.
- $S = \mathbf{R}[\mathbf{T}_1, \ldots, \mathbf{T}_{d+1}]$ and $\mathbf{L} = \ker(S \twoheadrightarrow \mathbf{R}[It])$ via $\mathbf{T}_j \mapsto a_j t$, where $\mathbf{R}[It]$ is the Rees algebra of I; note that $\mathbf{L}$ is a homogeneous ideal in the standard grading of S with $S_0 = \mathbf{R}$.
- L_i : $\mathbf{R}$–module generated by forms of $\mathbf{L}$ of degree i in $\mathbf{T}_j$'s. Thus L_i is the modules of syzygies of I^i.

The symmetric algebra $\mathrm{Sym}(I)$ and other noteworthy equations

- The degree 1 component of $\mathbf{L}$ is the ideal of entries of the matrix product

$$(L_1) = I_1([\mathbf{T}_1 \quad \cdots \quad \mathbf{T}_{d+1}] \cdot \varphi).$$

- (L_1) is the defining ideal of $\mathrm{Sym}(I)$.
- Assume that all forms a_i have the same degree n. The special fiber of I is the ring $\mathcal{F}(I) = \mathbf{R}[It] \otimes \mathbf{R}/\mathfrak{m}$. This is a hypersurface ring

$$\mathcal{F}(I) = k[T_1, \ldots, T_{d+1}]/(\mathbf{f}(\mathbf{T})),$$

where $\mathbf{f}(\mathbf{T})$ is an irreducible polynomial of degree $\mathrm{edeg}\,(I)$.

- $\mathbf{f}(\mathbf{T})$ is called the elimination equation of I, and may be taken as an element of $\mathbf{L}$. It is perhaps the most significant element of $\mathbf{L}$. Its degree is a divisor of n^{d-1}, with equality meaning that the mapping $\Psi_I : \mathbb{P}^{d-1} \dashrightarrow \mathbb{P}^{d-1}$ is birational.
- The elimination degree of I is $\deg(\mathbf{f}(\mathbf{T}))$, that is

$$\operatorname{edeg}(I) \quad = \quad \inf\{i \mid L_i \not\subseteq \mathfrak{m}S\}.$$

- One knows quite generally that $\mathbf{L} = (L_1) : \mathfrak{m}^\infty$. A secondary elimination degree is an integer r such that $\mathbf{L} = (L_1) : \mathfrak{m}^r$, i.e., an integer at least as large as the stabilizing exponent of the saturation.

Framework

$$0 \to T = \mathbf{L}/(L_1) \longrightarrow S/(L_1) = \mathrm{Sym}(I) \longrightarrow \mathbf{R}[It] \to 0$$

- $\mathrm{Sym}(I)$ is a Cohen-Macaulay ring of type at most $\dim \mathbf{R} - 1$: From a calculation with the approximation complex of I.
- T is the torsion part of $\mathrm{Sym}(I)$: Want to find r such that $\mathfrak{m}^r \cdot T = 0$ and the **R**-degrees of the generators of T:

$$\boxed{\mathbf{L} = (L_1) : \mathfrak{m}^r}$$

- Lacking a full set of generators for **L**, we want to find some of its distinguished generators. Moreover, we will seek conditions (certificates)/methods that guarantee the presence of new elements in **L**.

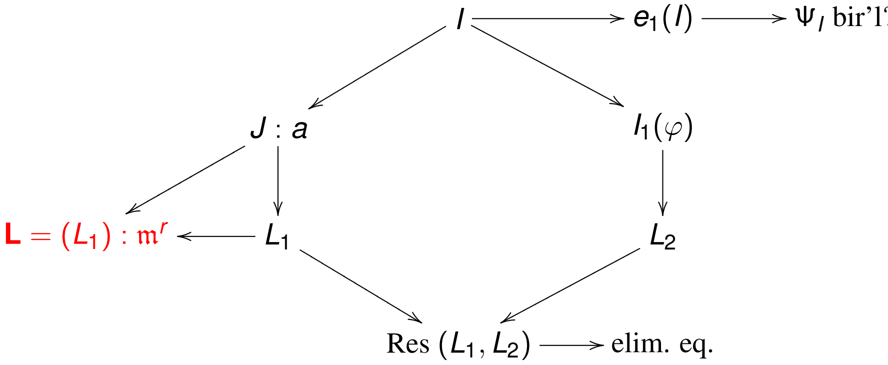
4 Hilbert Functions

Besides the syzygies of I, $\mathbf{f}(\mathbf{T})$ may be considered the most significant of the equations of I. Determining it, or at least its degree, is one of the main goals of this paper. The enablers, in our treatment, are four Hilbert functions associated to I, those of $\mathbf{R}/I$, $\mathbf{R}/J:a$, $\mathbf{R}/I_1(\varphi)$ and the Hilbert-Samuel function defined by I. Each encodes, singly or in conjunction, a different aspect of $\mathbf{L}$. For example, there are full equivalencies:

$$
\begin{aligned}
J : a &\Leftrightarrow L_1 \\
I_1(\varphi) &\Leftrightarrow L_2
\end{aligned}
$$

Hilbert functions and equations

In order to describe the other relationships between the invariants of the ideal I and its equations, we make use of the following diagram:



Elimination by Saturation

From the diagram

$$0 \to T = \mathbf{L}/(L_1) \longrightarrow S/(L_1) = \mathrm{Sym}(I) \longrightarrow \mathbf{R}[It] \to 0$$

and the fact that $\mathbf{L}/(L_1)$ is the torsion submodule of $\mathrm{Sym}(I)$, we have:

Proposition

If $0 \neq a \in I, \quad 0 \neq a \in I_1(\varphi)$, *then*

$$\mathbf{L} = (L_1) : a^\infty .$$

Elimination of Infinity

Theorem

Let $\mathbf{R} = k[x_1, \ldots, x_d]$ *and let* $I = (f_1, \ldots, f_{d+1})$ *be an ideal of finite colength, generated by forms of degree n. Suppose* $J = (f_1, \ldots, f_d)$ *is a minimal reduction, and set* $a = f_{d+1}$*. Let* ϵ *be the socle degree of* $\mathbf{R}/(J : a)$*, that is the largest integer m such that* $(\mathbf{R}/(J : a))_m \neq 0$*. If* $r = \epsilon + 1$*, then*

$$\mathbf{L} \cap \mathfrak{m}^r S \subset (L_1).$$

In particular,

$$\mathbf{L} = (L_1) : \mathfrak{m}^r.$$

The smallest value r for which $\mathbf{L} \cap \mathfrak{m}^r S \subset (L_1)$ will be called the secondary elimination degree of I, and noted r_I.

Example

Example

Let $R = k[x_1, x_2, x_3, x_4]$ and $\mathfrak{m} = (x_1, x_2, x_3, x_4)$.
Let $I = (x_1^3,\ \ x_2^3,\ \ x_3^3,\ \ x_4^3,\ \ x_1^2x_2 + x_3^2x_4)$,
$J = (x_1^3,\ \ x_2^3,\ \ x_3^3,\ \ x_4^3)$, and $a = x_1^2x_2 + x_3^2x_4$. Then

- Hilbert series of $(J : a)$: $1 + 4t + 9t^2 + 9t^3 + 4t^4 + t^5$.

A run with *Macaulay2* showed:

- $L = (L_1) : \mathfrak{m}^6$, exactly as predicted from the Hilbert function of $J : a$;
- The calculation yielded $\mathrm{edeg}\,(I) = 9$; in particular, Ψ_I is not birational.

Binary Ideals

- If $I = (a, b, c) \subset k[x_1, x_2]$ is generated by forms of degree n, $J : c = (\alpha, \beta)$, forms of degree p, q, $p + q = n$.
- The socle degree of $J : c$ is $p + q - 2 = n - 2$, thus $n - 1$ is a secondary elimination degree.
- $\mathbf{L} = (L_1) : \mathfrak{m}^{n-1}$, and all forms in $\mathbf{L} \cap \mathfrak{m}^{n-1} S \subset (L_1)$.

Relation type of binary ideals

Theorem

$\mathbf{L}$ *is minimally generated by forms* $\mathbf{f} = \sum_\alpha c_\alpha \mathbf{T}^\alpha$*, where* $\deg(\alpha) \leq n^4 - n^2 + 1$ *and* $\deg(c_\alpha) \leq n - 1$.

Proof. Reduce the presentation $0 \to \mathbf{L} \longrightarrow S \longrightarrow R[It] \to 0$ modulo I to get the exact sequence

$$0 \to \mathbf{L} \cap IS/I\mathbf{L} \longrightarrow \mathbf{L}/I\mathbf{L} \longrightarrow S/IS \longrightarrow \mathrm{gr}_I(R) \to 0$$

Since $I \subset \mathfrak{m}^r$, $\mathbf{L} \cap IS \subset (L_1)$. Thus the $\mathbf{T}$-degrees of the generators of $\mathbf{L}$ are controlled by those in (L_1) and the Castelnuovo-Mumford regularity of $\mathrm{gr}_I(R)$. Rossi-Trung-Valla gave an absolutely general bound, which in the case here is $n^4 - n^2 + 1$.

Bounding the Relation Type

Theorem

Let I be an almost complete intersection as above generated by forms of degree n. Then

$$\mathrm{reltype}(I) \leq (\alpha, \beta),$$

where $\alpha \leq \max\{r_I, n\}$ *and* $\beta = e(I)^{(d-1)!-1}(e(I)^2 - e(I)) + 1)$.

The value of β comes from the equality of the regularities of $R[It]$ and $\mathrm{gr}_I(R)$ ([Ooishi], [Trung], [Johnson-Ulrich]), and the bound of [Rossi-Trung-Valla] for the Castelnuovo-Mumford regularity of general Rees algebras.

Syzygetic Lemmas

- A persistent question is that of how to obtain higher degree generators from the syzygies of I. We will provide an iterative approach to generate the successive components of L:

$$L_1 \rightsquigarrow L_2 \rightsquigarrow L_3 \rightsquigarrow \cdots .$$

- This is not an effective process, except for the step $L_1 \rightsquigarrow L_2$. The more interesting development seems to be the use of the syzygetic lemmas to obtain $\delta(I) = L_2/S_1 L_1$ in the case of almost complete intersections. This is a formulation of the method of moving quadrics of several authors. The novelty here is the use of the Hilbert function of the ideal $I_1(\varphi)$ to understand and conveniently express $\delta(I)$. Such level of detail was not present even in earlier treatments of $\delta(I)$.

- The syzygetic lemmas are observations based on the Hilbert functions of I and of $I_1(\varphi)$ to allow a description of L_2. It converts the expression

$$\delta(I) = L_2/S_1L_1 = \mathrm{Hom}_{\mathbf{R}/I}(\mathbf{R}/I_1(\varphi), H_1(I)),$$

  where $H_1(I)$ is the canonical module of R/I (given by the syzygies of I), into a set of generators of L_2/S_1L_1 (note that $\delta(I)$ is the canonial module of $\mathbf{R}/I_1(\varphi)$).

- It is fairly effective in the case of binary forms, many cases of ternary and some quaternary forms, as we shall see. In these cases, out of L_1 and L_2 we will be able to write the elimination equation in the form of a resultant

$$\mathrm{Res}\,(L_1, L_2),$$

  or as one of its factors.

Sylvester Forms

- To get additional elements of L one often we make use of general Sylvester forms. Recall how these are obtained.
- Let $\mathbf{f} = \{f_1, \ldots, f_m\}$ be a set of polynomials in $L \subset S = \mathbf{R}[\mathbf{T}_1, \ldots, \mathbf{T}_n]$ and let $\mathbf{a} = \{a_1, \ldots, a_m\} \subset \mathbf{R}$. If $f_i \in (\mathbf{a})S$ for all i, we can write

$$\mathbf{f} = [f_1 \cdots f_m] = [a_1 \cdots a_m] \cdot \mathbf{A} = \mathbf{a} \cdot \mathbf{A},$$

  where $\mathbf{A}$ is an $m \times m$ matrix with entries in S.
- Since $\mathbf{a} \not\subset L$, then $\det(\mathbf{A}) \in L$.
- By an abuse of terminology, we refer to $\det(\mathbf{A})$ as a Sylvester form, or the Jacobian dual, of $\mathbf{f}$ relative to $\mathbf{a}$, in notation

$$\det(\mathbf{f})_{(\mathbf{a})} = \det(\mathbf{A}).$$

Sylvester Elimination

Proposition

Let $\mathbf{R} = k[x_1, \ldots, x_n]$, $\mathfrak{m} = (x_1, \ldots, x_d)$, and $\mathbf{f}_1, \ldots, \mathbf{f}_r$ be forms in $\mathbf{S} = \mathbf{R}[\mathbf{T}_1, \ldots, \mathbf{T}_m]$. Suppose the $\mathbf{R}$-content of the $\mathbf{f}_i$ (the ideal generated by the coefficients in $\mathbf{R}$) is generated by forms $\mathbf{a}_1, \ldots, \mathbf{a}_s$ of the same degree. Let

$$\mathbf{f} = [\mathbf{f}_1, \ldots, \mathbf{f}_r] = [\mathbf{a}_1, \ldots, \mathbf{a}_s] \cdot \mathbf{A} = \mathbf{a} \cdot \mathbf{A}$$

be the correspondng Sylvester decomposition. If $r \leq s$ and the first-order syzygies of $\mathbf{f}$ have coefficients in $\mathfrak{m}$, then $I_r(\mathbf{A}) \neq 0$.

- Note that $I_r(\mathbf{A})$ is generated by forms of $k[\mathbf{T}_1, \ldots, \mathbf{T}_m]$.
- When we apply it to forms in $(L_1) : (c)$, $\mathfrak{c} \subset \mathbf{R}$, $I_r(\mathbf{A}) \subset L$.

Proof.

- $I_r(\mathbf{A}) = 0$ means the columns of $\mathbf{A}$ are linearly dependent over $k[\mathbf{T}_1, \ldots, \mathbf{T}_m]$, thus for some nonzero column vector $\mathbf{c} \in k[\mathbf{T}_1, \ldots, \mathbf{T}_m]^r$

$$\mathbf{A} \cdot \mathbf{c} = 0$$

- Therefore

$$\mathbf{f} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{A} \cdot \mathbf{c} = 0$$

is a syzygy of the $\mathbf{f}_i$ whose content in not in $\mathfrak{m}$, against the assumption.

□

We refer to the condition on the content of syzygies as a syzygetic certificate for $I_r(\mathbf{A}) \neq 0$. This particular certificate will be called Sylvester certificate.

Binary Ideals

- Let $I \subset \mathbf{R} = k[x, y]$ be an (x, y)-primary ideal generated by three forms of degree n. Suppose that I has a minimal free resolution
$$0 \rightarrow \mathbf{R}^2 \xrightarrow{\varphi} \mathbf{R}^3 \longrightarrow I \rightarrow 0.$$
We will assume that the first column of φ has degree r, the other degree $s \geq r$. We note $n = r + s$.

- We denote by f and g the defining forms of the symmetric algebra of I, i.e., the generators of the ideal $(L_1) \subset \mathbf{S}$ in the earlier notation. We write this in the form $[f, g] = [\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3] \cdot \varphi$. In the standard bigrading, by assumption, f has bidegree $(r, 1)$, g bidegree $(s, 1)$. According to the Syzygetic Lemma, the component L_2 could be determined from $(L_1) : I_1(\varphi)$.

Certificates for binary ideals

In dimension two it is more convenient to get hold of a smaller quotient, $N = (L_1) : (x, y)^r$. The structure of this ideal gives a convenient

- N, being a direct link of the Cohen-Macaulay ideal $(x, y)^r$, is a perfect Cohen-Macaulay ideal of codimension two. The canonical module of S/N is generated by $(x, y)^r S/(f, g)$, so that its Cohen-Macaulay type is $r + 1$, according to Dubreil Theorem.
- Therefore, by the Hilbert-Burch theorem, N is the ideal of maximal minors of an $(r + 2) \times (r + 1)$ matrix ζ of homogeneous forms.

- Thus, $N = (f, g) : (x, y)^r$ has a presentation $0 \to S^{r+1} \xrightarrow{\zeta} S^{r+2} \longrightarrow N \to 0$, where ζ can be written in the form

$$\zeta = \left[\begin{array}{c} \sigma \\ -\,-\,-\,-\,-\,-\,- \\ \tau \end{array} \right],$$

with σ is a $2 \times (r+1)$ submatrix with rows whose entries are biforms of bidegree $(s-r, 1)$ and $(0, 1)$; and τ is an $r \times (r+1)$ submatrix whose entries are biforms of bidegree $(1, 0)$.

- Since $N \subset L_2$, this shows that in L_2 there are r forms $\mathbf{h}_i$ of degree 2 in the $\mathbf{T}_i$ whose $\mathbf{R}$-coefficients are forms in $(x, y)^{s-1}$.

- If $s = r$, write

$$[\mathbf{h}] = [\mathbf{h}_1 \ \cdots \ \mathbf{h}_r] = [x^{r-1} \ \ x^{r-2}y \ \ \cdots \ \ xy^{r-2} \ \ y^{r-1}] \cdot \mathbf{B},$$

  where $\mathbf{B}$ is an $r \times r$ matrix whose entries belong to $k[\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3]$.
- If $s > r$, collect the $s - r$ forms $\mathbf{f} = \{x^{s-r-1}f, x^{s-r-2}yf, \ldots, y^{s-r-1}f\}$ and write

$$[\mathbf{f}; \mathbf{h}] = [\mathbf{f}; \mathbf{h}_1 \ \cdots \ \mathbf{h}_r] = [x^{s-1} \ \ x^{s-2}y \ \ \cdots \ \ xy^{s-2} \ \ y^{s-1}] \cdot \mathbf{B},$$

  where $\mathbf{B}$ is an $s \times s$ matrix whose entries belong to $k[\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3]$.

Applications

A first consequence of this analysis is

Theorem

In both cases, $\det \mathbf{B}$ *is a nonzero polynomial of degree n.*

The next result provides a secondary elimination degree for these ideals.

Corollary

$\mathbf{L} = (L_1) : (x, y)^{n-1}$, *in particular* $I \cdot \mathbf{L} \subset (L_1)$.

Theorem

Let I be as above and $\beta = \det \mathbf{B}$*. Then* β *is a power of the elimination equation of I.*

Rest period

Take a breath

Balanced Ideals

For forms in dimension 3, or larger, we have not found any syzygetic certificate for fairly general classes of ideals. We are going to consider a restricted class and a certificate for it.

Definition

An $\mathfrak{m}$-primary ideal $I \subset R$ minimally generated by $d+1$ elements of the same degree is *s-balanced* if $I_1(\varphi) = \mathfrak{m}^s$, where φ is the matrix of syzygies of I.

Socle degree

Theorem

Let $R = k[x_1, \ldots, x_d]$ and I an almost complete intersection of finite colength generated by forms of degree n. If $I_1(\varphi) \subset \mathfrak{m}^s$, for s as large as possible, then:

(i) *The socle degree of $H_1(I)$ is $d(n-1)$;*

(ii) $\mathfrak{m}^{d(n-1)-s+1} = I\mathfrak{m}^{(d-1)(n-1)-s}$;

(iii) *Suppose I is s-balanced. Let $r(I)$ be the degree of the coefficients of L_2. Then*

$$r(I) = (d-1)(n-1) - s, \text{ and } R_{n+r(I)} = I_{n+r(I)}.$$

Well-balanced ideals

Corollary

Let $R = k[x_1, \ldots, x_d]$ be a ring of polynomials, and I an almost complete intersection of finite colength generated in degree n. Suppose that I is s–balanced.

(i) *If $d = 3$ and $s = n - 1$, then L_2 is generated by $\binom{s+1}{2}$ forms with coefficients of degree s and there are precisely n linear syzygies of degree $n - 1$, and $n \leq 7$;*

(ii) *If $d = 4$ and $s = n = 2$, then there are precisely 15 linear syzygies of degree 2.*

For those special cases, we have a straightforward analysis.

Suppose that I is $(n-1)$–balanced, where n is the degree of the generators of I. By Corollary 12(1), there are n linear forms $\mathbf{f}_i$, $1 \leq i \leq n$,

$$\mathbf{f}_i = \sum_{j=1}^{4} c_{ij}\mathbf{T}_j \in L_1,$$

arising from the syzygies of I of degree $n-1$. These syzygies, according to Dubreil Theorem, come from the syzygies of $J : a$, which by the structure theorem of codimension three Gorenstein ideals, is given by the Pfaffians of a skew-symmetric matrix Φ, of size at most $2n-1$. According to the Syzygetic Lemma, there are $\binom{n}{2}$ quadratic forms $\mathbf{h}_k$ $(1 \leq k \leq \binom{n}{2})$:

$$\mathbf{h}_k = \sum_{1\leq i\leq j\leq 4} c_{ijk}\mathbf{T}_i\mathbf{T}_j \in L_2,$$

with $\mathbf{R}$-coefficients of degree $n-1$.

Picking a basis for $\mathfrak{m}^{n-1}$ (simply denoted by $\mathfrak{m}^{n-1}$), and writing the $\mathbf{f}_i$ and $\mathbf{h}_k$ in matrix format, we have

$$[\mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{h}_1, \ldots, \mathbf{h}_{\binom{n}{2}}] \quad = \quad \mathfrak{m}^{n-1} \cdot \mathbf{B},$$

where $\mathbf{B}$ is the corresponding content matrix (see [2]). Observe that $\det \mathbf{B}$ is either zero, or a polynomial of degree

$$n + 2\binom{n}{2} = n^2.$$

It is therefore a likely candidate for the elimination equation.

Theorem

If $I \subset R = k[x_1, x_2, x_3]$ is a $(n-1)$-balanced almost complete intersection ideal generated by forms of degree n $(n \leq 7)$, then

$$\det \mathbf{B} \neq 0.$$

Proposition

Let $I \subset R = k[x_1, x_2, x_3]$ be an $(n-1)$-balanced almost complete intersection ideal generated by forms of degree n. Then

(i) *$2n-2$ is a secondary elimination degree of I.*

(ii) *If $(L_1) : \mathfrak{m}^{n-1} = (\mathbf{f}, \mathbf{h}) = (\mathbf{f}_1, \ldots, \mathbf{f}_n, \mathbf{h}_1, \ldots, \mathbf{h}_{\binom{n}{2}})$, then* $\det \mathbf{B}$ *is a power of the elimination equation.*

Quaternary Quadrics

- Let I be generated by 5 quadrics a_1, a_2, a_3, a_4, a_5 of R, $J = (a_1, a_2, a_3, a_4)$, and $a = a_5$. The analysis of this case is less extensive than the case of ternary ideals. Let us assume that $I_1(\varphi) = \mathfrak{m}^2$ – that is, the balancedness exponent equals the degree of the generators, a case that occurs generically in this degree.
- In this case, $\lambda(R/I) = 10$ and the Hilbert function of $H_1(I)$ is $(5, 4, 1)$. Note that $\nu(\delta(I)) = 4$.
- The last two graded components are of degrees 3 and 4. In degree 3 it leads to 4 forms $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$ in L_2, with linear coefficients:

$$[\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4] = [x_1, x_2, x_3, x_4] \cdot \mathbf{B}.$$

Theorem

Let $R = k[x_1, x_2, x_3, x_4]$ and I an $\mathfrak{m}$-primary almost complete intersection. If I is generated by quadrics and $I_1(\varphi) = \mathfrak{m}^2$, the polynomial $\det \mathbf{B}$*, defined by the equation* (1) *is divisible by the elimination equation of I. (*$\det \mathbf{B}$ *is not always irreducible.)*

Proposition

Let $R = k[x_1, x_2, x_3, x_4]$ and I an $\mathfrak{m}$-primary almost complete intersection. Suppose that I is generated by quadrics and $I_1(\varphi) = \mathfrak{m}^2$. Then

$$(L_1) : \mathfrak{m}^2 = \mathbf{L} \cap \mathfrak{m}S.$$

In particular,

$$\mathbf{L} = (L_1) : \mathfrak{m}^3.$$

Cautionary Remarks

- If I is balanced with $I_1(\varphi) = \mathfrak{m}$, there is only one new generator for L_2. If there are four linear syzygies, one assembles a resultant from L_1, $\mathrm{res}(L_1)$ to obtain the elimination equation, or a power of it.
- The following example shows that the elimantion is obtained as a resultant $\mathrm{res}(L_1, L_3)$.

Example

Let $I = (J, a)$, where $J = (x_1^2, x_2^2, x_3^2, x_4^2)$ and $a = x_1x_2 + x_2x_3 + x_3x_4 + x_1x_4$.

- $\nu(J : a) = 4$ (complete intersection of 2 linear equations and 2 quadrics)
- $I_1(\varphi) = \mathfrak{m}$ (i.e., 1-balanced, not 2-balanced)
- $\nu(Z_1) = 9$ (2 linear syzygies and 7 quadratic ones)
- Hilbert series of $R/(J : a) : 1 + 2t + t^2$.
- $\operatorname{edeg}(I) = 8 = 2^3$ (i.e., birational)
- The one step syzygetic principle does not work to get the elimination equation: One has:

$$\begin{cases} \mathbf{L} = (L_1, L_2, L_3, L_8), \\ \nu(L_1) = 9; \nu(L_2/S_1L_1) = 1; \nu(L_3/S_1L_2) = 2, L_8 = k\mathbb{P}, \end{cases}$$

  Nevertheless there are relationships among these numbers that are understood from the syzygetic

- Let f_1 and f_2 be in L_1 with linear coefficients, i.e.,
- $f_1 = (x_1 + x_3)(\mathbf{T}_2 - \mathbf{T}_4) + (x_2 - x_4)(-\mathbf{T}_5)$
- $f_2 = (x_1 - x_3)(-\mathbf{T}_5) + (x_2 + x_4)(\mathbf{T}_1 - \mathbf{T}_3)$
- Let $\mathbf{h} + L_1 S_1$ be a generator of $L_2/L_1 S_1$. We observed that, as a coset, $\mathbf{h}$ can be written as $\mathbf{h} = \mathbf{h}_1 + S_1 L_1$ and $\mathbf{h} = \mathbf{h}_2 + S_1 L_1$, with $\mathbf{h}_1$ and $\mathbf{h}_2$ forms of bidegree $(2, 2)$, with R-content contained in the contents f_1 and f_2, respectively.
- Write

  $$[f_1 \;\; \mathbf{h}_1] = [x_1{+}x_3 \;\; x_2{-}x_4]\mathbf{B}_1 \quad \text{and} \quad [f_2 \;\; \mathbf{h}_2] = [x_1{-}x_3 \;\; x_2{+}x_4]\mathbf{B}_2$$

  Then $\det \mathbf{B}_1$ and $\det \mathbf{B}_2$ form a minimal generating set of $L_3/S_1 L_2$.
- Write $[f_1 \;\; f_2 \;\; \det \mathbf{B}_1 \;\; \det \mathbf{B}_2] = [x_1 \;\; x_2 \;\; x_3 \;\; x_4]\mathbf{B}$. At the outcome $\det \mathbf{B} = \mathbb{P}$ is of degree 8, hence this is again birational.

Summary

- Explored methods to determine the equations of an ideal, focusing on almost complete intersections.
- Lacking effective methods to find the full set of equations, turned to an investigation of bounds to the degrees of the generators.
- Visited classical resultants to express some of the more important generators.
- This comes with caveats since they depend on generic properties. Cautionary examples abound.
- Questions in abstract Rees algebra theory are suggested.

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