MINORS OF SYMMETRIC AND EXTERIOR POWERS

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Abstract. We describe some of the determinantal ideals attached to symmetric, exterior and tensor powers of a matrix. The methods employed use elements of Zariski's theory of complete ideals and of representation theory.

Let R be a commutative ring. The determinantal ideals attached to matrices with entries in R play ubiquitous roles in the study of the syzygies of R–modules. In this note, we describe some of the determinantal ideals attached to symmetric, exterior and tensor powers of a matrix. The methods employed use elements of Zariski's theory of complete ideals and of representation theory, the results being sharper for rings containing the rationals.

Let R be an integral domain (or a field) and $\varphi : R^m \to R^n$ an R-linear map of rank r. It is an easy exercise to show that the d-th symmetric power $S^d(\varphi) : S^d(R^m) \to S^d(R^n)$ has rank $\binom{r+d-1}{d}$. Let $I_t(\varphi)$ denote the ideal generated by the minors (of a matrix representing φ). Since

$$\operatorname{rank}\varphi = \max\{r : I_r(\varphi) \neq 0\},$$

one can immediately determine the radicals of the ideals $I_t(S^d(\varphi))$, namely

$$\operatorname{Rad} I_t(S^d(\varphi)) = \operatorname{Rad} I_r(\varphi) \quad \text{if} \quad \binom{r+d-1}{d} \leq t < \binom{r+d}{d}.$$

Here we want to derive a more precise description of $I_t(S^d(\varphi))$, the ideals of minors of the exterior powers of φ, and the ideals of minors of a tensor product.

At least for the ideals associated with $\operatorname{rank}\varphi = n$ one has a very satisfactory result. To simplify notation we set $I(\varphi) = I_n(\varphi)$.

Suppose that $m = n$. Then the ideals $I(\varphi)$ and $I(S^d(\varphi))$ are principal, generated by the determinants of square matrices, and

$$(1) \qquad \det(S^d(\varphi)) = \det(\varphi)^s, \qquad s = \binom{n+d-1}{d-1},$$

1991 *Mathematics Subject Classification.* 13C40, 13B22.

as follows immediately by transformation to a triangular matrix (it is enough to consider a generic matrix over $\mathbb{Z}$, which is contained in a field). For non-square matrices we can replace $\det(\varphi)$ by $I(\varphi)$:

Theorem 1. *Let R be a commutative ring, and $\varphi : R^m \to R^n$ and $\psi : R^p \to R^q$ be R-linear maps. Set $s = \binom{n+d-1}{d-1}$ and $e = \binom{n-1}{d-1}$.*

(1) *Then*

$$I(\varphi \otimes \psi) \subset I(\varphi)^q I(\psi)^n, \qquad I(S^d(\varphi)) \subset I(\varphi)^s, \qquad I(\textstyle\bigwedge^d(\varphi)) \subset I(\varphi)^e,$$

with equality up to integral closure.

(2) *Suppose that R contains the field of rational numbers. Then equality holds in* (1).

Proof. For simplicity of notation we only treat the case of the symmetric powers in detail. That of the exterior powers is completely analogous. The tensor product requires slight modifications, which we will indicate below.

(1) The formation of $I(X)$ and $I(S^d(\varphi))$ commutes with ring extensions for trivial reasons. Thus it is enough to prove the inclusion for $R = \mathbb{Z}[X] = \mathbb{Z}[X_{ij} : i = 1, \dots, n,\ j = 1, \dots, r]$ and the linear map $\varphi : R^m \to R^n$ given by the matrix $X = (X_{ij})$. It is well-known (Trung [7]; also see Bruns and Vetter [3, (9.18)]) that the ideals $I(X)^k$ are primary with radical $I(X)$. It follows that $I(X)^k$ is integrally closed, and therefore

$$I(X) = \bigcap I(X)V$$

where V runs through the discrete valuation rings extending R. To sum up: we may assume that R is a discrete valuation ring.

Note that all ideals under consideration are invariant under base change in R^n and R^m. In fact, they are Fitting invariants of φ and $S^d(\varphi)$. By the elementary divisor theorem we can therefore assume that φ is given by a matrix with non-zero entries only in the diagonal positions (i,i), $I = 1, \dots, n$. Then $S^d(\varphi)$ is also given by a diagonal matrix, and it is an easy exercise to show that indeed $I(S^d(\varphi)) = I(\varphi)^s$ for such matrices φ.

As we have observed, equality holds for the ideals under consideration if R is a discrete valuation ring. This implies equality up to integral closure.

(2) We have to prove the inclusion opposite to that in (1), and it is enough to prove it for $R = \mathbb{Q}[X]$.

Since $\operatorname{rank} \varphi = n$, the linear map $S^d(\varphi)$ has rank $\binom{n+d-1}{d}$. Therefore the ideal under consideration is non-zero, and its generators have the same degree as those of $I(\varphi)^s$.

The group $G = \mathrm{GL}_m(\mathbb{Q}) \times \mathrm{GL}_r(\mathbb{Q})$ acts on R via the linear substitution sending each entry of X to the corresponding entry of

$$AXB^{-1}, \qquad A \in \mathrm{GL}_m(\mathbb{Q}),\ B \in \mathrm{GL}_r(\mathbb{Q}). \tag{2}$$

It is well-known that the $\mathbb{Q}$-vector space W generated by the degree rs elements of $I(X)^s$ is the irreducible G-representation associated with Young bitableaux of rectangular shape $s \times n$ (see De Concini, Eisenbud, and Procesi [4] or [3, Section 11]). So the desired inclusion follows if the ideal $I(S^d(\varphi))$ is G-stable.

This is not difficult to see. In fact let, σ be the automorphism induced by the substitution (2). We write $\sigma(\varphi)$ for the linear map which we obtain from φ by replacing each entry with its image under φ, i. e. $\sigma(\varphi) = \varphi \otimes \sigma$ for the ring extension $\sigma : R \to R$. Then

$$\begin{aligned}\sigma\left(I(S^d(\varphi))\right) &= I\left(\sigma(S^d(\varphi))\right) = I\left(S^d(\sigma(\varphi))\right)\\ &= I\left(S^d(\beta^{-1}\varphi\alpha)\right) = I\left(S^d(\beta)^{-1} \circ S^d(\varphi) \circ S^d(\alpha)\right)\\ &= I(S^d(\varphi))\end{aligned}$$

where α is the automorphism of R^m induced by the matrix A, and β the automorphism of R^n induced by B. The very last equation uses again that Fitting ideals are invariant under base change in the free modules.

For the tensor product the arguments above have to be modified at two places. We can of course assume that ψ is also given by a matrix Y of indeterminates. The first critical point is whether $I(X)^q I(Y)^n$ is again integrally closed as an ideal of $R = \mathbb{Z}[X, Y]$.

We can argue as follows: $\mathbb{Z}[X, Y]$ is a free $\mathbb{Z}$-module whose basis is given by the products $\mu\nu$ where μ is a standard monomial in the variables X_{ij} and ν is a standard monomial in the Y_{kl} (see [3, Section 4]). An element f of $\mathbb{Z}[X, Y]$ belongs to $I(X)^q$ if and only if $\mu \in I(X)^q$ for all the factors μ appearing in the representation of f as a linear combination of standard monomials, and a similar assertion holds with respect to $I(Y)^n$ and the factors ν. It follows immediately that $I(X)^q I(Y)^n = I(X)^q \cap I(Y)^n$. Since both $I(X)^q$ and $I(Y)^n$ are integrally closed, their intersection is integrally closed, too.

The other crucial question is whether the $\mathbb{Q}$-vector space generated by the degree rq elements of $I(X)^q I(Y)^n$ is an irreducible $G \times G'$-representation where $G' = \mathrm{GL}_p(\mathbb{Q}) \times \mathrm{GL}_q(\mathbb{Q})$ acts on $\mathbb{Q}[Y]$ in the same way as G on $\mathbb{Q}[X]$, and the action of $G \times G'$ on $K[X, Y] = K[X] \otimes K[Y]$ is the induced one. The answer is positive since $W \otimes W'$ is an irreducible $G \times G'$-representation if W and W' are irreducible for G and G' respectively. It is enough to test this after extending $\mathbb{Q}$ by $\mathbb{C}$, and then one can apply a classical theorem (for example, see Huppert [5, II.10.3]). □

Remark 2. (a) One can formulate an abstract version of the theorem as follows. Suppose that for each commutative ring R one has a functor F_R on the category of R-modules satisfying the following conditions:

(i) F_R commutes with ring extensions, i. e. $F_R(M\otimes S) = F_S(M\otimes R)$ for R-modules M (and similarly for R-linear maps) if $R \to S$ is a homomorphism of rings;
(ii) $F_R(R^n) = R^{\lambda(n)}$ for all n;
(iii) if V is a discrete valuation ring, then $I(F_V(\varphi)) = I(\varphi)^{\mu(m,n)}$ for a V-linear map $V^m \to V^n$;

then both parts (1) and (2) Theorem 1 hold accordingly.

The Schur functors [2], which generalize symmetric and exterior powers, satisfy these conditions.

(b) Especially the case of the symmetric powers is always suspicious to depend on characteristic, but we have not been able to find a counterexample to Theorem 1(2) in positive characteristics.

(c) Suppose that $F_R(\varphi)$ (as in (b)) is alternating when φ (is given by an alternating matrix). Then it makes sense to compare the Pfaffian ideals of φ (and ψ) with those of $S^d(\varphi)$, $\bigwedge^d(\varphi)$, and $\varphi\otimes\psi$.

For even n (the matrix) φ has a Pfaffian, and one can derive the formula analogous to equation (1) by transformation to an anti-diagonal matrix. For odd n, and more generally for lower order Pfaffians, one obtains variants of Theorems 1 and 3, using arguments analogous to those in the determinantal case. The neccessary representation theory is contained in [1].

As we will see in the following, the situation is much more complicated for lower order minors, and a description as precise as Theorem 1(2) seems to be out of reach. Nevertheless one can obtain reasonable upper and lower bounds from specialization to diagonal matrices.

Let us recall some notation and facts from [3] or [4]. A *Young diagram* (or partition) is a non-increasing finite sequence $\sigma = (s_1, \dots, s_u)$ of integers. We define the functions γ_j on the set of all Young diagrams by

$$\gamma_j(\sigma) = \sum_{i=1}^{u} \max(0, s_i - j + 1).$$

The set of Young diagrams is partially ordered by

$$\sigma \le \tau \quad \Longleftrightarrow \quad \gamma_j(\sigma) \le \gamma_j(\tau) \text{ for all } j.$$

Furthermore one sets

$$I^\sigma(\varphi) = \prod_{i=1}^{u} I_{s_i}(\varphi) \quad \text{and} \quad I^{(\sigma)}(\varphi) = \sum_{\tau \geq \sigma} I^\tau(\varphi).$$

If X is a matrix of indeterminates (over some ring of coefficients) we simply write I^σ for $I^\sigma(X)$ etc. If $\mathbb{Q} \subset R$, then

$$I^{(\sigma)} = I^\sigma; \tag{3}$$

see [3, (11.2)]. (This equation already holds if $m!$ or $n!$ is a unit in R.)

Suppose K is a field of characteristic 0 and X an $m \times n$ matrix of indeterminates. Then $K[X]$ splits into a direct sum

$$K[X] = \bigoplus_\sigma M_\sigma$$

of pairwise non-isomorphic irreducible $G = \mathrm{GL}_m(K) \times \mathrm{GL}_n(K)$-representations ($\sigma$ runs through all Young diagrams with at most $\min(m,n)$ columns). As a G-module, M_σ is generated by the doubly initial tableaux of shape σ, i. e. the product of the determinants of the $s_i \times s_i$-submatrices of X in its upper left corner, $i = 1, \dots, u$. The decomposition commutes with field extensions.

The third (and last) ideal associated with σ is I_σ, the ideal generated by M_σ. Evidently every G-stable ideal is a sum of ideals I_σ (see [3, Scetion 11]). If φ is a matrix over a ring $R \supset \mathbb{Q}$, then we may form $I_\sigma(\varphi)$ by substitution.

Suppose f is a monomial in the indeterminates $Y_1, \dots, Y_n$. Then f has a unique decomposition $f = f_1 \cdots f_u$ where each f_i is squarefree and $f_{i+1} \mid f_i$, $i = 1, \dots, u-1$. We set $s_i = \deg f_i$ and call $\sigma = (s_1, \dots, s_u)$ the *shape* of f.

Theorem 3. *Let R be a ring, and $\varphi : R^m \to R^n$ be an R-linear map. Moreover, let Σ be the set of the minimal elements (with respect to $\leq$) among the shapes of the monomials generating $I_r(S^d(Y))$ for a diagonal matrix Y of indeterminates (over $\mathbb{Z}$).*

(1) *Then*

$$I_r(S^d(\varphi)) \subset \sum_{\sigma \in \Sigma} I^{(\sigma)}(\varphi),$$

with equality up to integral closure.

(2) *If $\mathbb{Q} \subset R$, then*

$$\sum_{\sigma \in \Sigma} I_\sigma(\varphi) \subset I_r(S^d(\varphi)) \subset \sum_{\sigma \in \Sigma} I^\sigma(\varphi).$$

Proof. It is enough to prove the theorem for generic matrices X with $R = \mathbb{Z}[X]$ for (1) and $R = \mathbb{Q}[X]$ for (2).

Let us first observe that the inclusion in (1) follows from the same inclusion in (2), simply since $J = \sum_{\sigma\in\Sigma} I^{(\sigma)}(X)$ has a basis of standard monomials, and $\mathbb{Z}[X]/J$ is therefore $\mathbb{Z}$-torsionfree.

Now we prove the inclusion $I_r(S^d(X)) \subset J$ over $\mathbb{Q}[X]$. Note that $I_r(S^d(\varphi))$ is G-stable, as discussed in the proof of Theorem 1. Therefore it has a representation

$$I_r(S^d(\varphi)) = \sum_{\tau\in\mathrm{T}} I_\tau.$$

Evaluating the doubly initial tableau of shape τ on an $m \times n$ diagonal matrix yields a monomial of shape τ. Thus $\tau \geq \sigma$ for some $\sigma \in \Sigma$ and, hence, $I_\tau \subset I^{(\sigma)}$, as desired.

For the first inclusion in (2) we have to show that $\Sigma \subset \mathrm{T}$. Observe that a monomial in $I^\tau(Y)$ always has shape $\geq \tau$. Thus a monomial of shape $\sigma \in \Sigma$ can only be contained in $I_r(S^d(Y)) \subset \sum_{\tau\in\mathrm{T}} I^\tau(Y)$ if there exists $\tau \in \mathrm{T}$ with $\tau \leq \sigma$. But this implies $\sigma \in \mathrm{T}$.

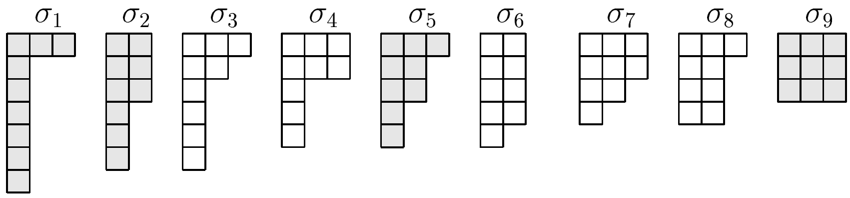
As in the proof of Theorem 1 it is enough to prove the assertion about integral closure for diagonal matrices of indeterminates over $\mathbb{Z}$. But then we are comparing two monomial ideals, and may pass to $\mathbb{Q}$, whence it is enough to show equality up to integral closure in (2). This however follows from [4, 8.1]: I^σ is the integral closure of I_σ. □

In the theorem one can replace symmetric powers by exterior powers and, more generally, establish a functorial version as indicated in Remark 2. A variant for tensor products can also be given.

We demonstrate by an example that both inclusions in Theorem 3(2) can be strict simultaneously. Choose a 3×3-matrix of indeterminates over $\mathbb{Q}$ and set $d = 3$ and $n = 3$. By inspection of the monomials that generate $I_3(S^3(Y))$ for a diagonal matrix Y of indeterminates one sees that, with the notation of the figure, $\Sigma = \{\sigma_1, \sigma_2\}$ is the right choice. We have computed the ideal $I_3(S^3(X))$ with Singular [6]. Its component in the lowest degree 9 has dimension 5610. The figure shows all the degree 9 Young diagrams $\geq \sigma_1$ or $\geq \sigma_2$. The dimensions of the representations M_τ can be computed by the hook formula, and it turns out that

$$I_3(S^3(X)) = I_{\sigma_1} + I_{\sigma_2} + I_{\sigma_5} + I_{\sigma_9}$$

since only the sum on the right hand side yields the correct dimension in degree 9. (Instead of comparing dimensions one could test which doubly initial Young tableau yield elements in $I_3(S^3(X))$.)



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