Mathematics 574 **Topics in Number Theory** **Spring 2019**

Numbers refer to problems in *The Arithmetic of Elliptic Curves* (second edition) by J. Silverman

Weierstrass equations, rational functions, and curves

1. Consider the projective plane curve $C : x^2y + xz^2 + y^3 + z^3 = 0$. Find a birational map of C to a curve given by a Weierstrass equation of form $y^2z = x^3 + Axz^2 + Bz^3$ with A, B integers. Show that C is a smooth curve over $\mathbf{Q}$.
2. Let C be a smooth curve isomorphic to the projective line. Show that for any finite set of distinct points $p_1, \ldots, p_k \in C$ and integers $m_1, \ldots, m_k$ the vector space of rational functions f on C regular on the complement of the points p_i and satisfying $ord_{p_i} + m_i >= 0$ is isomorphic to the vector space of rational functions associated to q with integer $\sum m_i$ and show that the latter space has dimension $\sum m_i + 1$. Apply this to show problem 2.5 in Silverman (where $(P) \sim (Q)$ for distinct points means that there is a rational function h with $ord_P(h) = 1, ord_Q(h) = -1$ which is regular at all points except Q).
3. 2.12
4. 2.15
5. 2.16 (take P to be a nonsingular point of C)