# The Riemann-Roch Theorem and Geometry, 1854-1914

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Abstract. The history of the Riemann-Roch Theorem, from its discovery by Riemann and Roch in the 1850's to its use by Castelnuovo and Enriques in from 1890 to 1914, offers one of the most instructive examples in the history of mathematics of how a result stays alive in mathematics by admitting many interpretations. Various mathematicians over the years took the theorem to be central to their researches in complex function theory, and in the study of algebraic curves and surfaces in a variety of algebraic and geometric styles. In surveying their interpretations and extensions of the theorem, the historian traces the creation of a general theory of complex algebraic curves and surfaces in the period, and uncovers lively agreements and disagreements. This paper provides an overview of the field; the Congress lecture will concentrate on the route from Riemann and Roch via Brill and Noether to Castelnuovo and Enriques. For reasons of space a number of the better-known developments have been omitted. One may consult Dieudonné [1976].

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## 1 Curves

In the 1850s (see his [1857] and Laugwitz [1996]) Riemann put together a theory of complex functions defined on some 2-dimensional domain, which might be any simply-connected domain, or the whole complex $z$-sphere (what we call the Riemann sphere) or a finite covering of the $z$-sphere branched over some points (what we call a Riemann surface). He showed how to define such a function with poles on a patch using his version of the Dirichlet principle. His motivation was the example of algebraic curves, and the outstanding topic of abelian integrals.

He established the existence of complex functions on a surface with no boundary by the Riemann inequality - his contribution to the Riemann-Roch Theorem. His imprecise argument retains its heuristic value. He supposed the surface was $(2p+1)$-fold connected, which means that it is rendered simply-connected by $2p$ cuts, when it forms a $4p$-sided polygon. He showed that there are $p$ linearly independent everywhere holomorphic functions defined inside the polygon by considering what would happen if the real parts of their periods all vanished (using the Dirichlet principle again). Later he showed that the differentials of these functions are everywhere-defined holomorphic integrands. Then he specified $d$ points at which the function may have simple poles, again imposing the condition that the functions jump by a constant along the cuts. Now he argued that to create functions with only simple poles and constant jumps one took a sum of $p$ linearly independent functions with no poles plus functions of the form $\frac{1}{z}$ at one of the specified points, and added a constant term. The resulting expression depends linearly on $p + d + 1$ constants. The jumps therefore depend linearly on $p + d + 1$ constants, and there are $2p$ of them to be made to vanish (if the function is single-valued as required). So there will be non-constant meromorphic functions when $p + d + 1 - 2p \geq 2$, i.e. $d > p$. This result, today called the Riemann inequality, says there is a linear space of complex functions of dimension $h^0 \geq d + 1 - p$, and this contains non-constant functions as soon as $d + 1 - p > 1$, or $d > p$.

Roch was a gifted student of Riemann who died of tuberculosis in 1866 aged only 26. He was able to interpret analytically the difference $d + 1 - p$ as the dimension of a certain space of holomorphic integrands, those that vanish at some of the points where the function may have poles. This implies that the difference $h^1 = h^0 - (d+1-p)$ is an analytically meaningful quantity. In Roch's terminology: if a function $w$ has $d$ simple poles, and if $q$ linearly independent integrands can vanish at these poles, then $w$ depends on $d-p+q+1$ arbitrary constants (Roch [1865]).

Riemann showed that any two meromorphic functions on an algebraic curve are algebraically related, whence a theorem establishing that a 'Riemann surface' branched like an algebraic curve over the Riemann sphere can be mapped into the projective plane. This established a close relationship between intrinsic curves and embedded curves, and Riemann showed that any two polynomial equations for the same algebraic curve are birationally related (the variables in one equation for the curve are rational functions of the variables in any other equation for the curve). He also used his inequality to calculate that the dimension of the moduli space of algebraic curves of genus $p > 1$ is $3p - 3$.

The approach of Riemann and Roch was aimed at extending complex function theory. It made abundant use of the Dirichlet principle (which Riemann attempted to prove; his proof was refuted by Prym in 1870). But by directing attention to a theory based on a topological concept of connectivity, Riemann opened the way to elucidate intrinsic properties of curves independent of their embeddings in the plane.

Riemann died in 1866. His eventual successor at Göttingen was Clebsch, who had already pioneered the application of Riemann's ideas to geometry in his [1863]. His initial response to these ideas had been to find them very hard - the topological nature of a Riemann surface was difficult to grasp and the

Dirichlet principle appeared confusing - and Roch's paper struck him as almost incomprehensible. So he initially defined the genus of a curve as the number of linearly independent holomorphic integrands on it. If the curve is non-singular, these, he observed, are integrands of the form $\phi dz/\frac{\partial F}{\partial w}$ where $\phi(z,w)$ is of degree at most $n-3$. The condition on the degree of $\phi$ arises by considering what happens at infinity, and was dealt with by passing to homogeneous coordinates. If the curve has a $k$-fold point (it passes $k$ times through a point) then the curve $\phi = 0$ is required to pass $k-1$ times through that singular point (such curves are called adjoint curves). In his book of 1866 with Paul Gordan, Clebsch restricted his attention to curves having only double points and cusps, for which a purely algebraic definition of the number $p$ (called the genus by Clebsch) is possible: $p = \frac{1}{2}(n-1)(n-2) - d - r$ where $n$ is the degree of the defining equation, $d$ is the number of double points and $r$ the number of simple cusps.

The response of Riemann's former student Prym was harsh. He wrote to Casorati (2 December 1866): "They would never have dared publish the foreword in Riemann's lifetime. The attempt to base function theory on algebra can be regarded as completely useless . . . . On the contrary, algebra is an outcome of function theory and not the other way round. (In Neuenschwander, [1978], p. 61). But Clebsch was a charismatic teacher, Klein [1926, p. 297] called him divinely inspired in that respect, and when he too died young, in 1872, he left behind a vigorous group of mathematicians who were to become the custodians of the Riemann-Roch Theorem. They regarded him as having led German mathematicians into the newer geometry and algebra - precisely the subjects, one might note, upon which Gauss did not work.

Prominent among them were Brill and Noether, and their critique of Clebsch-Gordan was that it had not gone far enough in embracing algebra. For them algebra was the source of rigour, and moreover, in Brill's opinion Riemann's work on the Riemann-Roch Theorem was in a form foreign to geometry. This was a sound, critical response, but the price was high: the very definition of genus became entangled with the nature of the singular points a plane curve might have, and the invariance of genus under birational transformations now had to be proved. Clebsch and Gordan had given such a geometric proof by means of a subtle elimination process, which Brill and others wanted to simplify.

The first problem is to define the multiplicity of a singular point on an algebraic curve, the second is to show that by suitable birational transformations any curve can be reduced to one having only what were called ordinary singularities, that is, singular points where all the tangent directions are distinct. Such a point was said to be of multiplicity $k$ if there are $k$ branches at that point. Noether broached the first of these topics in his paper of 1873 with a theorem which gave conditions for a curve that passes through the common points of two curves with equations $F = 0$ and $G = 0$ to have an equation of the form $AF + BG = 0$, where $A$, $B$, $F$, and $G$ are polynomials in the complex variables $x$ and $y$. It is indicative of the subtleties involved that the English mathematician F.S. Macaulay in the

1890s was among the first to pay scrupulous attention to the cases where the tangent directions are not separated.

The second problem is also difficult. It must be shown that a birational transformation can be found to simplify any given singularity, which is not obvious, and then, since the transformation necessarily introduces new singular points on the transformed curve, it must be shown that these can be made ordinary singularities. The consensus in the literature as to when this was achieved is as late as Walker's (unpublished) Chicago thesis of 1906. There is a significant papers on the topic by Bertini in 1888, and Bliss made it the subject of a Presidential address as late as 1923.

Brill and Noether were the first to call the Riemann-Roch Theorem by that name, in their [1874]. They took from Clebsch the idea that it was to be studied geometrically, that is, in terms of a linear family of adjoint curves. It followed from their definition of the genus (a generalisation of the Clebsch-Gordan definition to a curve having arbitrary ordinary singularities) that the number of free coefficients in the equation for an adjoint curve of degree $n-3$ is $p-1$. The total number of intersection points of the curve and its adjoint apart from the multiple points is $2p-2$, so at most $p-1$ of these are determined by the rest, equivalently, at least $2p-2-(p-1)=p-1$ can be chosen arbitrarily. It follows that $q$, the dimension of the space of adjoint curves of order $n-3$ that cut out a set of $Q$ points, satisfies the inequality $q \geq Q-p+1$.

Using induction on $q$ and $Q$, Brill and Noether proved the converse: there is a family of dimension q of adjoint curves of degree $n-3$ that cut out a set of $Q$ points, provided $q \geq Q-p+1$. This is their version of the Riemann inequality. They now came to their version of the Riemann-Roch Theorem, which they stated in terms of what they called special families. By definition a special family satisfies the strict inequality $q > Q-p+1$. The Brill-Noether version of the Riemann-Roch Theorem then says: If an adjoint curve of order $n-3$ is drawn through a special set of $Q$ points in a $q$-dimensional family of points, for which $q=Q-p+1+r$ (where $0<r<p-1$), then this curve meets the given curve in $2p-2-Q=R$ further points that themselves belong to a special set of $R$ points in an $r$-dimensional family, where $r=R-p+1+q$.

Their strong preference for algebra and geometry over function theory was criticised by Klein in his [1892]. Relations between Klein and the followers of Clebsch became strained as he moved in the late 1880s to adopt the mantle of Riemann and became his true successor at Göttingen. His enthusiasm for intuitive geometry clashed with their preference for the certainties of algebra.

For Brill and Noether, the Riemann-Roch Theorem was a theorem about families of plane curves. The first to use higher-dimensional geometry in this context were L. Kraus (who had studied under Klein and Weierstrass and died at the age of 27) and E.B. Christoffel, although credit has usually been given to

Noether. All started from the observation that an algebraic curve of genus $p > 1$ has a $p$-dimensional space of holomorphic 1-forms. While Christoffel and Noether pursued the analytic implications of taking a basis for these 1-forms, say $\omega_1, \ldots, \omega_p$, Kraus [1880] thought of the $p$-tuple $(\omega_1(z), \ldots, \omega_p(z)) = (f_1(z)dz, \ldots, f_p(z)dz)$ as giving a map from the curve to a projective space of dimension $p-1 : z \longrightarrow [f_1(z), \ldots, f_p(z)]$. That this map is well-defined (independent of the coordinate system used) and is a map into projective space (the $f_i$ never simultaneously vanish) was fudged by Kraus, (a modern simple proof uses the Riemann-Roch Theorem). That the degree of the map is $2p-2$ was explicit in Riemann's work. As Kraus saw, the case where the curve is hyper-elliptic also causes problems: here one gets a 2-1 map from the curve to the Riemann sphere. The novelty of Kraus's insight, which Klein appreciated, is the emphasis on higher-dimensional geometry. Whenever there is such a map, questions about the curve, or whole families of curves, are reduced to questions in projective geometry.

First Dedekind and Weber [1882] then Hensel and Landsberg (see Gray [1997]) took up the study of algebraic curves via their associated function fields. Landsberg's [1898a] gave a new proof of the Riemann-Roch theorem, as he put it 'in full generality and without birational transformations'. In the same issue of *Mathematische Annalen* Lansdberg also formulated and proved what he called an analogue of the Riemann-Roch Theorem in the theory of algebraic numbers, and observed that Hilbert had told him that an analogous result held for algebraic number fields. In 1902 Hensel and Landsberg published their joint book on the subject, which was to be the foundation of subsequent work in this direction.

The weak point of this approach is that it does not generalise automatically to algebraic surfaces. Nonetheless in his [1909] (corrected and simplified in his [1910a, b]) Heinrich Jung was able to extended the ideas of Hensel and Landsberg to cover function fields in two variables, and in this way he was able to obtain a Riemann-Roch theorem within the arithmetic tradition (see Gray [1994b]). As he pointed out, his proof was not that different from the Italian one, except that it also applied to divisors that were not integral, which in his view was an improvement.

The markedly algebraic approaches just described were different in spirit from those adopted by Klein and Poincaré, who hewed more closely to the ideas first elaborated by Riemann. The uniformisation theorem, conjectured by Poincaré and Klein in 1881 and eventually proved by

Poincaré and Koebe in 1907, (see Gray [1994a]) opened another route, if one could count the free constants in the Fuchsian functions having at most $m$ poles on a given Riemann surface. The first to try was a former student of Klein's, Ernst Ritter, who made a spirited attempt in his [1894] to connect Klein's work to Poincaré's. Ritter was led to what he called an extended Riemann-Roch Theorem not for functions but for pairs of automorphic forms of particular kinds, and formulated for fractional divisors (a concept first introduced by Klein in his lectures [1892, p.65]). Ritter died age 28, but his ideas were taken up by Robert Fricke and incorporated into the second volume of Fricke-Klein [1912]. Hermann Weyl, in his famous [1913], proved both the usual Riemann-Roch Theorem and what he called Ritter's extended Riemann-Roch Theorem. The first to give a proof of the Riemann-Roch Theorem using the uniformisation theorem was probably Osgood, who communicated such a proof to his Harvard colleague Coolidge in 1927 and later published it in the second volume of his *Funktiontheorie*, 1929.

## 2 SURFACES

In the wake of work by Cayley and Clebsch (see Gray [1989]) Noether defined what became known as the arithmetic genus of a surface of degree $n$ in his [1871]. It was a number, $p_a$, obtained by counting coefficients, which was related to the dimension of the space of adjoint surfaces of order $n-4$ passing $(i-1)$-times through each $i$-fold curve of $F$ and $(k-2)$-times through each $k$-fold point. Zeuthen had shown it was a birational invariant, and so one which would survive attempts to resolve the singularities. In his [1875], Noether defined the surface genus, later called the geometric genus, $p_g$, as the actual number of linearly independent surfaces of degree $n-4$ adjoint to a surface $F$ of degree $n$. He called the genus of the intersection of $F$ with an adjoint surface the linear genus and showed that surfaces with small values of these genera yielded immediately to classification, as surfaces defined by a polynomial equation of a certain degree with such-and-such double curves and multiple points.

Then in a short paper of 1886 Noether gave the first statement of a Riemann-Roch theorem for algebraic surfaces. Although hopelessly flawed, Noether's mistakes give a good indication of the difficulties inherent in the new subject. Noether took a curve $C$ of genus $\pi$ on a surface $F$, and supposed it belongs to an $r$-dimensional linear system, $|C|$, of curves of the same order, and that $C$ meets a generic curve of this system in a set, $G_s$, of $s$ points (called the characteristic series on $C$). This set of points belongs to a linear series on the curve $C$ of dimension $r-1$. If $\rho$ denotes the dimension of the space of adjoints of degree $n-4$, that also pass through $C$, then Noether's Riemann-Roch theorem asserts that

$$r \geq p_g + s - \pi - \rho + 1,$$

where $p_g$ is the geometric genus of the surface $F$. For, by the Riemann-Roch theorem on the curve $C$, there is a linear system $|C'|$ residual to $|C|$ that cuts $C$ in point group consisting of $2\pi - 2 - s$ points. This residual linear system has dimension $d = r - s + \pi - 1$, and arises by cutting $C$ with adjoint surfaces that pass through a fixed $G_s$. These include a fixed surface through the $G_s$ and a linear system of free surfaces adjoint to $F$ of dimension $p_g - \rho$. Therefore, said Noether,

$$r - s + \pi - 1 \geq p_g - \rho.$$

As Enriques and Castelnuovo showed, Noether's account rests on two crucial, and doubtful, statements. First, the application of the Riemann-Roch theorem on $C$ assumed that the characteristic series $G_s$ was complete (i.e. of maximal dimension), but this not obvious, and indeed is not always true. One can only say that $d = r - s + \pi - 1 + \delta$ , for some $\delta \geq 0$. Second, Noether's claim that $d = p_g - \rho$, is again neither obvious nor always true. One can only say that the dimension $d$ is $p_g - \rho + \eta$, where $\eta \geq 0$. Consequently one only has

$$r = s - \pi + p_g + 1 - \rho - \delta + \eta,$$

which is a disaster, because the correction terms $\delta$ and $\eta$, each non-negative, enter with opposite signs, and not even an inequality can be disentangled from the correct formula. In the absence of a Noether *Nachlass*, we may never know why Noether offered only this brief, and flawed, sketch.

Another approach to the study of algebraic surfaces was initiated by the Italian mathematician Veronese, who in his [1881] used the method of projection and section to show how curves and surfaces in the plane or in 3-space with singularities could profitably thought of a non-singular objects in a higher-dimensional space; the singularities were the result of the projection of the object into 3-space. Veronese's insight, together with that of Kraus as taken up by Klein, suggested to Corrado Segre that the best approach to surfaces would be to study them birationally, and to look for families of curves sufficiently well behaved to yield an embedding of the surface in some suitable projective space. So Segre advocated a third approach to the Riemann-Roch Theorem, also algebrao-geometrical, but with the emphasis on higher-dimensional projective geometry, which was to prove characteristically Italian. On this approach all birational images of a surface in any projective space were treated equally.

The aim became to find systems of canonical curves on an algebraic surface that yield embeddings in some projective space. Canonical curves $K$ on the surface should be cut out by appropriate adjoint surfaces (as in Noether's approach). The adjoint, $A(C)$, of a curve $C$ should be the sum $C + K$. A suitable generalisation of the Riemann-Roch Theorem should apply to the surface and a curve $C$ or the maximal linear system $|C|$ to which $C$ belongs and evaluate dimensions of linear systems of curves. In particular, if the dimension of the space of canonical curves (or some multiple of them) is large enough, the adjoint surfaces will yield an embedding of the surface in projective space.

However, as Enriques observed in his first major paper, his [1893], Noether's definition of an adjoint surface invokes the degree, so it is projective but not birational. Another definition of the terms 'adjoint' and 'canonical' must be sought. Moreover, linear families of curves on the surface will yield maps to projective space, but if the curves have base points (points common to all the curves), the image of the surface that they provide will have new singularities (the base points will 'blow up' into curves). Similarly components common to all curves of a family may blow down to points. So a way must be found of controlling, and ideally eliminating, these exceptional curves.

In an interesting split in the development of the theory, Italian algebraic geometers offered definitions that had nothing to do with holomorphic integrands, whereas the study of single and double integrals on an algebraic surface (1- and 2-forms) was energetically taken up by the French, notably Picard but also Humbert (see Houzel [1991]). The algebraic and transcendental theories were developed in parallel, with each side reading the other's work, but not merged.

When Castelnuovo and Enriques began their work, little was known about the nature of algebraic surfaces, and there was no method available for the resolution of their possible singularities (one was later developed in Jung [1908]). Much of their work behind the scenes is documented in the recently published letters of Enriques to Castelnuovo (see Bottazzini et al, [1996]). They came to favour a characterisation of surfaces in terms of integers, generalising the arithmetic and geometric genera, and the crucial result that gave them control over these numbers was their formulation of a Riemann-Roch Theorem.

To produce birationally invariant definitions in his [1893], Enriques excluded irregular surfaces (those for which the geometric genus exceeds the arithmetic genus, such as ruled surfaces) and surfaces of genus 0. For regular algebraic surfaces of genus greater than zero he could give quite general conditions that ensured that the characteristic series was complete. Under these conditions he considered a curve $C$ of genus $\pi$ on the surface that belongs to a linear system of curves of dimension $r$, such that $C$ meets a generic curve of $|C|$ in $s$ points. If the system $|C|$ is not contained in the canonical system he supposed that through the points common to two curves $|C|$ there passed a space of adjoint curves of dimension $2p+\omega$; he called the non-negative number $\omega$ the super-abundance of $|C|$. If the system $|C|$ is contained in the canonical system and the residual system has dimension $i-1$, the space of adjoint curves has dimension $2p+\omega-i$. He then established the first Riemann-Roch Theorem for algebraic surfaces:

$$r = s - \pi + p_g + 1 + \omega - i,$$

by an ingenious application of the Riemann-Roch theorem on the curve $C$. His friend Castelnuovo then showed how irregular surfaces could be treated, in his [1896].

Enriques soon became dissatisfied with the arithmetic and geometric genera, because they did not characterise surfaces. In 1896 (see Bottazzini et al [1996, p. 278]) Enriques considered a tetrahedron in $\mathbf{CP}^3$ and observed that there was a surface of degree 6 which had the edges of the tetrahedron as its double curves. Its adjoint surfaces must be of degree 6 - 4 = 2, and must pass through the double curves of the surface. But plainly there is no quadric surface through the 6 edges of a tetrahedron. However, there is a surface of degree $2(n-4) = 2.(6-4) = 4$ which passes twice through the edges of the tetrahedron: the surface composed of the four planes that form the faces of the tetrahedron. So the surface of degree 6 has no adjoint surface and its genus is zero, but it does have what is called a bi-adjoint surface and its bi-genus, the dimension of the space of bi-adjoint surfaces, is $P_2$ is 1, not zero. In his [1896] Castelnuovo showed that a surface with arithmetic and geometric genera equal to zero and bi-genus $P_2 = 0$ is indeed a rational surface. This was the first birational characterisation of a surface. It also marks the moment when the so-called plurigenera decisively enter the analysis. The $i^{th}$ plurigenus, $P_i$, is defined as one more than the dimension of the $i^{th}$ multiple of the canonical system, $|iK|$. In their [1901] Castelnuovo and Enriques used the Riemann-Roch Theorem to obtain lower bounds on the plurigenera (see below).

Enriques' [1896] marks a considerable advance on his [1893] in its level of generality. Irregular surfaces could now be treated, because of recent discoveries by Castelnuovo in his [1896], and a Riemann-Roch Theorem proved about them. The characteristic property of a canonical curve was now that it was a residual curve of any linear system $|C|$ with respect to the adjoint system $A(C)$. Enriques pointed out that this indirect definition had the advantage of being independent of the nature of the fundamental curves of $|C|$ which were therefore subject to no restriction. In his opinion, this made it most appropriate to a birational theory of surfaces.

In their [1901] Castelnuovo and Enriques made notable simplifications to the theory, when they showed that a non-ruled surface can be transformed to one without exceptional curves (curves obtained as the blow-up of points under a birational transformation). This established the existence and uniqueness of what became called minimal models. For surfaces without exceptional curves, Castelnuovo's formula for the plurigenera applied to linear system $|C|$ of genus $\pi$ and degree $n$, for which $n < 2\pi - 2$ (the genus of $|C|$ is the genus of a generic member of $|C|$, the degree the number of points in the generic intersection of two members of $|C|$). It asserts that

$$P_i \geq p_a + \frac{1}{2}i(i-1)(p^{(1)} - 1) + 1,$$

for all $i > 1$. The number $p^{(1)}$ is Noether's linear genus of the surface, but with a new birational definition that applies where Noether's does not, as Castelnuovo and Enriques pointed out. The formula indicates that the cases $p^{(1)} > 1$ and $p^{(1)} \leq 1$ will be very different. In fact, the plurigenera can grow in essentially four ways: they might all be 0, they might be 0 or 1, they might grow linearly with $i$ or quadratically with $i$. This distinction lies at the heart of the subsequent classification of algebraic surfaces.

Castelnuovo and Enriques now gave a preliminary classification of surfaces, characterising rational and ruled surfaces in terms of the values of $p^{(1)}$ and the plurigenera. In 1905 Enriques characterised ruled surfaces in similar terms as the surfaces for which $p_g = 0 = P_4 = P_6$. In his [1906] Enriques characterised his sextic surface birationally as the only surface for which $p_a = 0 = P_3$, $P_2 = 1$. This characterised the class of surfaces nowadays called Enriques surfaces. In his [1906] with Castelnouvo (published as an appendix in the second volume of Picard and Simart's book) and again in his [1907b] Enriques analysed surfaces for which

$p^{(1)} = 1$, and connected the values of $p^{(1)}$ and $p_g$ to growth of the plurigenera.

In his [1914a, b] Enriques turned back to the study and classification of algebraic surfaces. He reported in more detail in his essay with Castelnuovo published in the *Encyklopädie der Mathematischen Wissenschaften* for 1914. The behaviour of the plurigenera led him to argue now that the crucial feature was the value of $P_{12}$. If $P_{12} = 0$ the surface was ruled; if $P_{12} = 1$ then $p^{(1)} = 1$ and many of the surfaces discovered by Enriques and Severi, Picard, and Bagnera and de Francis belong in this family. If $P_{12} > 1$ and $p^{(1)} > 1$ then the surface has effective canonical and pluri-canonical curves of some positive order (which in turn meant that it could be embedded in some projective space). Or rather, a model of the surface containing no exceptional curves could be mapped birationally onto its image in a projective space of an appropriate dimension. The classification is actually somewhat finer, but it is clear that the broad outlines of the classification are provided by numbers determined, in fair part, by the Riemann-Roch Theorem.

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