Tensor products of algebras over a field K

Recall that an associative K-algebra is a ring A such that the field K is a subring of the center of A. All algebras in this note will be associative.

The basic property of the tensor product $A \otimes_K B$ is that for every K vector space C

$$Hom(A \otimes B, C) \simeq Hom(A, Hom(B, C))$$

When K is replaced by a commutative ring R and A, B, C by R-modules this property is satisfied by the tensor product of R-modules. The case of vector spaces can be described using bases, so it is somewhat simpler than the case of modules over a ring. We initially only need tensor products for vector spaces.

Given bases a_α, b_β of A, B as vector spaces over K, the tensor product $A \otimes_K B$ is the vector space with basis given by symbols $a_\alpha \otimes b_\beta$. Given an element $\phi \in Hom(A \otimes B, C)$ consider the vector space homomorphism which sends a_α to the homomorphism $\psi \in Hom(B, C)$ sending b_β to $\phi(a_\alpha, b_\beta)$. It is easy to verify that this is a natural isomorphism of $Hom(A \otimes B, C) \simeq Hom(A, Hom(B, C))$.

For $v = \sum x_\alpha a_\alpha \in A, w = \sum y_\beta b_\beta \in B$ we define $v \otimes w = \sum_{\alpha,\beta} x_\alpha y_\beta a_\alpha \otimes b_\beta$. When A, B are K-algebras, then $A \otimes_K B$ has an algebra structure defined by $(v \otimes w)(v' \otimes w') = vv' \otimes ww'$.

Example 1: Let $K[x]$ vector space of polynomials with coefficients in K. The map sending the vector space $K[x] \otimes_K L$ to $L[x]$ defined by $p(x) \otimes l \to lp(x)$ is easily seen to be an isomorphism of vector spaces by checking using the basis x^i of $K[x]$. Thus $K[x] \otimes_K L$ is an algebra over L.

Example 2: Let $f(x) \in K[x]$ be a polynomial. Then $K[x]/(f(x)) \otimes L \simeq L[x]/(f(x))$ since the isomorphism above identifies the subspace $(f(x)) \otimes_K L$ in $K[x] \otimes L$ with the ideal that $f(x)$ generates in $L[x]$.

Example 3: The tensor product of a number field E with an extension $L/\mathbf{Q}$ is isomorphic to a product of fields, since E can be expressed as $\mathbf{Q}[x]/(f(x))$ for an irreducible polynomial $f(x) \in \mathbf{Q}[x]$ by the primitive element theorem and then the previous example together with the Chinese Remainder Theorem show that if $f(x) = \prod f_j(x)$ is the factorization of $f(x) \in L[x]$ then $E \otimes_Q L \simeq L[x]/(f(x)) \simeq \prod L[x]/(f_j(x))$. (Since the fields have characteristic 0, there are no repeats among the $f_j(x)$).

Example 4: If L/K is a Galois extension of number fields with Galois group G, then $L \otimes_k L = \prod_{g \in G} L$. This follows from Example 3 by noting that if $L = \mathbf{Q}[x]/(f(x))$ for an irreducible polynomial $f(x) \in \mathbf{Q}[x]$ and L/K is Galois, then L contains all roots of $f(x)$.