Problem Set 7.

1. Show that the maximal order in the pure cubic number field $\mathbf{Q}(2^{1/3})$ has trivial Picard group .

2.
   a) Suppose that R is a commutative ring in which all nonzero prime ideals are maximal and every ideal is equal to a product of powers of prime ideals. If R has finitely many prime ideals, use the Chinese remainder theorem to show that R is a principal ideal ring (that is all ideals are principal). Give an example of such an R which is not a field.

   b) Use part a) to show that given an ideal I of the maximal order O_K of a number field and an nonzero element $a \in I$ there exists an element $b \in I$ such that $I = aO_K + bO_K$.

3. Let $\mathcal{O}$ be an order in a number field K and let J be a nonzero ideal in this order. Show that every invertible ideal is equivalent to an ideal I which is prime to J (that is $I + J = \mathcal{O}$) by carrying out the following:

   a) Let I be a nonzero finitely generated proper ideal in an integral domain R. Use Nakayama's lemma (if I is an ideal in R a ring, M finitely generated R-module, then $IM = M$ implies there is some $z \in I$ such that $zm = m$ for all $m \in M$) to show that the ideals $I, I^2, I^3, \cdots$ are distinct.

   b) Let P be a maximal ideal in an order $\mathcal{O}$ in a number field and let I be an invertible ideal in $\mathcal{O}$. Show that there exists an integer $t \geq 0$ such that $I \subset P^t$ but $I \not\subset P^{t+1}$. Remark: The Krull Intersection Theorem states that in any Noetherian integral domain the intersection of the powers of a proper ideal equals 0. In the case that we are working in an order we can use a) to avoid the general Intersection Theorem.

   c) Let I, P, t be as in b). Show that there exists an element $\alpha \in I$ such that $I + P^{t+1} = \alpha\mathcal{O} + P^{t+1}$, that is the image of I in $\mathcal{O}/P^{t+1}$ is a principal ideal. Hint: essentially the same proof as Problem set 6.3.i except for passing to the quotient instead of inverting elements outside P

   d) Let $P_1, \ldots, P_r$ be the maximal ideals containing J. Let I be an invertible ideal. Let t_i be the greatest power such that $I \subset P^{t_i}$ and let $I + P_I^{t_i+1} = \alpha_i\mathcal{O} + P^{t_i+1}$. Show by the Chinese Remainder Theorem that there exists $\beta \in I$ such that $\beta - \alpha_i \in P_i^{t_i+1}$ and βI^{-1} is an ideal prime to J.

e) Let J be the conductor of $\mathcal{O}$, that is the largest $\mathcal{O}_{\mathcal{K}}$ ideal contained in $\mathcal{O}$. Show that $\mathrm{Pic}(\mathcal{O})$ is isomorphic to the group of fractional ideals prime to J modulo principal fractional ideals prime to J and is generated by prime ideals prime to J.

f) Show that there is a bijection between prime ideals of $\mathcal{O}$ prime to the conductor of $\mathcal{O}$ and prime ideals of the maximal order prime to the conductor. Show that the natural map $\mathrm{Pic}(\mathcal{O}) \to \mathrm{Pic}(\mathcal{O}_{\mathcal{K}})$ given by $I \to I\mathcal{O}_{\mathcal{K}}$ is surjective by using parts d) and e) above and Problem 6.4.