Problem Set 6.

1. Show that the maximal order in $\mathbf{Q}(\sqrt{6})$ is a principal ideal domain by using the Minkowski bound to show every class of fractional ideals is represented by an ideal containing 2 and then finding all such ideals.
2. Consider the field $K = \mathbf{Q}(\sqrt{-6})$.
   a) Show that every ideal class for the maximal order of K is represented by an ideal of norm at most 3.
   b) Show that in the ring of integers of K
   $$(2) = (2, \sqrt{-6})^2, \qquad (3) = (3, \sqrt{-6})^2.$$
   c) Find all ideals of $\mathbf{O}_K$ of norm 2 and 3.
   d) Show that $\mathbf{O}_K$ is not a principal ideal domain.
   e) Show that K has Picard group of order 2 by using the above.
   f) Find principal ideals A and B so that
   $$A(2, \sqrt{-6}) = B(3, \sqrt{-6})$$
3. Let $\mathcal{O}$ be an order in a number field K and define the conductor of the order $\mathcal{O}$ to be the $\mathcal{O}_K$ submodule of $\mathcal{O}$ given by $c(\mathcal{O}) = (\mathcal{O} : \mathcal{O}_K) = \{x \in K | x\mathcal{O}_K \subset \mathcal{O}\}$. Note that $c(\mathcal{O})$ is the largest $\mathcal{O}_K$ module contained in $\mathcal{O}$. We say a ideal I of $\mathcal{O}$ is prime to an ideal J of $\mathcal{O}$ if $I + J = \mathcal{O}$.
   i) For any prime P of $\mathcal{O}$ let $\mathcal{O}_P$ be the subring of K given by expressions r/s for $r \in \mathcal{O}$ and $s \notin P$. Show that there is a unique maximal ideal $P\mathcal{O}_P$ in $\mathcal{O}_P$. Show that if I is an invertible ideal in $\mathcal{O}$ then $I\mathcal{O}_P$ is a principal ideal in $\mathcal{O}_P$. Hint: Write $1 \in II^{-1}$ as a sum of elements in $\mathcal{O}$ of the form $x_i y_i, x_i \in I, y_i \in I^{-1}$. Some summand is not in P, say $x_1 y_1$. Show that $I\mathcal{O}_P = x_1 O_P$.
   ii) Show that the prime ideals P in $\mathcal{O}$ which contain $c(\mathcal{O})$ are not invertible $\mathcal{O}$-ideals. (Hint: Show that if P is invertible in $\mathcal{O}$ then any ideal of $\mathcal{O}_P$ is of the form $(P\mathcal{O}_P)^k$ and is principal. Use this to show that any algebraic integer in K is an element of $\mathcal{O}_P$. Use the denominators of members of a integer basis for O_K considered as elements of $\mathcal{O}_P$ to construct an element of $c(\mathcal{O})$ which is not in P when P is invertible).
4. Show that if P is a prime ideal in an order $\mathcal{O}$ such that $P + c(\mathcal{O}) = \mathcal{O}$ then $P\mathcal{O}_K$ is a prime ideal in $\mathcal{O}_K$. (Hint: Show that PO_K is a proper ideal of $\mathcal{O}_K$ and that the natural map from $\mathcal{O}/P$ to $\mathcal{O}_K/P\mathcal{O}_K$ is surjective.)