Problem Set 12.

Remark: For K a number field, we abuse terminology by referring to the units in $\mathbf{O}_K$ as "units of K". By "fundamental units of K" we of course mean a basis for $\mathbf{O}_K^*/(\mathbf{O}_K^*)_{tor}$ as a free Abelian group.

1. Suppose that a and b are square free positive integers greater than 1. Show that the units of $\mathbf{Z}[\sqrt{a}, \sqrt{-b}]$ are the same as the units of $\mathbf{Z}[\sqrt{a}]$.
2. Find fundamental units in $\mathbf{Q}(\sqrt{d})$ for d= 3,5,6,7,10,30.
3. Show that $2 - \sqrt[3]{7}$ is a fundamental unit in $\mathbf{Q}(\sqrt[3]{7})$.
4. Let $K = \mathbf{Q}(\sqrt[3]{5})$.
   a) Show that $u_0 = 41 + 24\sqrt[3]{5} + 14\sqrt[3]{25}$ is a unit of K and that the real embedding of u_0 has absolute value less than 125.
   b) Show that the ideal generated by $2 - \sqrt[3]{5}$ has norm 3.
   c) Show that if u is a unit of K with the absolute value of its real embedding between 1 and 12, then $u(2 - \sqrt[3]{5})$ must be of the form $a + b\sqrt[3]{5}$ with a and b integers satisfying $|b| < 2, |a| < 3$. Hint: Compute upper bounds for the absolute values of the imbeddings of $u(2 - \sqrt[3]{5})$ and use this to effectively estimate the size of its coefficients when expressed as a combination of 1, $\sqrt[3]{5}$, $\sqrt[3]{25}$.
   d) Show using c) (or any other way you wish) that u_0 is a fundamental unit of K.
5. Find all integers x, y which solve the equation $3x^2 - 4y^2 = 11$ by introducing a suitable order $\mathbf{O}$ in a number field and analyzing it.