Mathematics 552 **Algebra II** **Spring 2021**

When is $a \otimes b$ equal to zero in the tensor product $A \otimes_R B$?

Let R be a ring, A a right R-module, B a left R-module generated by b_α. Suppose that an element $\sum a_\alpha b_\alpha$ equals 0 in $A \otimes_R B$ (only finitely many a_α are nonzero elements of A). Consider the exact sequence mapping the free left R-module F generated by symbols g_α onto B by sending g_α to b_α, with kernel U:

$$0 \to U \to F \to B \to 0.$$

Tensoring with A is right exact, so the sequence

$$A \otimes_R U \to A \otimes_R F \to A \otimes_R B \to 0$$

is exact. The element $\sum a_\alpha \otimes g_\alpha$ in the free module $A \otimes_R F$ maps to $\sum a_\alpha \otimes b_\alpha = 0$ in $A \otimes_R B$ by assumption, so it is in the image of $A \otimes_R U$ by exactness. Thus there exist $a'_\beta \in A, u_\beta \in U$ such that $\sum a_\alpha \otimes g_\alpha = \sum_\beta a'_\beta \otimes u_\beta$. Since $U \subset F$ there exist exist $r_{\alpha\beta}$ such that $u_\beta = \sum_\alpha r_{\alpha\beta} g_\alpha$ with $\sum_\alpha r_{\alpha\beta} b_\alpha = 0$. Thus

$$\sum a_\alpha \otimes g_\alpha = \sum_\beta a'_\beta \otimes \sum_\alpha r_{\alpha\beta} g_\alpha = \sum_\alpha (\sum_\beta a'_\beta r_{\alpha\beta}) \otimes g_\alpha.$$

Since F is a free R-module, $A \otimes_R F$ is the direct sum of copies of A, and the two expressions above are equal if and only if the coefficient $a_\alpha = (\sum_\beta a'_\beta r_{\alpha\beta})$

This proves the interesting direction in the following proposition.

Proposition. *Let A, B, R, b_α be as in the first paragraph. Then the element $\sum_\alpha a_\alpha \otimes b_\alpha$ equals 0 in $A \otimes_R B$ if and only if there exist $a'_\beta \in A, r_{\alpha\beta} \in R$ such that*

$$a_\alpha = \sum_\beta a'_\beta r_{\alpha\beta} \text{ and } \sum_\alpha r_{\alpha\beta} b_\alpha = 0$$

Proof.

The existence of such $a'_\beta \in A, r_{\alpha\beta} \in R$ was established above when $\sum a_\alpha \otimes b_\alpha$ equals 0 in $A \otimes_R B$.

Conversely, if such $a'_\beta \in A, r_{\alpha\beta} \in R$ satisfying the two equations of the proposition exist, then

$$\sum_\alpha a_\alpha \otimes b_\alpha = \sum_\alpha (\sum_\beta a'_\beta r_{\alpha\beta}) \otimes b_\alpha = \sum_\beta a'_\beta \otimes (\sum_\alpha r_{\alpha\beta} b_\alpha) = 0$$

Exercise: Use this to prove that when B is an abelian group the element $1 \otimes b = 0$ in $\mathbf{Q} \otimes B$ if and only if there is a nonzero integer m such that $mb = 0$ in B.