Mathematics 552 **Algebra II** **Spring 2021**

Week 6 Semisimple rings, the Artin-Wedderburn Theorem and Group Representations
Jacobson II: 5.1, 5.2, 5.3, 5.5, 5.6

1. Show that if R is a semisimple ring and M is a finitely generated left R-module, then $End_R(M)$ is a semisimple ring.
2. Jacobson II 5.3.1
3. Jacobson II 5.3.3 (Take G to be a finite group and take all representations of G to be finite dimensional. Then the representations of G are homomorphisms to groups of matrices. Check that isomorphism of matrix representations is equivalent to the existence of an invertible matrix A with entries in K such that $A\rho_1(g) = \rho_2(g)A$ for all $g \in G$. Use a basis for K/F (which may be infinite dimensional) to write A as a K-linear combination of r matrices A_i with entries in F and show the matrices A_i satisfy the same relation as A does when multiplying the matrices $\rho_i(g)$. Show that $det(X_1A1 + \cdots X_rA_r)$ is not the zero polynomial since A has an inverse, and that when F has infinitely many elements (Jacobson I Theorem 2.19) there are $x_i \in F$ such that $A' = \sum x_iA_i$ has entries in F and is invertible and $A\rho_1(g) = \rho_2(g)A$. The theorem is true in the case of F finite as well but requires more work.)
4. Let G be a finite group and consider representations of G on finite dimensional complex vector spaces.
   a) Show that the element $t = \sum_{g\in G} g$ in $\mathbf{C}[G]$ is in the center of the group algebra, hence by Schur's lemma acts by a scalar on any irreducible representation V. Show that if the subspace $W = tV \subset V$ is nonzero, then V is the trivial representation. Show that $\chi_V(t) = 0$ if V is not the trivial representation, and $\chi_V(t) = |G|$ if V is the trivial representation. Show that if V, W are irreducible representations then $\chi_{V^*\otimes W}(t)$ is 0 or $|G|$ according as V, W are not isomorphic or are isomorphic. Deduce that the characters of irreducible representations of G form an orthonormal family of functions with respect to the pairing $< \phi, \psi >= (1/|G|) \sum_{g\in G} \phi(g^{-1})\psi(g)$.
   b) Show that if a representation V of G decomposes as a sum of irreducibles $V = \oplus m_iV_i$ with V_i, V_j not isomorphic when $i \neq j$, then $< \chi_V, \chi_V >= \sum m_i^2$. Show that V is irreducible if and only if $< \chi_V, \chi_V >= 1$.
   c) Show that if $W = \oplus m_iV_i$ is the decomposition of W into distinct irreducibles, then $< \chi_W, \chi_{V_i} >= m_i$. This, together with b) allows one to check from characters if a representation is irreducible, and if so how many times it appears in another representation.

5. The complex representations of S_4, the symmetric group on four letters.

    a) Find the conjugacy classes in S_4, find all normal subgroups, and determine how many irreducible complex representations S_4 has.

    b) Show that the group G of rotations preserving a cube permutes the 4 diagonals joining opposite corners of the cube and that this gives an injective group homomorphism from G to S_4. Show that G acts transitively on the 8 vertices of the cube and compute the stabilizer in G of a vertex. Show that the group G of rotations is isomorphic to S_4

    c) Show that the group G of rotations of the cube maps onto the group of permutations of the lines joining centers of opposite faces and that S_3 is a quotient of G. Use this to find some irreducible representations of S_4 and to determine the dimension of the remaining irreducible representations.

    d) Show that the natural representation of G as linear maps of R^3 is irreducible

    e) Let W be the representation of G on the vector space of complex functions on the faces of the cube. Compute the character of W and express W as a sum of irreducible representations.

6. Jacobson II 5.5.7

7. Jacobson II 5.6.1

8. Jacobson II 5.6.4