Mathematics 552 **Algebra II** **Spring 2021**

Week 2 The Dehn invariant, the Cauchy-Binet formula
Jacobson I:7.1-7.2, Jacobson II: 1.1-1.3 , 3.1, 3.7-3.9

1. Let A and B be abelian groups.
   a) Let m be a positive integer and let $a \in A$ be divisible by m in the sense that $a = ma'$ for some $a' \in A$. Show that $a \otimes b = a' \otimes mb$ in $A \otimes B$.
   b) Let m and a be as in part a). Let $b \in B$ be an element of an abelian group B such that $mb = 0$ in B. Show that the element $a \otimes b$ is the identity element of $A \otimes B$.
   c) Let A be an abelian group such that A is divisible in the sense that $A = mA$ for all positive integers m. Suppose that B is a torsion abelian group (that is, all elements of B have finite order). Show that $A \otimes B = 0$. Find an example of a nontrivial abelian group A such that $A \otimes A = 0$.
   d) Let S be the set of collections of finitely many rectangles in $\mathbf{R}^2$ with sides parallel to the coordinate axes. We say that a collection of such rectangles is equivalent to another if using only vertical and horizontal cuts (finite in number) and translation the rectangles in one collection can be cut up and taped together along sides to produce the rectangles in the second collection. Show that we can coordinatize such collections up to equivalence by the abelian group $\mathbf{R} \otimes \mathbf{R}$
   e) Show that if α_i are real numbers which are linearly independent in the rational vector space $\mathbf{R}$ then the element $\sum_i r_i \otimes \alpha_i = 0$ in $\mathbf{R} \otimes \mathbf{R}$ if and only if all $r_i = 0$
   f) Show that a rectangle with a vertical side of length $\sqrt{2}$ and horizontal side of length $1/\sqrt{2}$ cannot be cut apart by a finite number of horizontal and vertical cuts and translated and rejoined along edges to form a square.
2. Show that in $\mathbf{R} \otimes \mathbf{R}/\pi\mathbf{Z}$ we have that for any nonzero integer m that $l \otimes \alpha/m = l/m \otimes \alpha$. Show that given an element $z \in \mathbf{R} \otimes \mathbf{R}/\pi\mathbf{Z}$ there is a finite set S of real numbers (perhaps empty) such that $z = \sum_{s \in S} l_s \otimes s$ such that the union of S and π is a linearly independent set over the rational field. Show that z is zero in the tensor product if and only if all $l_s = 0$. Show that $\mathbf{R} \otimes \mathbf{R}/\pi\mathbf{Z}$ is a torsion free abelian group.
3. Use the formula which computes the $k \times k$ minors of the product of a matrix A of size $m \times n$ and a matrix B of size $n \times p$ in terms of $k \times k$ minors of A and B in the special case $m = p = 2$ to prove Lagrange's identity: given elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ of a commutative ring R then

$$(\sum_{l=1}^{n} a_l^2)(\sum_{l=1}^{n} b_l^2) - (\sum_{l=1}^{n} a_l b_l)^2$$

is a sum of squares in R. (Note this generalizes the Cauchy-Schwarz inequality for the case R the real numbers).