Mathematics 552 **Algebra II** **Spring 2021**

Week 12 Simple Algebras and the Brauer Group
Jacobson II: 4.6, 4.7

1. The statements below will prove: Let K be a field and let D_1, D_2 be division algebras over K of dimensions n_1, n_2 (the centers may be larger than K). If n_1, n_2 are relatively prime, then $D_1 \otimes_K D_2$ is a division algebra. (a generalization of Jacobson II 4.6.9)

   a) Show that for any simple K-algebras A_1, A_2 there is a simple algebra B and a surjective K-algebra homomorphism $A_1 \otimes A_2 \to B$. (Hint: take the quotient by maximal two-sided ideal of $A_1 \otimes A_2$).

   b) For division algebras of finite dimension over K consider an algebra B constructed in a) as a vector space under $D_1 \otimes 1$. Show that $dim_K B = dim_{D_1} B dim_K D_1 \leq dim_K D_1 dim_K D_2$.

   c) If $dim_K D_1$ is relatively prime to $dim_K D_2$ show that $dim_K B = dim_K D_1 dim_K D_2$ so that $D_1 \otimes_K D_2$ is a simple algebra.

   d) If $D_1 \otimes_K D_2$ is a simple algebra, let L be an irreducible ideal of the simple ring $A = D_1 \otimes_K D_2 = M_r(E)$, where the division algebra $E = (End_A(L))^{opp}$. Then $A = L^r$ as an A-module. L is a module over $D_1 \simeq D_1 \otimes 1$ and $D_2 \simeq 1 \otimes D_2$. Compute the dimension of A as vector space over D_1 and D_2 and show that r divides the dimensions of D_1, D_2 as vectors spaces over K. Conclude that if these dimensions are relatively prime, then $r = 1$, establishing the statement at the beginning of this problem.

2. Jacobson II 4.6.10

3. Jacobson II 4.6.11 (Use Skolem-Noether).