A minimum of category theory in two pages

Category: A class of objects $obj(\mathcal{C})$, disjoint sets of morphisms $\mathrm{Mor}_{\mathcal{C}}(A,B)$ between objects A,B and associative composition gf of $f \in \mathrm{Mor}_{\mathcal{C}}(A,B), g \in \mathrm{Mor}_{\mathcal{C}}(B,C)$ and distinguished morphism $1_A \in \mathrm{Mor}_{\mathcal{C}}(A,A)$ such that $f1_A = f, 1_B g = g$.

*Functo*r from category $\mathcal{C}$ to category $\mathcal{D}$: an assignment to each object A of C and object $F(A)$ and to each morphism $f \in \mathrm{Mor}_{\mathcal{C}}(A,B)$ a morphism $F(h) \in \mathrm{Mor}_{\mathcal{C}}(F(A),F(B))$ such that $F(gf) = F(g)F(f)$.

A morphism $f \in \mathrm{Mor}_{\mathcal{C}}(A,B)$ is an *isomorphism* if it has an inverse under composition, an *epimorphism* if $gf = hf$ implies $g = h$ and a *monomorphism* if $fg = fh$ implies $g = h$ (corresponding to isomorphism, surjection and injection in the categories $Set, Grp, R-Mod$)

An *initial object* of a category $\mathcal{C}$ is an object A such that $\mathrm{Mor}_{\mathcal{C}}(A,B)$ is a one element set for all objects B, a *terminal object* is an object Z such that $\mathrm{Mor}_{\mathcal{C}}(B,Z)$ is a one element set for all objects B , and a *zero object* is an object which is initial and terminal.

A *natural transformation* η of functors F_1, F_2 from category $\mathcal{C}$ to $\mathcal{D}$ is an assignment to each object A of $\mathcal{C}$ a morphism $\eta_A \in \mathrm{Mor}_{\mathcal{D}}(F_1(A), F_2(A))$ such that for each morphism $f \in \mathrm{Mor}_{\mathcal{C}}, \eta_B \circ F_1(f) = F_2(f) \circ \eta_B$. A *natural isomorphism* of functors is a natural transformation that has a two sided inverse natural transformation.

Categories $\mathcal{C}, \mathcal{D}$ are*isomorphic* if there exist functors F from $\mathcal{C}$ to $\mathcal{D}$ and G from $\mathcal{D}$ to $\mathcal{C}$ such that FG and GF equal the identity functors on $\mathcal{D}, \mathcal{C}$ respectively. If the we replace the word equal by the terms “are naturally isomorphic to” we say the categories are *equivalent*.

Diagram in category C: A functor D from index category $\mathcal{I}$ to $\mathcal{C}$, for example the constant diagram which sends all objects in the index category to the same object N in $\mathcal{C}$ and all morphisms to the identity morphism 1_N.

A *cone* over a diagram D is a natural transformation from a constant diagram to D (*cocone* : to a constant diagram from D)

Limit of a diagram D is a terminal object in category of cones over D , a *colimit* is an initial object in the category of cocones over D.

Examples: If D is a category with two objects and no morphisms between them a cocone is a diagram with maps from objects A,B to Y and an initial object is an object called the coproduct which has morphisms from A,B and for any other Y having morphisms from A and B they are obtained via a map from the coproduct to Y. In the category of modules this is the direct sum $A \oplus B$. In other words, to give a pair of maps from A,B to an object is the came as giving a map of the coproduct to the object. The limit of this diagram in a category is the product $A \times B$ which has maps to A, B and any other map of an object Y to A and B factors through a map of $Y \to A \times B$.

Adjoint functors: We say $F : \mathcal{C} \to \mathcal{D}$ and $G : D \to C$ are adjoint if $\mathrm{Mor}_{\mathcal{D}}(FC, D) = \mathrm{Mor}_{\mathcal{C}}(C, GD)$ and there is a natural isomorphism of $\mathrm{Mor}_{\mathcal{D}}(F\_, \_)$ and $\mathrm{Mor}_{\mathcal{C}}(\_, G\_)$ as functors from $\mathcal{C}^{op} \times \mathcal{D}$ to SET. E say F is *left adjoint to* G and G is *right adjoint.* Note left adjoints preserve terminal objects, right adjoints preserve initial objects.

Theorem: Left adjoints preserve colimits, right adjoints preserve limits:

Proof: Let CoL be a colimit of a diagram D. Apply F to the diagram D and the cocone of it with vertex CoL. This is a cocone over the diagram FD with vertex F(CoL). Given any other cocone over FD with vertex Y, by the adjoint property we have elements of $\mathrm{Mor}(FD_i, Y) = \mathrm{Mor}(D_i, GY)$. Thus the object GY is the vertex of a cocone for D, hence there is a unique map of the cone with vertex CoL to GY, and by adjointess a unique morphism of F(CoL) to Y. Hence F(Col) is a colimit for the diagram FD. The proof for colimits is similar except for reversing the morphisms appropriately.