

### The set of $k$ -dimensional subspaces in an $m$ -dimensional vector space

Let  $K$  be a field and  $M$  a vector space over  $K$  of dimension  $m$ . For an integer  $k$  with  $0 \leq k \leq m$  let  $G(k, M)$  be the set of  $k$ -dimensional subspaces of  $M$ . We also write  $G(k, m)$  for the set  $G(k, K^m)$  and refer to it as the Grassmannian of  $k$ -planes in  $M$  in honor of Grassmann who introduced exterior algebras.

**Theorem.** *There is a one-to-one map of the set  $G(k, M)$  to the set of one dimensional subspaces in  $\Lambda^k(M)$  spanned by decomposable vectors of the form  $v_1 \wedge \cdots \wedge v_k$ .*

**Proof.**

Suppose that  $V$  is a  $k$ -dimensional subspace of  $M$  with basis  $v_1, \dots, v_k$ . Then  $v_1 \wedge \cdots \wedge v_k \in \Lambda^k(M)$  is decomposable. If  $v'_1, \dots, v'_k$  is another basis of  $V$  then the isomorphism  $\phi$  of  $V$  with itself obtained by sending  $v_i$  to  $v'_i$  induces the map multiplication by  $\det(\phi)$  on  $\Lambda^k(V)$  so that  $v_1 \wedge \cdots \wedge v_k = \det(\phi) v'_1 \wedge \cdots \wedge v'_k$ . Thus changing the basis does not alter the line associated to  $V$  so the map is well defined.

Given  $\omega \in \Lambda^k(M)$  consider the linear map  $\phi_\omega(v) = v \wedge \omega$  from  $V$  to  $\Lambda^{k+1}V$ . Let  $w_1, \dots, w_r$  be a basis for the kernel of this map and extend this linearly independent set to a basis  $w_1, \dots, w_n$  of  $M$ . Write  $\omega = \sum_I a_I w_I$  where  $I$  is a set with  $k$  elements  $i_1 < i_2 < \cdots < i_k$  from the integers  $1, \dots, m$  and  $w_I = w_{i_1} \wedge \cdots \wedge w_{i_k}$ . Since  $w_1$  is in the kernel of  $\phi_\omega$  we have  $w_1 \wedge \omega = \sum_I a_I w_1 \wedge w_I = 0$ . Thus  $a_I = 0$  if  $1 \notin I$ , so that each  $w_I$  appearing with nonzero coefficient is of the form  $w_1 \wedge w_{I-1}$  so that  $\omega = w_1 \wedge \omega_1$  for some  $\omega_1 \in \Lambda^{k-1}(M)$ . Similarly  $\omega$  is divisible by  $w_1 \wedge \cdots \wedge w_r$  so that  $r \leq k$  and if  $r = k$  then  $\omega$  is decomposable. Thus given any decomposable element  $\omega = v_1 \wedge \cdots \wedge v_k \in \Lambda^k(M)$  the kernel of  $\phi_\omega$  is the  $k$ -dimensional subspace of  $M$  spanned by  $v_i$ , so a decomposable  $\omega$  determines a unique subspace  $V$ , showing that the map is one-to-one.

The coordinates of  $\omega$  with respect to the basis  $v_I$  are called the Plucker coordinates of the subspace  $V$ . So the one dimensional subspaces spanned by decomposable elements coordinatize the set  $G(k, M)$ . If  $A$  is the  $k \times m$  matrix expressing a basis of  $V \subset M$  in terms of a chosen basis of  $M$  then the coefficient  $A_I$  of  $v_I$  in this expression is the determinant of the submatrix formed by columns indexed by elements of  $I$ .

**Proposition.** *There exists a collection of polynomials in the variables  $x_I$  indexed by all  $I$  an ordered list of  $k$  elements chosen from  $[1, m]$  such that an element  $\omega = \sum_I a_I v_I \in \Lambda^k(V)$  is decomposable if and only if the coordinates  $a_I$  are zeros of each polynomial in the collection.*

**Proof.**

Let  $v_i$  be a basis of  $M$ . An element  $\omega = \sum_I a_I v_I$  has rank  $\phi_\omega$  at least  $n - r$  by the rank-nullity formula, with equality if and only if  $\omega$  is decomposable. The matrix of  $\phi_\omega$  involves elements of the ring generated by  $a_I$ , and the condition to be rank =  $n - r$  is that all larger minors vanish, giving a collection of polynomials in  $a_I$  with simultaneous zero set precisely the coordinates of the decomposable elements.