MATHEMATICS 552 ALGEBRA II SPRING 2021

1.

1.1. Introduction. These notes deal with Artinian rings, those in which any descending chain of ideals stabilizes.

We begin with some results for arbitrary rings R.

Definition. *The Jacobson radical $J(R)$ of a ring R is the intersection of the kernels of all irreducible representations of R.*

The Jacobson radical is a two sided ideal, since every kernel is. If I is a nilpotent left ideal which is not in J , say $I^k = 0$ and V is a simple module such that $IV \neq 0$ then $IV \subset V$ is a nonzero submodule which by simplicity equals V. Then $V = IV = I^2V = \cdots = I^kV = 0$ gives a contradiction, so every nilpotent ideal is in the Jacobson radical.

We will have use of the following result:

Lemma. *(Nakyama's Lemma) Let M be a finitely generated left module over a ring R and let J be the Jacobson radical of R. If $JM = M$ then $M = 0$.*

Proof. Suppose that M is a nonzero finitely generated left R-module. Any chain of proper left R-submodules in M has an upper bound (the union, which is proper because otherwise the finite number of generators for M would be contained in a proper submodule appearing in the chain, which is impossible) so by Zorn's lemma there is a maximal proper left R-submodule $M' \subset M$. Since M/M' is a simple module by maximality of M' it is annihilated by the Jacobson radical, so that $J(M/M') = 0$ which is equivalent to $JM \subset M'$ is a proper subset of M so that $JM \neq M$.

□

1.2. Artinian rings. Any decreasing chain of left R-modules in an Artinian ring R stabilizes, and thus by Zorn's lemma there exist a minimal nonzero left ideal in any nonzero ideal.

Theorem. *Let A be an Artinian ring. The Jacobson radical J of A is a nilpotent ideal.*

Proof. The decreasing chain of ideals $J \supset J^2 \supset J^3$ stabilizes so there is some k such that $J^k = J^{k+1} = J^{k+2} = \cdots$. If $J^k \neq 0$ consider the set $\mathcal{S}$ of left ideals L such that $J^kL \neq 0$. Since A is Artinian every decreasing chain in $\mathcal{S}$ has a minimum element so by Zorn's Lemma there is a minimum ideal L_m in $\mathcal{S}$ such that $J^kL_m \neq 0$. There exists $l \in L_m$ such that $J^kl \neq 0$ so $L = Rl$ by minimality. Then $0 \neq J^kRl \subset L_m$ so by simplicity $J^kRl = Rl$. By Nakayama's lemma applied to the finitely generated module $M = Rl$ we find that $h = 0$, contradicting that $J^kRh \neq 0$. Hence $J^k = 0$ and J is a nilpotent ideal.

□

Since the Jacobson radical contains all nilpotent left ideals, in an Artinian ring A the Jacobson radical is the maximal nilpotent ideal.

Theorem. *Let A be an Artinian ring. The A is semisimple if and only if the Jacobson radical $J(A) = 0$.*

Proof. If A is semisimple then it is isomorphic to a product $\prod_1^r M_{n_i}(D_i)$ for division rings D_i. The only two sided ideals are products of subcollections of the $M_{n_i}(D_i)$ which are not nilpotent because they contain nonzero idempotents.

Conversely suppose the Jacobson radical $J = 0$. Then there are no nilpotent left ideals, so that if I is a simple left ideal, $I^2 = I$. The map $\phi_a : I \to I$ given by $\phi_a(x) = xa$ is an A-module homomorphism of the simple A-module I to itself. Since $I^2 = I$ there is some $a \in A$ such that ϕ_a is an isomorphism onto I and thus

an element $e \in I$ such that $ea = a$. Then $\phi_a(e^2 - e) = eea - ea = 0$ so $e^2 = e$. Every element $z \in A$ can be written $z = ze + (z - ze)$ and $ze \in I, (z - ze)e = 0$. So $R = I + \{r \in R | re = 0\}$ and since ϕ_e is an isomorphism the sum is direct. Thus every simple ideal in R has a complement in R. WE can write $R = I \oplus M_1$ for an ideal M_1, which contains a simple ideal I_2. Writing $R = I_2 \oplus M_2$ we say that $M_1 = I_2 + (M_2 \cap M_1)$ so that $R = I_1 \oplus I_2 \oplus (M_2 \cap M_1)$. Continuing in this fashion gives a descending chain of ideals $(M_1 \cap M_2 \cap M_3 \cap \dots$ which stabilizes at some $M_1 \cap \cap M_k$ which is a simple ideal IK and $R = I_1 \oplus \dots \oplus I_k$ shows that R is a semisimple ring.

□

Corollary. *Let A be an Artinian ring with Jacobson radical A. Then the ring $A/J(A)$ is semisimple.*

Proof. Suppose that $\bar{I} \subset R/J$ is a nonzero nilpotent left ideal, say $\bar{I}^k = 0$. Then there is an ideal I in R properly containing J such that $I^k \subset J$. Since $J^l = 0$ for some l we have $I^{kl} subset J^l = 0$ so that I is nilpotent and strictly larger than J, contradicting that J is the largest nilpotent left ideal in A.

□

Theorem. *Let A be a left Artinian ring. Then A is left Noetherian.*

Proof. Let J be the Jacobson radical of A. It is nilpotent, so there exists k such that $J^k = 0$.

Consider the exact sequence

$$0 \to J^j/J^{j+1} \to R/J^{j+1} \to R/J^j \to 0$$

of R-modules. Since submodules and quotient modules of Artinian modules are Artinian all modules that appear are Artinian. Since R/J is semisimple and J^j/J^{j+1} is an Artinian R/J module it is a finite direct sum of simple ideals, so both $R/J, J^j/J^{j+1}$ are Noetherian modules. Using the exact sequence for $j = 1$ shows that R/J^2 is Noetherian. Inductively the exact sequence then shows that R/J^t is Noetherian for all t, in particular for $t = k$ where $R/J^k \simeq R$.

□

Note that Noetherian does not imply Artinian, as the ring of integers shows.

Recall finite direct sums of simple modules are both Artinian and Noetherian since submodules are finite sums of simple modules, and in proper containment the number of summands strictly increases.