1.

1.1. Introduction. The main goal of these notes is to prove

Theorem. *(Artin-Wedderburn) Let R be a nonzero ring which as a left R-module is a direct sum of irreducible modules. Then R is a finite product of matrix rings $M_{n_i}(D_i), i = 1, \ldots, r$ for division rings D_i with r, n_i, D_i essentially unique.*

This appears as the structure theorem on page 203 in Jacobson II. Since Jacobson actually wants to prove a theorem about more general rings his treatment in 4.1-4.4 includes much more than what is needed to prove the result. In these notes we take a more direct route to the result to emphasize that it depends only on a small number of definitions, lemmas and propositions.

Throughout these notes let R be a ring (not necessarily commutative). It is worthwhile to keep in mind standard examples of rings such as fields, division rings, principal ideal domains, matrix rings, group rings, algebras over a field etc. so that we have examples for various definitions and propositions. By R-module we mean a left module over the ring R unless otherwise noted.

These notes stick closer to the historical development than Jacobson's treatment. The basic idea is to use modules over a ring R as "measuring devices" to find out about the structure of the ring. The irreducible modules, which have no proper nonzero submodules, can be regarded as basic building blocks to construct other modules via direct sums. We call a module completely reducible (or semisimple) if it is a direct sum of irreducible submodules. The result we are aiming for is a description of rings R which are completely reducible as modules over themselves.

1.2. Simple objects in a category. Often in mathematics we study objects by trying to understand the least complicated objects of the type considered and try to understand the more complicated objects in terms of the least complicated. We consider an object in a category to be simple if any epimorphism from the object is either an isomorphism or a map to a zero object in the category. In the category of groups this gives the usual notion of a simple group as one having no nontrivial normal subgroups, in the category of rings a simple ring is one with only 0 and the full ring as two sided ideals (since the image of an epimorphism is a quotient ring). In an abelian category (where every monomorphism is the kernel of its cokernel) we see that a simple object can also be described as one which has only two subobjects, 0 and the object itself. The category of rings (or groups) also shows why we don't take the dual concept by considering objects in which monomorphisms to the object are 0 or isomorphisms : since the polynomial ring $\mathbf{Z}[x]$ maps to every ring, such a ring would have to be quotient of the polynomial ring (in group we would get only the cyclic groups).

It depends on the category whether simple objects can be used in someway to understand more general objects. For finite groups understanding simple groups is only the first step in understanding all finite groups. In certain abelian categories we can try to construct objects as direct sums of simple objects.

In the category of modules (or more generally in an abelian category) a simple object is is one with no nontrivial subobjects . In certain categories, more general objects may be analyzed in terms of simple objects. In an abelian category one could ask if any object was a coproduct of simple objects. In other categories other operations such as products might be used to build up more complicated objects.

1.3. Irreducible modules.

Definition. *A module M is irreducible (or simple) if and only if $M \neq 0$ and the only submodules are $0, M$.*

Since the set of multiples Rm is a left submodule for all $m \in M$, every simple module is isomorphic to a quotient module R/I where I is a maximal (left) ideal of R. A homomorphism of a simple module M to

another module N is either 0 or injective since the kernel is a submodule, so that every nonzero homomorphism from a simple module is an isomorphism to its image. Similarly a homomorphism from any module N' to a simple module M is either 0 or surjective, since the image is a submodule. This proves the following

Lemma. *(Schur's Lemma) A homomorphism from a simple module M to a simple module N is either 0 or an isomorphism. Hence the endomorphism ring of a simple module M is a division ring.*

1.4. **Completely reducible modules.**

Definition. *A module which is the direct sum of simple submodules is called semisimple (or completely reducible).*

Lemma. *Let M be an R-module and let I be an index set. The following are equivalent:*
1) M is a sum of irreducible submodules of M
2) M is a direct sum of irreducible submodules of M
3) For each submodule $N \subset M$ there exists a submodule $N' \subset M$ such that $N \oplus N' = M$.

Proof. Suppose that $M = \sum M_\alpha$ and all M_α are irreducible, For $J \subset I$ let $M(J) = \sum_{\alpha \in I} M_\alpha$. If $J_1 \subset J_2 \subset \cdots I$ is a chain of subsets such that $M(J_i)$ is a direct sum, and let J be the union of J_i. Then $M(J)$ is clearly generated by the set of $M_\alpha, \alpha \in J$. For distinct $\alpha, \beta \in J$ there exists i such that $\alpha, \beta \in J_i$, and since $M(J_i)$ the direct sum of $M_\alpha, \alpha \in J_i$ the intersection $M_\alpha \cap M_\beta = 0$. So every chain among subsets J' such that $M(J'))$ is a direct sum has an upper bound. By Zorn's lemma there is a maximal subset $I' \subset I$ such that $M(I') = \oplus_{\alpha \in I'}$. If for some β $M\beta \cap M_\alpha = 0$ for all $\alpha \in I'$ then $M(I' \cup \{\beta\})$ would be a direct sum, contradicting the maximality of I', so no such β exists. Thus for some $\alpha \in I'$ $M\beta \cap M_\alpha = M_\alpha$ so that $M(I') = M$ establishing that 1 implies 2. Further the irreducible modules exhibiting the complete reducibility of M are chosen from the original M_α.

If N is a submodule of a direct sum $M = \oplus M_{\alpha \in I}$ of irreducible submodules M_α consider subsets such that $N + M(J)$ is a direct sum of the modules $N, M_\alpha, \alpha \in J$. As before, for any chain $J_1 \subset J_2 \subset \cdots I$ of subsets such that $N + M(J)$ is a direct sum has an upper bound, so by Zorn's lemma there is a maximal I' I such that $N + M(I')$ is a direct sum of the modules $N, M_\alpha, \alpha \in I'$. If some M_β did not intersect N or some M_α then $N + M(I' \cup \{\beta\}$ would be a direct sum, contradicting the maximality of I' . Thus every M_β is contained in N or $M(I')$, so that $M = N \oplus M(I')$. Taking $N' = M(I')$ gives 3.

For every nonzero element $m \in M$ the map $\phi : R- > Rm$ exhibits the submodule $Rm \simeq R/ker\phi$ and submodules of Rm correspond to ideals of R under ϕ. The kernel is contained in a maximal left ideal I so that $Im \subset Rm$ is a maximal submodule the module Rm/Im is simple. By 3 $M = Im \oplus N'$ for some submodule N'. Thus $Rm = Im \oplus (N' \cap Rv)$. Since Im is maximal in Rm $N' \cap Rm$ must be an irreducible submodule in Rm so when 3 holds every nonzero submodule contains an irreducible submodule. Let $N \subset M$ be a submodule and let N_0 be the sum of all irreducible submodules of N. By 3, $M = N_0 \oplus N_0'$ so that $N = N_0 \oplus (N_0' \cap N)$. If $N_0' \cap N \neq 0$ it contains a simple submodule which appears in N_0, contradicting that it is a direct sum. Hence $N_0 = N$ and establishes 1 by taking $N = M$.

□

By the remarks preceeding Schur's lemma in section 1.3 a quotient of a completely reducible module M satisfies 1, so is completely reducible, and the simple modules appearing are isomorphic to some simple module appearing in the direct sum giving M. Every submodule of a completely reducible module is isomorphic to a quotient module by 3, so is completely reducible.

Definition. *A ring R is semisimple if and only if it is completely reducible as a left module over itself.*

Since every module is a quotient of a free module, all modules over a semisimple ring R are completely reducible, and every irreducible R-module is isomorphic to an ideal in R.

The only simple module for a field or division ring is isomorphic to the the ring itself. Any module over a division ring D is free, hence a sum of simple modules so that every division ring is semisimple. For commutative rings the simple modules are quotient rings by maximal ideals, and hence are fields, so the completely reducible modules are direct sums of such fields. In the matrix ring $M_n(D)$ of $n \times n$ matrices over a division ring D the left ideal L_j of matrices with all columns except the j-th consisting of zeros is

simple since any nonzero matrix in L_j can be multiplied on the left by a permutation matrix to obtain a matrix in L_j with nonzero entry in the first row, and then left multiplying by matrices that perform standard row operations shows that the matrices E_{ij} with only nonzero entry a 1 in the i-th row, j-th column are all in L_j and these generate L_j as a module over D. The matrix ring is the direct sum of the modules L_j so $M_n(D)$ is semisimple and all simple modules over $M_n(D)$ are isomorphic to $L_j \simeq D^n$. The ideal of right multiples of any nonzero element in $M_n(D)$ contains a nonzero element of some L_j (if the element has i, j entry nonzero multiply by E_{ij}) so any nonzero two-sided ideal contains some L_j. Multiplication on the right by permutation matrices permutes the columns of a matrix, so a nonzero two-sided ideal contains all L_j and hence equals $M_n(D)$, so that $M_n(D)$ is a simple ring.

The following shows that we may know that a ring is semisimple without explicitly knowing the simple modules for the ring:

Theorem. *(Maschke's Theorem) Let G be a finite group and let k be a field. If the order of G is prime to the characteristic of k. Then any left module V over the group ring $k[G]$ is completely reducible.*

Proof. We check property 3 in the first lemma in these notes and show every $k[G]$-submodule $W \subset V$ has a complementary submodule. Every $k[G]$ module is a vector space over k. Since W is a $k[G]$- submodule by complete reducibility of k-modules there exists a k-module W' such that as k vector spaces $V = W \oplus W_1$. Let $\pi : V \to W$ be projection on the first factor. This may not be a $k[G]$-module homomorphism, but the average $\phi(v) = 1/|G| \sum_{g \in G} g(\pi(g^{-1}v))$ makes sense as a k-linear map because G is finite of order prime to the characteristic. It is a $k[G]$-module map since $\phi(hv) = h\phi(v)$ and ϕ is identity on W. Writing $v = \pi(v) + (v - \pi(v))$ shows the exact sequence of $k[G]$-modules $0 \to \ker \phi \to V \to W \to 0$ splits and hence $V = \ker \phi \oplus W$ is completely reducible. □

1.5. **Structure of semisimple rings.** We begin with three lemmas valid for any ring R:

Lemma. *(Orthogonality lemma) Let L be an irreducible left ideal of an arbitrary ring R, and let M be an irreducible R-module and $m \in M$. The homomorphism $\phi(l) = lm$ from $L \to M$ is either 0 or an isomorphism. Thus the module $LM = 0$ if L, M are not isomorphic.*

Proof. $Lm = 0$ or $Lm = M$ by irreducibility of M. Since L is irreducible, in the latter case ϕ is injective, and so gives an isomorphism of L, M . If $LM \neq 0$ then for some $m \in M$ we have $Lm \neq 0$ so that $L \simeq M$.

□

Lemma. *For any ring R the ring $End_R(R)$ of R-module endomorphisms of R as a left module over itself is isomorphic to the opposite ring R^{opp} to R.*

Proof. If $\phi : R \to R$ is an R-module homomorphism then $\phi(r) = \phi(r1) = r\phi(1)$ is determined by right multiplication by the element $\alpha_\phi = \phi(1) \in R$. The composition $\phi \circ \tau(m) = m\alpha_\tau\alpha_\phi$, so $\phi \to \alpha_\phi$ is ring homomorphism $End_R(R) \to R^{opp}$ and is clearly an isomorphism.

□

Lemma. *Let R be any ring and let M be an R-module isomorphic to a direct sum of a finite number of isomorphic submodules M_i, say $M \simeq N^n$. Then $End_R(M)$ is isomorphic to the matrix ring $M_n(End_R(N))$.*

Proof. Write elements of N^n as column vectors with entries from M. Let $\phi \in End(M)$. Given $m \in M_j$ denote the $i-th$ component of $\phi(m)$ by $\phi_{ij}(m)$ Then $\phi_{ij} \in End_R(M)$. For $m_k \in M_k$ we have that $\phi(m_1 + \ldots + m_n)$ is the result of applying the matrix formed by ϕ the to the column vector formed by m_k . This establishes the isomorphism.

□

We use the above lemmas to see how the simple modules for a semisimple ring lead to a description of the ring structure.

Lemma. *A semisimple ring is a direct sum of finitely many simple left ideals $L_1, \ldots, L_k$. Every simple R-module is isomorphic to some L_i.*

Proof. Since $R = \oplus L_\alpha$ where L_α are simple left ideals of R, there exists finitely many $e_i \neq 0$, each in some L_{α_i} such that $1 = e_1 + \ldots e_k$. Thus $R = Re_1 \oplus \cdots \oplus \ldots Re_k$ with $Rei = L_{\alpha_i}$ simple. □

Lemma. *Let $R = L_1 \oplus \cdots \oplus L_n$ be a semisimple ring. Partition the set $S = \{L_1, \ldots L_n\}$ into r equivalence classes S_i under isomorphism, and let R_i denote the direct sum of modules in the class S_i. Then $R = R_1 \oplus \cdots \oplus R_k$ as a direct sum of submodules, each R_i a direct sum of all modules in the equivalence class S_i. We have $R_i R_j = 0$ if $i \neq j$ and $R_i R_i = R_i$ so that R_i are semisimple rings such that $R = \prod_1^r R_i$.*

Proof. Since $R = \oplus_1^n L_i$ it also equals $\oplus_1^r R_i$. The orthogonality lemma shows that $R_i R_j = 0$ if $i \neq j$. Let $1 = e_1 + \cdots + e_n$ with $e_i \in R_i$. Since $e_i e_j = 0$ for $i \neq j$ we get $e_i = 1e_i = e_i^2$ and $R_i = Re_i = e_i R$. Every simple R_i-submodule of R_i is an R-submodule of R. Thus R_i is a ring, a two-sided ideal of R and as a ring $R = \prod R_i$.

□

We will use that $R_i \simeq End_{R_i}(R_i)^{opp}$ to compute R_i.

For a first example, suppose R itself is a simple module. For any nonzero $r \in R$ we have $Rr = R$ so that there is a left inverse s to r. Thus all elements in R have left inverses. Taking a left inverse t of s we get $tsr = r = t$ so that R is a division ring (a ring such that all nonzero elements are invertible). The lemmas at the beginning of this section will be used to study R that are semisimple modules which are not simple modules.

Theorem. *(Artin-Wedderburn) A semisimple ring R as an R-module is isomorphic to a finite direct sum $\oplus L_\alpha$ of simple ideals L_i. Let $L_1, \ldots, L_r$ be representatives for the isomorphism classes of simple ideals in R, and let R_i be the direct sum of all L_α such that $L_\alpha \simeq L_i$ so that $R_i \simeq \oplus_1^{n_i} L_i$. The R-submodules R_i are the minimal two-sided ideals in R and subrings of R and the ring R is the product ring $\prod_1^r R_i$. R_i is isomorphic as a ring to a matrix ring $M_{n_i}(D_i)$ for division rings $D_i = End_R(L_i)^{opp}$.*

Proof. It follows from the previous lemmas that R_i are rings and $R = \prod_1^r R_i$. By the orthogonality lemma L_i is a simple R_i module and every left ideal I in R_i is a left ideal of R, $End_R(I) = End_{R_i}(I)$ and I is simple as an R-module if an only if it is simple as an R_i-module. The lemmas at the beginning of this section show that $End_R(R_i) = End_{R_i}(R_i) \simeq R_i^{opp} = End_R(L_i^{n_i})$. The latter ring is isomorphic to $M_{n_i}(End_{R_i}(L_i))$ and by Schur's lemma $End_R(L_i)$ is a division ring. Thus $R^{opp} \simeq \prod End_{R_i}(R_i) \simeq \prod M_{n_i}(End_R(L_i))$ and $End_R(L_i)$ is a division algebra. Thus R_i is isomorphic to a matrix algebra over D_i, the opposite algebra to the division algebra $End_R(L_i)$ provided by Schur's lemma. Since matrix rings over a division ring are simple rings, R_i are minimal two sided ideals in R (hence are unique)

□

1.6. Uniqueness of expression via the Artin-Wedderburn theorem.

Lemma. *If a finite direct sum of simple R-modules $\oplus_1^r L_i$ is isomorphic to another finite direct sum of simple R-modules $\oplus_1^s M_j$ then r=s and after reordering $L_i \simeq M_i$.*

Proof. Let L_i, M_j be simple R-modules and $f : \oplus_1^r L_i \to \oplus_1^s M_j$ be an isomorphism. Let $L = \oplus_2^r L_i$. Then $L1 \simeq \oplus_1^r L_i/(L) \simeq \oplus_1^s M_j/f(L)$. The left side is irreducible so all but one quotient $M_j/(M_j \cap f(L))$ vanishes. Renumber the M_j so that $M_1/(M_1 \cap f(L) \simeq M_1 \simeq L_1$ and then $M_2 \oplus ...M_s = f(L)$ is an isomorphism with smaller number of modules on the left side. By induction on the minimum of r,s the theorem is proved.

□

Theorem. *If $R = \prod_1^r M_{n_i}(D_i) \simeq \prod_1^{r'} M_{n_i'}(D_i')$ as rings then $r = r'$, and the indices can be reordered such that $n_i = n_i'$ and $D_i \simeq D_i'$ as rings.*

Proof. As a sum of simple modules R is isomorphic to $\oplus n_j D_j^{n_j}$ and to $\oplus n_j' D_j'^{n_j'}$ so by the preceeding lemma we can reorder so that $D_j^{n_j} \simeq D_j^{n_j'}$ and $n_j = n_j'$ and $r = s$. The simple R-module $D_j^{n_j}$ has endomorphism ring $End_R(D_j^{n_j}) \simeq D_j^{opp}$ so we obtain that $D_j^{opp} \simeq D_j'^{opp}$ so that the expressions as products of matrix rings are essentially unique.

□