Mathematics 503 **Complex Analysis** **Fall 2017**

Week 4 workshop problems

1. Let $f(z) = \sum a_n z^n$ be a power series with radius of convergence R. Show Gutzmer's 1888 inequality for $0 < \rho < R$:

$$\sum |a_n|^2 \rho^{2n} \leq sup_{|z|=\rho} |f(z)|^2.$$

Show that this implies the Cauchy inequality for absolute values of derivatives. When can equality occur in Gutzmer's inequality?
Hint: Consider $f(z)f(\bar{z})/z$ as the series $\sum a_n a_m z^n \bar{z}^m$ and check that this can be integrated termwise on the path $|z| = \rho$. Apply Cauchy's theorem to show only the terms with $m = n$ give nonzero contributions and use the integral to give an upper bound for the result.

2. Show that the following functions are holomorphic on the set of nonzero complex numbers. Which functions can be analytically continued to entire functions?.

a) e^z/z^{10}

b) $(e^z - 1)^2/z^2$

c) $z/(e^z - 1)$

d) $(\cos(z) - 1)/z^2$