Mathematics 503 **Complex Analysis** **Fall 2017**

Week 3 workshop problems

For problems for consideration during the workshop I prefer to have problems to discuss besides the weekly assigned homework problems. If time is left over after discussing these workshop problems the homework problems due that week can be discussed.

1. Let a, b be positive real numbers. Let α, β be ellipses passing through the points $(-a, 0), (a, 0)$.

   a) Show using Cauchy's theorem that
   $$\int_\alpha \frac{1}{z}\, dz = \int_\beta \frac{1}{z}\, dz.$$

   b) By appropriate choice of ellipses in part a) show
   $$\int_0^{2\pi} \frac{1}{a^2 cos^2(t) + b^2 sin^2(t)}\, dt = \frac{2\pi}{ab}.$$

2. Suppose that a complex function $f(z)$ defined on open sets Ω_1, Ω_2 in $\mathbf{C}$ has a holomorphic primitive F_i on Ω_i satisfying $F_i'(z) = f(z), z \in \Omega_i$. Show that if $\Omega_1 \cap \Omega_2$ is connected then $f(z)$ is the derivative of a holomorphic function on $\Omega_1 \cup \Omega_2$. If the connectedness assumption is removed is the conclusion still true?

3. Let m, n be integers. Use the partial fraction decomposition and the Cauchy formula for derivatives to compute
   $$\int_{|z|=3} z^n (z-1)^m\, dz$$

4. (This problem was on the previous workshop problem set. If you did not compute the answer $\pi(e-1)/2$ that Cauchy computed analyze the suggested contour integral.) Apply Cauchy's theorem to the function
   $$e^{e^{iz}}/z$$
   and the portion of an annulus between circles of radius R and ϵ in the first quadrant to determine (as Cauchy did) the value of
   $$\int_0^\infty \frac{e^{cos(x)} sin(sin(x))}{x}\, dx$$

5. (This problem was on the previous workshop problem set). Apply Cauchy's theorem to rectangles with vertices $(-T, 0), (T, 0), (T, 1), (-T, 1)$ with circles of radius ϵ centered at $0, 2\pi i$ removed to show that for $0 < a < 1$
   $$\int_{-\infty}^\infty \frac{e^{ax}}{(1 - e^x)}\, dx = \frac{\pi}{\tan(a\pi)}$$
   Be sure to explain what you mean by the improper integral above.